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THE THEORY OF DETERMINANTS in the HISTORICAL ORDER OF DEVELOPMENT

THE THEORY OF DETERMINANTS

IN THE
HISTORICAL ORDER
OF DEVELOPMENT

BY
THOMAS MUIR

FOUR VOLUMES BOUND AS TWO
VOLUME THREE: THE PERIOD 1861 TO 1880.
VOLUME FOUR: THE PERIOD 1880 TO 1900.

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VOLUME THREE

THE PERIOD 1861 TO 1880.

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IN MEMORY OF

M. (B.) M.

PREFACE.

SAVE to a minute extent the present volume, unlike the two preceding it, has not had the advantage of appearing in a preliminary form under the auspices of the Royal Society of Edinburgh. This change of procedure has in no wise been due to a falling off in good will on the part of the Society, to whose helping and encouraging hand I am lastingly indebted, but to the widespread untoward circumstances of the years 1914–18. On the other hand, the delay brought about by the same cause has made repeated revision possible, so that there has been practically an equal opportunity for securing that fullness and accuracy of treatment which have been aimed at from the outset.

The bibliographical groundwork is of course still the five 'Lists of Writings' previously referred to. A sixth that was published in 1916,* containing the results of a final gleaning, has also been fully taken advantage of.

During the period dealt with there has been a still further increase in the number of specialized determinant forms, and of these there has been a proportionate number that have attracted a considerable amount of general attention. Forms of the latter character have each been given separate consideration, and as a consequence the number of chapters in the volume has suffered a corresponding increase. By this course the inevitable scrappiness of a chronologically arranged report of isolated papers is somewhat diminished,

* *Quart. Journ. of Math.*, xlvii. pp. 344–384.

an improvement in readableness is secured, and the book made more readily useful as a work of reference. It is hoped, for example, that the specialist in *skew* determinants who takes up in succession the fourteenth, ninth, tenth chapters of volumes I., II., III. respectively, will find a fairly connected and interesting account of the development of his particular branch of the subject, and also that the general student will obtain with increased ease any information he may desire on some single point of passing interest to him.

The fourth volume, bringing the record up to the end of the nineteenth century, is nearly complete in manuscript. My hope is to complete it and to include in it a detailed index to the whole work.

Rondebosch, South Africa,
23*rd June*, 1918.

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CHAPTER I.

DETERMINANTS IN GENERAL, FROM 1860 TO 1880.

THE number of writings to be considered under this heading amounts to about one hundred and eighty (180), and the number of writers to about one hundred and twenty (120), these being almost three times the corresponding numbers for the twenty-year period immediately preceding. A striking new feature is the large proportion of text-books,—more than a third—there being over sixty (60) of them dealing with determinants alone and several others in which the subject forms an important part. It is also noteworthy that for the first time text-books and other expositions of a really *elementary* character make their appearance, Germany becoming especially productive from the time that the subject began to be introduced into her higher schools. As a further evidence of this increase of interest attention may be called to the fact that, besides the usual number of works in English, French, German and Italian, we have now not only as before three or four publications in Russian, but also contributions in eight of the less widely spread languages of Europe,—Czech, Polish, Dutch, etc.

TERQUEM, O. (1860).

[Origine première des déterminants. *Nouv. Annales de Math.*, xix.,—*bulletin de bibliogr.* . . . pp. 27–33.]

Attention is here drawn to the work of Leibnitz as disclosed in 1850 by the publication of the second volume of the first Abtheilung of his *Mathematische Schriften* (see *Hist.*, i. pp. 6–10). In France and elsewhere determinants were still viewed as having originated with Cramer.

CAYLEY, A. (1860).

[Recent terminology in mathematics. *English Encyclopaedia*, v. pp. 534–542 ; or *Collected Math. Papers*, iv. pp. 594–608.]

The terms here explained are those connected with what we have seen called by Sylvester (1852) the *Calculus of Forms* and by Salmon (1859) *Modern Higher Algebra*. The list includes about two dozen, but some of them, like *determinant*, *quantic*, *invariant*, include quite a number of subordinates.

It recalls the 'Glossary' given by Sylvester at the end of his memoir on Syzygetic Relations (1853).

ISANDER, F. (1860) : OLIVER, J. E. (1860) : BALTZER, R. (1861) : FERRERS, N. M. (1861) : TODHUNTER, I. (1861) : TAIT, P. G. (1861).

[Inledning till determinant-theorien. Dissert. 23 pp. Upsala.]

[A treatise on determinants. Chap. i. Introductory. *Runkle's Math. Monthly*, iii. pp. 86–90.]

[Théorie et applications des déterminants, avec l'indication des sources originelles. Traduit de l'allemand par J. Houel. xii + 235 pp. Paris.]

[An elementary treatise on Trilinear Coordinates. . . . (see pp. 58–71), xii + 154 pp., Cambridge. Second edition, 1866 (see pp. 59–72, 170, 177), xiv + 182 pp. London.]

[An elementary treatise on the Theory of Equations (see pp. 220–258), viii + 279 pp., London. Second edition, 1867 (see pp. 256–297). viii + 318 pp. London.]

[On determinants. A chapter of elementary Algebra. *O. C. and D. Messenger of Math.*, i. pp. 25–37.]

The appearance of these six different expositions in the space of two years proves the increased interest which had come to be taken in the subject. Isander's is a carefully written introduction based on Cauchy's memoir of 1812. Oliver's is a mere fragment. Houel's Baltzer is nothing more than a close translation of the original. Ferrers' is a short chapter simply and clearly written

for beginners. Todhunter's plan and style are quite similar to Ferrers', and his three chapters are therefore still more helpful. Tait's also resembles Ferrers', and is of the same extent.

STOCKWELL, J. N. (1860, July).

[On the resolution of symmetrical equations with indeterminate coefficients. (*Gould's*) *Astron. Journ.*, vi. pp. 145–149.]

Stockwell's subject being the equation of the secular inequalities, he opens with a very short account of determinants. His mode of writing the final expansion of $|00\cdot11\cdot22|$, $|00\cdot11\cdot22\cdot33|$, ... is that which would be reached by using Cauchy's rule of 1841 ; for example,

$$|00\cdot11\cdot22|$$

$$=00\cdot11\cdot22-21\cdot12\cdot00-20\cdot02\cdot11-10\cdot01\cdot22+21\cdot10\cdot02+20\cdot01\cdot12.$$

In specifying the number of terms of each type in the expansion, he gives for the number of terms in a zero-axial determinant of the 3rd, 4th, 5th, ... orders the expressions :

$$1\cdot3-1,$$
$$(1\cdot3-1)\,4+1,$$
$$\{(1\cdot3-1)\,4+1\}5-1,$$
$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

CAYLEY, A. (1860, December).

[Note on the theory of determinants. *Philos. Magazine*, (4), xxi. pp. 180–185 ; or *Collected Math. Papers*, v. pp. 45–49.]

Cayley's 'mode of arrangement of the developed expression of a determinant' is in its result the same as Stockwell's, and is not therefore essentially different from Cauchy's of the year 1841 ; as written, however, his development is more compendious than Cauchy's, and deserves attention. For all three it is necessary that the elements of the determinant be specified by their row and column numbers.

He denotes

	11, 22, ... ;	12·21,	12·23·31,	13·32·21, ...
by	|1|, |2|, ... ;	|12|,	|123|,	|132|, ...,

and, when by reason of multiplication two of the upright lines come together, he leaves one out, writing for example

$$|12|2|123| \quad \text{for} \quad |12|\cdot|2|\cdot|123|.$$

The developments of $\Sigma(\pm 11\cdot 22)$, $\Sigma(\pm 11\cdot 22\cdot 33)$, ..., he then writes

$$\begin{array}{lll} +|1|2| & +|1|2|3| & +|1|2|3|4| \\ -|1\ 2|, & -|1\ 2|3| & -|1\ 2|3|4| \\ & +|1\ 2\ 3|, & +|1\ 2\ 3|4| \\ & & +|1\ 2|3\ 4| \\ & & -|1\ 2\ 3\ 4|, \ldots, \end{array}$$

the sign in any case being determined by taking -1 as a factor for every compartment containing an even number of umbrae. He omits to note, however, that he also intends a sign of summation. Σ say, to be prefixed mentally in each case : for example,

$$-|12|3|4| \quad \text{should be} \quad -\Sigma|12|3|4|,$$

as it stands for the six-termed aggregate

$$-12\cdot 21\cdot 33\cdot 44 - 13\cdot 31\cdot 22\cdot 44 - \ldots - 34\cdot 43\cdot 11\cdot 22.$$

We may remark that in the notation of his paper of the year before (*Hist.*, ii. pp. 469–470)—a paper of which this is a natural extension—Cayley would have given for the next case the expression

$$\begin{array}{l} 11\ 22\ 33\ 44\ 55 - \Sigma 12\ 21\cdot 33\ 44\ 55 + \Sigma 12\ 23\ 31\cdot 44\ 55 \\ +\Sigma 12\ 21\cdot 34\ 43\cdot 55 - \Sigma 12\ 23\ 34\ 41\cdot 55 - \Sigma 12\ 23\ 31\cdot 45\ 54 \\ +\Sigma 12\ 23\ 34\ 45\ 51 : \end{array}$$

whereas he now gives

$$\begin{array}{l} |1|2|3|4|5| \\ -\Sigma|1\ 2|3|4|5| \\ +\Sigma|1\ 2\ 3|4|5| \\ +\Sigma|1\ 2|3\ 4|5| \\ -\Sigma|1\ 2\ 3\ 4|5| \\ -\Sigma|1\ 2\ 3|4\ 5| \\ +\Sigma|1\ 2\ 3\ 4\ 5|, \end{array}$$

which, besides its compactness, has the advantage of bringing into evidence the part played in the matter by the partitions of the order-number of the determinant.

Further, there should be noted the difference between this and the same author's development of the year 1847 (*Hist.*, i. p. 42).

The two agree as far as the third line only: the fourth and fifth lines above are combined into one in the earlier development, the cofactor of |5| appearing there as a zero-axial determinant; and a similar substitute appears for the aggregate of the sixth and seventh lines.

SMITH, H. J. S. (1861).

[On systems of linear indeterminate equations and congruences. *Transac. R. Soc. London*, cli. pp. 293–326; or *Collected Math. Papers*, i. pp. 367–409.]

As the equations dealt with are for solution in integral numbers, the subject of this memoir belongs strictly to the Theory of Integers. There are, however, several points of contact with Determinants, just as in the case of previous papers of the same kind by Hermite (1849), Bazin (1854), Heger (1856).

Thus, he resuscitates Cayley's row-by-column multiplication of an n-line determinant by an array of n rows and $n+h$ columns (1843, *Hist.*, ii. p. 16), using, for example, the statement

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \cdot \left\| \begin{matrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{matrix} \right\| = \left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix} \right\|$$

to stand for six equalities of the type

$$|a_1 b_2| \cdot |c_1 d_2| = |x_1 y_2|.$$

In this connection, too, he draws attention to a necessary distinction by saying that $|a_1 b_2|$ is here *postmultiplied* * by $|c_1 d_2|$, and that $|c_1 d_2|$ is *premultiplied* by $|a_1 b_2|$.

The main subject of the memoir was continued in three papers published under the conjoint title "Arithmetical Notes" in the *Proceed. London Math. Soc.*, iv. (1873), pp. 236–253.

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* In Cayley's other theorem of 1843 the determinant is *premultiplied* by an array, for example,

$$\left\| \begin{matrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{matrix} \right\| \cdot \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 \\ \rho_1 & \rho_2 & \rho_3 & \rho_4 \end{vmatrix} = \left\| \begin{matrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \end{matrix} \right\|$$

which stands for six equalities of the type

$$\|c_1 d_4\| \cdot \|\lambda_1 \mu_4\| = |\xi_1 \eta_2|,$$

the two factors on the left being now, not determinants, but 2-by-4 arrays.

ZANNOTTI, M. (1861).

[Elementi della Scienza del Calculo. 2ª Parte, Elementi di Algebra. x+570 pp. Napoli.]

The Third Section of the First Book, extending to 120 pages, is headed, 'Teorica dei Determinanti.' It is professedly due to Trudi, and, so far as it goes, does not differ materially from Trudi's text-book of the following year.

JORDAN, C. (1861).

[Mémoire sur le nombre des valeurs des fonctions, *Journ. de l'Éc. Polyt.*, xxii. cah. 38, pp. 113–194.]

The title here is similar to that of the memoir which Cauchy * wrote as an introduction to his great monograph of 1812 on determinants (*Hist.*, i. pp. 93–94). The writer's purpose, however, is quite different, his subjects being *substitutions*, *matrices* (not so called), and *groups*. The nearest approach to the field of determinants is in the fifth chapter (pp. 179–187, 119), which concerns the question of 'the number of linear substitutions,' and which is occupied with a proof that *the number of systems of values of* a_{rs} *satisfying the condition*

$$|a_{1n}| \gtrless 0 \quad (mod.\ p)$$

is

$$p^{\frac{1}{2}n(n-1)}(p-1)(p^2-1)\ldots(p^n-1),$$

where p *is a prime.*

GRASSMANN, H. (1861).

[Die Ausdehnungslehre. xii+388 pp. Berlin.]

This is not the second part which was promised in 1844, but a recast of the whole work. 'Systems of units,' like Cauchy's algebraic keys, are now defined; and sums of multiples of the

* Besides the important writers referred to by Jordan we may note Despeyrous, more especially as he was about this time also studying another cognate subject, namely, *permutations*. (See *Liouville's Journal*, 2e sér. x. pp. 55–64, vi. pp. 417–439, x. pp. 177–202).

members of such a system are called 'extensive Grösse.' Determinants are no longer ignored. Early in the chapter on the 'Combinatorial Product' (chap. iii. pp. 31–107) the ordinary definition and the Σ notation are given; and thereafter the functions are repeatedly referred to. They are even found to be needed for the statement of theorems in the 'Ausdehnungslehre' itself, for example, the theorem on the inner product of two magnitudes each of the m^{th} 'Stufe' and consisting of m simple factors. On p. 130 this is expressed in the form

$$[abc\ldots|a'b'c'\ldots] = \text{Determ.}\begin{cases}[a|a'] & [a|b'] & [a|c']\ldots \\ [b|a'] & [b|b'] & \\ [c|a'] & [c|b'] & \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\end{cases}$$

The solution of a set of simultaneous linear equations which formerly came at the end of a chapter on 'Outer Multiplication' now occupies a similar position in chapter iii. In this matter, too, there is a new procedure which is of considerable interest when brought into comparison with the old. Using as multipliers with the equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right\}$$

the 'extensive magnitudes' $e_1, e_2, \ldots, e_n$, whose 'combinatorial product' is 1, and performing addition, we obtain an equation which may be written

$$x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = \beta,$$

if we put

$$\alpha_1 \text{ for } a_{11}e_1 + a_{21}e_2 + \ldots + a_{n1}e_n, \text{ etc.}$$

This equation, Grassmann says, takes the place of the given equations, and in order to find the value of any one of the x's, x_1 say, we apply the factor $\alpha_2\alpha_3\ldots\alpha_n$ to both sides, and on account of the vanishing of $[\alpha_2\alpha_2\alpha_3\ldots\alpha_n]$, $[\alpha_3\alpha_2\alpha_3\ldots\alpha_n]$, … the result is

$$x_1[\alpha_1\alpha_2\ldots\alpha_n] = [\beta\alpha_2\alpha_3\ldots\alpha_n].$$

BAEHR, G. F. W. (1861).

[Propriété des coefficients du binôme et théorème Sylvester sur les déterminants. *Nouv. Annales de Math.*, xx. pp. 417–420.]

The theorem in question is Sylvester's unimportant second lemma of 1852, Oct. (*Hist.*, ii. p. 77).

COINTE, I. L. A. LE (1861).

[Mémoire sur les déterminants cramériens ou résultants algébriques. *Annali di Mat.*, iv. pp. 233–245.]

This so-called memoir, lengthy and tiresomely precise, ends only in a rule like Reiss' (1829) and Cantor's (1855) for telling the sign of a permutation when the ordinal number of the permutation is known.

PAINVIN, L. (1862).

[Mémoire sur les tétraèdres. *Nouv. Annales de Math.*, (2) i. pp. 267–286, etc.]

The memoir opens with a preliminary theorem in determinants (pp. 267–271) which, if stated in the form

$$\left\{\text{adjug}\,|\,a_{1n}\,|\right\}^2 \;=\; \text{adjug}\left\{|\,a_{1n}\,|^2\right\},$$

would justly be viewed as merely a case of a theorem of Cauchy's of the year 1812 (*Hist.*, i. p. 121). It must be noted, however, that what Painvin establishes is something more important, namely, the identity of the two determinants 'élément à élément,' that is to say, the identity of their matrices. (Cf. *Hist.*, ii. p. 95.)

TRUDI, N. (1862).

[Teoria de' Determinanti, e loro applicazioni. xii + 268 pp. Napoli.]

This is a text-book similar in scope to those of Brioschi and Baltzer, but intended for less advanced readers. The explanations are consequently fuller, the illustrations are more numerous, the less simple theorems are broken up for gradual absorption, and to deductions that are self-evident are given formal enunciations.

The result of course is that considerably more space is occupied. The scheme of arrangement resembles Baltzer's, there being a first part devoted to the theory and a second part to the so-called applications. Whereas, however, Trudi apportions to the parts 112 and 156 pages respectively, the like apportionment in Houel's Baltzer is 63 and 172; further, on account of the unequal treatment of geometry 250 pages of Trudi's 268 are algebraical as against 170 of Baltzer's 235.

As an instructive illustration of the growing solicitude for the reader's convenience, we may point out that whereas Spottiswoode (1851) merely notes that

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} A+B & B \\ B+C & C \end{vmatrix},$$

and Brioschi (1854) with equal brevity extends the property to a general n-line determinant, and Baltzer (1857) at little more length but with more of a teacher's skill provides a formal enunciation, like Jacobi's original, and two helpful examples, Trudi on the other hand not only prolongs Baltzer's treatment to half a page, but follows this up usefully with a whole page of application to the computation of a single arithmetical determinant of the fourth order, illustrating in particular how the order is reducible when an element of a line is a submultiple of each of the other elements. Similarly, instead of the few lines which Spottiswoode and Baltzer give to Jacobi's theorem

$$d|a_{1n}| = \sum_{h,k} \mathrm{A}_{h,k} da_{h,k}$$

Trudi devotes two pages (pp. 53–55), and is the first to put the development in the afterwards familiar form of the sum of n determinants.

Save this marked fullness and clearness of explanation there is not much fresh to be noted so far as general determinants are concerned. A number of additional expressions of more or less importance are by definition given a special sense for the sake of definiteness or convenience; for example (p. 9), 'matrici simili,' 'linee omologhe,' etc., and (p. 17) 'caratteristica,' which is used in connection with a minor to denote the sign-index, that is to say the integer ν where the sign-factor of the minor is $(-1)^{\nu}$. In the

most prominent definition of all, namely, that of a *determinant*, he is the first to take both row-numbers and column-numbers into account in fixing the sign of a term, his words being (p. 12), " dando ad ogni prodotto il + o il −, secondochè le due correspondenti permutazioni di orizzontali e di verticali sono di una stessa classe, o di classe diversa ; o in altri termini, secondochè il numero totale delle loro inversioni è pari o impari." A result which he reaches when dealing with 'inversions' is also worth noting, namely, *If any permutation of* 1, 2, 3, . . . , *n be divided into any two parts, thus,*

$$a_1,\ a_2,\ \ldots,\ a_m | a_{m+1},\ \ldots,\ a_n,$$

and p+q+r *be the number of inverted pairs in it,* p *being the number in the first part,* q *the number in the second, and* r *the number which the individuals of the first part make with those of the second, then*

$$\begin{aligned} r &= (a_1 - 1) + (a_2 - 2) + \ldots + (a_m - m) \\ &= \Sigma a - \tfrac{1}{2}m(m+1). \end{aligned}$$

When, however, we get beyond the section on skew determinants (p. 94), we come on matter that is worthy of closer attention, the subject most conspicuous for fresh treatment being that dealt with by us under the heading Bigradients.

BATTAGLINI, G. (1862, 1871).

[Nota sui determinanti. *Rendic. dell' Accad. . . . Napoli*, i. pp. 101–112 ; or *Giornale di Mat.*, ix. pp. 136–144.]

The problem here attempted to be solved is the finding of an m-line determinant equal to the sum of the m-line minors of an n-line determinant. The results being burdened with exceptions are a little disappointing. One is that if

$$\sigma_{rs} = a_{rs} - a_{r,s+n-m} - a_{r+n-m,s} + a_{r+n-m,s+n-m},$$

then $|\sigma_{11}\sigma_{22}\ldots\sigma_{mm}|$ is equal to the sum of all the m-line minors of $|a_{1n}|$ excepting those whose complementaries contain two row-numbers or two column-numbers which are congruent with respect to $n-m$.

We may note for ourselves that the important case of this where $m = n-1$ and where there are no exceptions is readily deduced from

the theorem suggested by Routh (1864). Thus, the sum of the signed primary minors of $|\, a_1 b_2 c_3 d_4 \,|$ being known to be

$$-\begin{vmatrix} . & 1 & 1 & 1 & 1 \\ 1 & a_1 & a_2 & a_3 & a_4 \\ 1 & b_1 & b_2 & b_3 & b_4 \\ 1 & c_1 & c_2 & c_3 & c_4 \\ 1 & d_1 & d_2 & d_3 & d_4 \end{vmatrix},$$

we would only have to perform the operations

$$\text{col}_2 - \text{col}_3, \quad \text{col}_3 - \text{col}_4, \quad \text{col}_4 - \text{col}_5,$$
$$\text{row}_2 - \text{row}_3, \quad \text{row}_3 - \text{row}_4, \quad \text{row}_4 - \text{row}_5,$$

to obtain

$$\begin{vmatrix} a_1 - a_2 - b_1 + b_2 & a_2 - a_3 - b_2 + b_3 & a_3 - a_4 - b_3 + b_4 \\ b_1 - b_2 - c_1 + c_2 & b_2 - b_3 - c_2 + c_3 & b_3 - b_4 - c_3 + c_4 \\ c_1 - c_2 - d_1 + d_2 & c_2 - c_3 - d_2 + d_3 & c_3 - c_4 - d_3 + d_4 \end{vmatrix},$$

which is Battaglini's form. It may also be pointed out that the elements in the latter form are three-line minors of the former.

SMITH, H. J. S. (1862).

[Report on the theory of numbers. Part IV. *Report . . . . British Assoc. . . . .* xxxii. (Cambridge), pp. 503–526; or *Collected Math. Papers*, i. p. 229.]

In § 105, p. 504, Smith denotes a determinant, whose elements are double-suffixed, by placing between two upright lines a single element with variable suffixes, and then appending an indication of the extent of the variability. For example, the determinant

$$\sum\left(\pm \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}\cdots\frac{\partial y_n}{\partial x_n}\right) \quad \text{or} \quad \left|\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}\cdots\frac{\partial y_n}{\partial x_n}\right|$$

he writes

$$\left|\frac{\partial y_\alpha}{\partial x_\beta}\right| \qquad \alpha = \beta = 1, 2, \ldots, n.$$

Four years later Kronecker made similar use of $|\, b_{ik} \,|$. (See *Crelle's Journ.*, lxviii. p. 276.)

CATALAN, E. (1863).

[Une propriété des déterminants. *Mém. . . . Soc. roy. des sci.* (Liège), (2) xii. pp. 218–221 ; or *Mélanges math.*, i. pp. 218–221.]

Taking the set of equations

$$\left.\begin{array}{rcrcrcl} x_1|b_2c_3| & + & x_2|a_2c_3| & + & x_3|a_2b_3| & = & 0, \\ x_1|b_1c_3| & + & x_2|a_1c_3| & + & x_3|a_1b_3| & = & 0, \\ x_1|b_1c_2| & + & x_2|a_1c_2| & + & x_3|a_1b_2| & = & |a_1b_2c_3| \end{array}\right\}$$

and comparing the evident solution

$$x_1,\ x_2,\ x_3 = a_3,\ -b_3,\ c_3$$

with the solution obtained in the ordinary way by means of determinants, Catalan is able to arrive at results already well-known regarding the adjugate.

MIKELLI, A. (1863).

[Dei Determinanti. pp. (?). Mantua.]

SALMON, G. (1863, 1866, 1868).

[Vorlesungen zur Einführung in die Algebra der linearen Transformationen. Deutsch bearbeitet von Dr. Wilhelm Fiedler. viii+271 pp. Leipzig.]

[Lessons introductory to the modern higher algebra. 2nd edition. viii+296 pp. Dublin.]

[Leçons d'algèbre supérieure. Traduit de l'anglais par M. Bazin. Augmenté de notes par M. Hermite. xii+247 pp. Paris.]

Fiedler improves the original by giving additional illustrations and other matter, the number of 'lessons' on determinants being increased from three to four. Salmon himself does likewise, but increases the number of lessons to six, marking for the first time the importance of "the reduction and calculation of determinants," by assigning thereto a separate lesson. Bazin, on the other hand, makes considerable curtailments.

In none of the three does anything really fresh appear in regard to the general theory.

RUBINI, R. (1863–1867).

[Elementi di Algebra. 2ª ediz. 381 pp. Napoli. 3ª ediz. iv+295 pp. 1866.]

[Complementi agli Elementi di Algebra. Napoli, 1863. 2ª ediz. viii+484+iv. pp. 1867.]

Rubini was a skilled expositor. The third edition of the 'Elementi,' a superior text-book, contains in two chapters (vii. pp. 217–230, x. pp. 249–292) of nine lessons an excellent account of the subject, occupying about twice the space given by Todhunter. In the 'Complementi' there are an equally helpful lesson on elimination (xxxi. pp. 244–258), a second (xliii. pp. 337–343) on Jacobians and Hessians, and other 'applications' less important.

ROUTH, E. J. (1864).

[Note on the invariants of the equation of the second degree. *Quart. Journ. of Math.*, vi. pp. 270–278.]

The carelessly worded statement (p. 273), that "*the determinant*

$$\begin{vmatrix} f & n & m & l \\ n & g & k & l \\ m & k & h & l \\ l & l & l & . \end{vmatrix}$$

is the sum of the minors of all the terms of the determinant

$$\begin{vmatrix} f & n & m \\ n & g & k \\ m & k & h \end{vmatrix}$$"

seems to be the first approximation to a later formulated theorem regarding the sum of the signed primary minors of any determinant.

BALTZER, R. (1864).

[Theorie und Anwendungen der Determinanten. 2te vermehrte Auflage. vi+224 pp. Leipzig.]

The enlargement referred to in the title would have amounted to about forty pages if the former size of page had been retained,

and ten of these would have belonged to the 'Theorie.' Only a few of the additions, however, are new.

One of these is Gordan's mode of verifying the extended multiplication-theorem (§ 4. 8, § 5. 3). This will be fully understood from its application to one of the simplest cases, namely, the case of the two arrays,

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}, \qquad \begin{matrix} h_1 & h_2 & h_3 \\ k_1 & k_2 & k_3 \end{matrix}.$$

Putting the left-hand member of the desired identity in the form

$$\begin{vmatrix} a_1h_1+a_2h_2+a_3h_3 & a_1k_1+a_2k_2+a_3k_3 & a_1 & a_2 & a_3 \\ b_1h_1+b_2h_2+b_3h_3 & b_1k_1+b_2k_2+b_3k_3 & b_1 & b_2 & b_3 \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{vmatrix},$$

he transforms this into

$$\begin{vmatrix} . & . & a_1 & a_2 & a_3 \\ . & . & b_1 & b_2 & b_3 \\ -h_1 & -k_1 & 1 & . & . \\ -h_2 & -k_2 & . & 1 & . \\ -h_3 & -k_3 & . & . & 1 \end{vmatrix},$$

and then uses Laplace's expansion-theorem to obtain

$$|a_1b_2|\cdot|h_1k_2| \;+\; |a_1b_3|\cdot|h_1k_3| \;+\; |a_2b_3|\cdot|h_2k_3|,$$

as Sylvester did in 1852 (*Hist.*, ii. pp. 199–200 ; also p. 57).

A second result worth noting is a contribution of Kronecker's (§ 4. 7) to the subject of Sylvester's 'homaloidal law.' Following on Cayley's work of 1843 (*Hist.*, ii. p. 15), Sylvester had asserted in 1850 (*Hist.*, ii. p. 51) that *by making* $(n-k+1)^2$ *of the* k*-line minors of an* n*-line determinant vanish all will vanish.* What Kronecker now adds is that *the* $(n-k+1)^2$ *vanishing minors must have a common non-evanescent primary minor*; for example, all the four-line minors of $|a_1b_2c_3d_4e_5|$ will vanish if

$$|a_1b_2c_3d_4| = 0, \qquad |a_1b_2c_3d_5| = 0,$$
$$|a_1b_2c_3e_4| = 0, \qquad |a_1b_2c_3e_5| = 0,$$

and

$$|a_1b_2c_3| \neq 0.$$

It is just worth noting also that in illustration of a well-known elementary property which we have spoken of under Trudi (1862), there is given (p. 16) the excellent example

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \frac{1}{x} \begin{vmatrix} a_1x+a_2y+a_3z & a_2 & a_3 \\ b_1x+b_2y+b_3z & b_2 & b_3 \\ c_1x+c_2y+c_3z & c_2 & c_3 \end{vmatrix},$$

with its bearing on the question of the solution of a set of simultaneous linear equations.

BABCZYŃSKI (1864–5); ZAJACZKOWSKI, W. (1865–6).

[O determinantach. (In his lectures on *Higher Alg.*, pp. 289–320.)]

[O wyznacznikach. (In his lectures on *Analyt. Geom.*, pp. 72–88.)]*

MEILINK, B. (1865): POKORNÝ, M. (1865): DÖLP, H. (1866).

[Over de determinanten. Acad. proefschrift. viii+61 pp. Leiden.]

[Determinanty a vyšší rovnice. iv+135 pp. V Praze.]

[Die Determinanten als Gegenstand des Gymnasialunterrichts, Sch. Progr. Giessen.]

Meilink's degree-dissertation is a 60-page exposition influenced in parts by Brioschi's text-book. In Pokorný's book the section (pp. 1–40) devoted to the subject resembles Todhunter's sketch in extent and somewhat also in character. Dölp's school-program is of slighter interest.

HORNER, J. (1865).

[Notes on determinants. *Quart. Journ. of Math.*, viii. pp. 157–162.]

Horner's first result is a slight extension of Chio's of the year 1853, and therefore is a companion to Hermite's lemma of the year 1849, both of the latter being viewable as giving a reduction of a determinant to one of the next lower order. Taking $|a_{11}a_{22}\ldots a_{nn}|$ Horner produces a column of 1's by dividing the rows in order by $a_{1q}, a_{2q}, \ldots$; next he subtracts the p^{th} row from each of the others. thus obtaining a determinant of the $(n-1)^{\text{th}}$ order: then by multi-

* Neither of these works have I seen.

plication he clears each row of fractions; and finally he adjusts his multipliers and divisors. For example, n, p, q being 4, 2, 3, the result is

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \frac{(-1)^5}{b_3^2} \begin{vmatrix} |a_1b_3| & |a_2b_3| & |a_4b_3| \\ |c_1b_3| & |c_2b_3| & |c_4b_3| \\ |d_1b_3| & |d_2b_3| & |d_4b_3| \end{vmatrix}.$$

We may note that both Horner's result and Hermite's are viewable also as a case of the theorem regarding a minor of the second compound (see under Spottiswoode, 1853, *Hist.*, ii. pp. 201–205).

ZMURKO, L. (1866).

[Beitrag zur Theorie des Grössten und Kleinsten der Functionen mehrerer Variablen, nebst . . . *Denkschr. . . . Akad. d. Wiss.* (Wien), Math.-Nat. Cl., xxvii. (2), pp. 63–82.]

This is a curiously belated effort. In § 3 (pp. 76–80), which is headed 'Ueber die Combinatorische Determinante,' the elementary properties of the subject are dealt with as if new, nothing, however, but the notation

$$\begin{pmatrix} 1 & \cdots & n \\ & & \\ n & & \end{pmatrix}$$

being fresh. Two pages (pp. 80–82) are then given to a known property of the Wronskian.

RENSHAW, A. (1866).

[Identities, their composition and relation to determinants. *O. C. and D. Messenger of Math.*, iv. pp. 34–36.]

[Easy method of deducing consecutively the permutations of the first n integers. *O. C. and D. Messenger of Math.*, iv. pp. 36–37.]

Renshaw obtains the final expansion of a determinant by using the recurrent law of formation in the cyclic form adopted by Vandermonde and Cayley: both papers are unimportant.

DODGSON, C. L. (1866).

[Condensation of determinants, being a new and a brief method for computing their arithmetical values. *Proceed. R. Soc. London*, xv. pp. 150–155.]

Dodgson's condensation-process for the evaluation of determinants whose elements are arithmetical has a different basis from the theorems of Hermite (1849) and Chio (1853). Stated in the form of a rule it is : *Compute every two-line minor consisting of four adjacent elements in the given determinant* A, *and form with the results in order a square array* B : *compute every similar two-line minor of* B, *performing however this time the additional operation of division by the particular element of* A *which entered into the composition of the four elements of the minor in question, and form with the quotients in order a new array* C : *thereafter let successive arrays be computed in the same way as* C *until an array is reached with only one element : this element is the value of the original determinant.* For example, the given determinant being

$$\begin{vmatrix} -2 & -1 & -1 & -4 \\ -1 & -2 & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{vmatrix},$$

the series of derived arrays is

$$\begin{matrix} 3 & -1 & 2 \\ -1 & -5 & 8 \\ 1 & 1 & -4, \end{matrix} \qquad \begin{matrix} 8 & -2 \\ -4 & 6, \end{matrix} \qquad -8.$$

and the value required is -8.

The rule is stated to be dependent on Jacobi's theorem regarding a minor of the adjugate, and is formally proved for determinants of the 3rd and 4th orders. Care is also taken to show how the difficulty caused by the appearance of a zero as one of the divisors can be obviated, and how the labour of computation may be minimized when a number of related determinants have to be evaluated, as happens in solving a set of simultaneous linear equations.

We may note for ourselves in passing that by applying the rule to a determinant with general elements we may obtain not only

the necessary justification for its use but a fuller and better understanding of the whole process. Thus, the given determinant being

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix},$$

the successive derived arrays are

$$\begin{matrix} |a_1b_2| & |a_2b_3| & |a_3b_4| & |a_4b_5| \\ |b_1c_2| & |b_2c_3| & |b_3c_4| & |b_4c_5| \\ |c_1d_2| & |c_2d_3| & |c_3d_4| & |c_4d_5| \\ |d_1e_2| & |d_2e_3| & |d_3e_4| & |d_4e_5| \end{matrix},$$

$$\begin{matrix} |a_1b_2c_3| & |a_2b_3c_4| & |a_3b_4c_5| \\ |b_1c_2d_3| & |b_2c_3d_4| & |b_3c_4d_5| \\ |c_1d_2e_3| & |c_2d_3e_4| & |c_3d_4e_5| \end{matrix},$$

$$\begin{matrix} |a_1b_2c_3d_4| & |a_2b_3c_4d_5| \\ |b_1c_2d_3e_4| & |b_2c_3d_4e_5| \end{matrix}, \qquad |a_1b_2c_3d_4e_5|.$$

In this way we see that the determinants of the intermediate arrays are not equivalents of the given determinant: indeed, if the given determinant be of the n^{th} degree in the elements, the second is of the degree $(n-1)2$, the third of the degree $(n-2)3$, the fourth of the degree $(n-3)4$, and so on by a rise and a subsequent fall until the degree $1\cdot n$ is reached.

TAIT, P. G. (1866): MARTIN, H. (1867).

[Note on determinants of the third order. *Proceed. R. Soc. Edinburgh*, vi. pp. 59–61.]

[Note on Professor Tait's 'quaternion path' to determinants of the third order. *Proceed. R. Soc. Edinburgh*, vi. pp. 121–125.]

As the scalar portion of the product of three vectors

$$ix+jy+kz, \quad ix'+jy'+kz', \quad ix''+jy''+kz''$$

is the negative of the determinant

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix},$$

it is natural that workers with quaternions should have properties of three-line determinants suggested to them. Tait draws attention

to this, and having given half-a-dozen instances is followed by Martin seeking to show how they can be dealt with by means of the ordinary laws of determinants.

The only property of the six that has an appearance of freshness is one which resembles Catalan's of the year 1846 (*Hist.*, ii. p. 38). We may formulate it for ourselves as follows : *If from any* n*-line determinant* Δ *we form another* Δ' *such that the* r^{th} *row of* Δ' *is the sum of all the rows of* Δ *except the* r^{th}, *then*

$$\Delta' = (-1)^{n-1}(n-1)\Delta.$$

For example,

$$\begin{vmatrix} x'+x'' & y'+y'' & z'+z'' \\ x''+x & y''+y & z''+z \\ x+x' & y+y' & z+z' \end{vmatrix} = 2 \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}.$$

As Martin points out, this is readily established by performing the operations

$$\begin{aligned} &\text{row}_1+\text{row}_2+\ \ldots\ +\text{row}_n, \quad \text{row}_1 \div (n-1), \\ &\text{row}_2-\text{row}_1, \quad \text{row}_3-\text{row}_1, \quad \ldots\ , \\ &\text{row}_1+\text{row}_2+\ \ldots\ +\text{row}_n. \end{aligned}$$

SARDI, C. (1867).

[Nuova dimostrazione del prodotto di due matrici. *Giornale di Mat.*, v. pp. 174–177.]

The proof is not essentially different from Gordan's as given by Baltzer (1864), the steps of procedure being the same but in reverse order, and being therefore suitably illustrated by the footnote at the end of our account of Sylvester's paper of 1852, Dec. (*Hist.*, ii. p. 200). Sardi's has more appearance of being the result of a suggestion from the equatement of the two denominators in Spottiswoode's proof of 1851.

HANKEL, H. (1867).

[Vorlesungen über die complexen Zahlen und ihre Functionen. I. Theil, Theorie der complexen Zahlensysteme. xii+196 pp. Leipzig.]

What concerns us here is the first five pages (pp. 119–124) of the seventh chapter. They recall vividly Cauchy's memoir of

twenty years earlier on 'algebraic keys' (*Hist.*, ii. pp. 43–46) and the relative portions of Grassmann's work of 1861.

The 'alternating units' $i_1, i_2, \ldots, i_n$ being such that

$$i_r i_s = -i_s i_r, \qquad i_r i_r = 0,$$

'alternating numbers' are defined as sums of multiples of

$$i_1, i_2, \ldots, i_n.$$

Taking the product of two such numbers, n being made 4 for shortness' sake, we readily see that

$$(a_1 i_1 + a_2 i_2 + a_3 i_3 + a_4 i_4)(b_1 i_1 + b_2 i_2 + b_3 i_3 + b_4 i_4)$$
$$= |a_1 b_2| i_1 i_2 + |a_1 b_3| i_1 i_3 + |a_1 b_4| i_1 i_4 + |a_2 b_3| i_2 i_3 + |a_2 b_4| i_2 i_4 + |a_3 b_4| i_3 i_4,$$

whence it follows that if the two numbers be denoted by α, β, we have

$$\alpha\beta = -\beta\alpha, \qquad \alpha\alpha = 0;$$

in other words, that alternating numbers obey the same multiplication-laws as the units on which they are based. Multiplying the product by a third number

$$c_1 i_1 + c_2 i_2 + c_3 i_3 + c_4 i_4, \quad \text{or } \gamma, \text{ say,}$$

we have

$$\alpha\beta\gamma = |a_1 b_2 c_3| i_1 i_2 i_3 + |a_1 b_2 c_4| i_1 i_2 i_4 + |a_1 b_3 c_4| i_1 i_3 i_4 + |a_2 b_3 c_4| i_2 i_3 i_4.$$

Similarly

$$\alpha\beta\gamma\delta = |a_1 b_2 c_3 d_4| i_1 i_2 i_3 i_4,$$

and

$$\alpha\beta\gamma\delta\epsilon = 0.$$

The penultimate result is the basis of the proposition that *a general determinant of the* nth *order can by means of alternating numbers be resolved into* n *linear factors*,—a proposition from which, as Cauchy originally pointed out, all the fundamental properties of determinants are readily deducible. For example, by reason of the identity

$$\alpha\beta\gamma\delta = \alpha\beta \cdot \gamma\delta,$$

we have

$$\begin{aligned} |a_1 b_2 c_3 d_4| = {} & |a_1 b_2| \cdot |c_3 d_4| - |a_1 b_3| \cdot |c_2 d_4| + |a_1 b_4| \cdot |c_2 d_3| \\ & + |a_2 b_3| \cdot |c_1 d_4| - |a_2 b_4| \cdot |c_1 d_3| \\ & + |a_3 b_4| \cdot |c_1 d_2|, \end{aligned}$$

thus showing how Laplace's expansion-theorem in all its generality may be obtained.

In a historical note (p. 140) Hankel makes definite reference to Grassmann and Cauchy, although perhaps too sparingly in the case of the latter.

NATANI, L. (1867).

[Substitution, lineare. *Hoffmann's Math. Wörterbuch*, vi. pp. 605–685.]

The subjects dealt with in this concise article correspond closely with those of such a book as Salmon's *Modern Higher Algebra*, and about 30 pages of it (pp. 606–637) strictly concern determinants, all the results given being carefully formulated.

REISS, M. (1867).

[Beiträge zur Theorie der Determinanten. viii + 113 pp. Leipzig.]

It is only the first section of this long and important memoir that comes up here for consideration. As its title 'Ueber erweiterte Determinanten' implies, it concerns theorems of the kind known at a later date as 'extensionals' and exemplified in the writings of Desnanot (1819), Schweins (1825), Reiss himself (1839), and others.

In the matter of notation Reiss is no longer peculiar. His symbolism for a determinant is now professedly the same as Sylvester's, and his mode of specifying a sum of products of pairs of determinants is not far removed from that of Schweins. A closer imitation of the latter,—whom, by the way, he never mentions, —would not have been a disadvantage: and instead of Sylvester he might more appropriately have mentioned Desnanot for the additional reason just given. Thus, the expansion of a five-line determinant in terms of the minors formed from the 2nd and 4th rows and the minors formed from the 1st, 3rd and 5th rows is indicated by

$$si\begin{pmatrix}2&4&1&3&5\\1&2&3&4&5\end{pmatrix}\sum_{12345}^{\beta_1\beta_2}\epsilon\begin{pmatrix}2&4\\\beta_1&\beta_2\end{pmatrix}\begin{pmatrix}1&3&5\\\beta_3&\beta_4&\beta_5\end{pmatrix},$$

where, firstly, $\begin{pmatrix} 2 & 4 \\ \beta_1 & \beta_2 \end{pmatrix}\begin{pmatrix} 1 & 3 & 5 \\ \beta_3 & \beta_4 & \beta_5 \end{pmatrix}$ is the typical product of two complementary minors: secondly, the accessories to the $\sum$ imply that for β_1, β_2 are to be taken any two integers from 1, 2, 3, 4, 5; and, thirdly, the remaining items are connected with the determination of sign.

In the way of new contributions the first point to be noted is that, whereas preceding writers had dealt with the 'extension' of only special cases of Laplace's expansion-theorem, Reiss gave the full generalization of the theorem. For example, while Desnanot showed that the identities

$$\begin{aligned} |a_1b_2c_3d_4\ldots| &= a_1|b_2c_3d_4\ldots| - a_2|b_1c_3d_4\ldots| + \ldots, \\ 0 &= |a_1b_2||a_3b_4| - |a_1b_3||a_2b_4| + |a_1b_4||a_2b_3|, \end{aligned}$$

could be 'extended,' Reiss affirmed and sought to show that the theorem

$$\begin{pmatrix} a_1 & a_2 \ldots a_n \\ a_1 & a_2 \ldots a_n \end{pmatrix} = \sum_{a_1 \ldots a_n}^{\beta_1 \ldots \beta_k} \epsilon \begin{pmatrix} a_1 \ldots a_k \\ \beta_1 \ldots \beta_k \end{pmatrix}\begin{pmatrix} a_{k+1} \ldots a_n \\ \beta_{k+1} \ldots \beta_n \end{pmatrix}$$

was a mere preliminary to

$$\begin{pmatrix} a_1 \ldots a_n\, t_1 \ldots t_m \\ a_1 \ldots a_n\, \tau_1 \ldots \tau_m \end{pmatrix}\begin{pmatrix} t_1 \ldots t_m \\ \tau_1 \ldots \tau_m \end{pmatrix} = \sum_{a_1 \ldots a_n}^{\beta_1 \ldots \beta_k} \epsilon \begin{pmatrix} a_1 \ldots a_k\, t_1 \ldots t_m \\ \beta_1 \ldots \beta_k\, \tau_1 \ldots \tau_m \end{pmatrix}\begin{pmatrix} a^{k+1} \ldots a_n\, t_1 \ldots t_m \\ \beta_{k+1} \ldots \beta_n\, \tau_1 \ldots \tau_m \end{pmatrix}$$

and that the like held in regard to all vanishing aggregates of products of pairs of determinants. He also takes the further step of 'extending' that form of Laplace's expansion-theorem which gives $\begin{pmatrix} a_1 & a_2 \ldots a_n \\ a_1 & a_2 \ldots a_n \end{pmatrix}$ as an aggregate of products of more than two minors, affirming that if the number of minors in each product be r the left-hand member of the identity is

$$\begin{pmatrix} a_1 & a_2 \ldots a_n & t_1 \ldots t_m \\ a_1 & a_2 \ldots a_n & \tau_1 \ldots \tau_m \end{pmatrix}\begin{pmatrix} t_1 \ldots t_m \\ \tau_1 \ldots \tau_m \end{pmatrix}^{r-1}.$$

More important, however, is the fact that having set himself to inquire what the left-hand member of Laplace's expansion-theorem would become if each first factor in the right-hand member were

'extended' by $\begin{pmatrix} t_1 \dots t_m \\ \tau_1 \dots \tau_m \end{pmatrix}$, he arrives at Schweins' theorem for transforming an aggregate of products of pairs of determinants into another aggregate of similar kind (*Hist.*, i. pp. 165–172), and then advances to the further conclusion that this theorem also can be 'extended.' Further, he notes that a theorem of Sylvester's in compound determinants (*Hist.*, ii. pp. 60–61) is an instance * of 'extension,' and that it may be deduced from Jacobi's theorem regarding a minor of the adjugate. Lastly, he gives what he calls an application of Sylvester's and Jacobi's theorems to functions of the form

$$\sum e_a A_a c_{a_1} \dots c_{a_n},$$

where the c's are determinants, the A's are coefficients independent of the c's, the e's are signs, and the function as a whole is, like the individual determinants, homogeneous in the elements. Such a 'doppelt-homogene Function,' if it vanishes identically, he finds has two interesting properties, namely, (1) it may be 'extended' and still vanish, (2) the identity will continue to hold if the determinants be replaced by their coefficients in another determinant to which they all belong as minors. It is also carefully noted that the same properties belong to the difference of two such functions if they be identical.

It is much to be regretted that these theorems of Reiss' received no attention from his contemporaries. His work is not even mentioned by any of the German text-books giving bibliographical references, for example, Baltzer's editions of 1870, 1875, 1881 and Günther's of 1875, 1877. Had it been otherwise, the generalisations known as the Law of Complementary Minors and the Law of Extensible Minors would have been formulated much earlier than they were.

The second and third Sections of the memoir (pp. 25–99) deal with *compound* determinants, and the fourth Section (pp. 100–113) with geometrical applications.

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* As we have already seen, Bazin's theorem of 1851 is another instance: also Chio's of 1853; indeed the latter may be viewed as an extreme case of Sylvester's (see *Hist.*, ii. pp. 206–207, 80).

DODGSON, C. L. (1867).

[AN ELEMENTARY TREATISE ON DETERMINANTS, with their application to simultaneous linear equations and algebraical geometry. viii+143 pp. London.]

This is a text-book quite unlike all its predecessors. Professedly its main aim is logical exactitude. In pursuance thereof all definitions, conventions, axioms, propositions and corollaries are carefully formulated, labelled and numbered: every step in a train of reasoning is kept scrupulously separate from its antecedent and consequent, and in order to guard against possible contamination of the reasoning illustrative examples are relegated to the footnotes. Clearness is also sought to be promoted by the use of more than the ordinary variety of printers' type, and precision by the introduction of new terms and symbols. One of the latter, namely,

$$r\backslash s,$$

which is used for the element in the $(r, s)^{\text{th}}$ place, is naturally of frequent occurrence. The consequence is that the pages, though broad-margined and well-printed, present an unwonted and rather bizarre appearance.

Leaving out twenty-two propositions (chap. vii., viii.) in coordinate geometry, we find that there remain forty-eight connected with algebra. Of the latter twenty-three (chap. iii., iv.) concern linear equations, and thirteen (chap. v., vi.) the evanescence of 'blocks' (that is, arrays), leaving only twelve (chap. i., ii.) devoted to permutations and determinants.

The most important chapters are iii., iv., v., vi., the subjects which they concern being treated in more detail than in any previous text-book.

Chapters v. and vi. (pp. 63–75) concern arrays, and are in form related to one another as are iii. and iv., which concern equations, chap. vi. giving the tests for evanescence of an array just as chap. iv. gives the tests for consistency of a set of equations. Chapter v. is in two sections, the one dealing with 'data resulting in evanescence,' and the other with 'consequences of evanescence.' We shall confine ourselves to one proposition of chap. v. and two of chap. vi. For this purpose it must be premised that an 'oblong'

array is one of length m and breadth n, where $m > n$: and that the lines of m elements are called the 'longitudinals' and the lines of n elements the 'laterals' of the array. The propositions,—which should be viewed in connection with Kronecker's of 1864,—are (pp. 67, 73–74) :

(1) *If one of the* k*-line minors of an array does not vanish, but all the minors of the* $(k+1)^{th}$ *order which include the said minor do vanish, then every other minor of the* $(k+1)^{th}$ *order must vanish also.*

(2) *If an oblong array have one of its secondary minors non-evanescent, a necessary and sufficient test for the array itself being evanescent is that of its primary minors each one be evanescent that contains the said secondary minor.*

(3) *A necessary and sufficient test for an array being evanescent is that either every element of it be zero, or that two or more of its longitudinals form an array provable to be evanescent by the first test.* Proof of part of the second may be given by way of sample. Taking the array

$$\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ e_1 & e_2 & e_3 \end{matrix}$$

with the data

$$|\, a_1 b_2 \,| \neq 0,$$

$$|\, a_1 b_2 c_3 \,| = 0, \qquad |\, a_1 b_2 d_3 \,| = 0, \qquad |\, a_1 b_2 e_3 \,| = 0,$$

Dodgson in effect affirms that two magnitudes x and y can be found such as to satisfy the equations

$$\left.\begin{aligned} a_1x+a_2y+a_3 &= 0 \\ b_1x+b_2y+b_3 &= 0 \\ c_1x+c_2y+c_3 &= 0 \\ d_1x+d_2y+d_3 &= 0 \\ e_1x+e_2y+e_3 &= 0 \end{aligned}\right\}.$$

For, he says, by reason of data (1) and (2) the first three equations are consistent : and similarly data (1) and (3) may be used : and

data (1) and (4). The entire set is thus consistent, and the initial array consequently evanescent.*

The freshest of the appendices is the fifth (pp. 128–133), which deals with the problem of constructing a determinant of monomial elements which will vanish simultaneously with a given algebraic expression of two or more terms. Here the fundamental theorem is that *if in an* n*-line determinant all the elements of a* p*-by-*q *array be multiplied by* x, *and all the elements of the complementary array be divided by* x, *the determinant is multiplied by* x^{p+q-n}. Taking then the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix},$$

he in the first place divides the $(2, 2)^{\text{th}}$ element by itself and multiplies each of the elements of its complementary minor by the same: secondly, he divides the $(3, 3)^{\text{th}}$ element and multiplies the elements of its complementary minor by i: thirdly, he multiplies the 2^{nd} and 3^{rd} rows by b and c respectively: and fourthly, he multiplies the 2^{nd} and 3^{rd} columns by d and g respectively. The result of all this is

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = bdi \cdot ceg \begin{vmatrix} aei & bdi & ceg \\ bdi & bdi & bfg \\ ceg & cdh & ceg \end{vmatrix},$$

where the elements of the determinant on the right are five of the terms of the determinant on the left. It is consequently suggested to construct from any six-termed expression

$$\alpha+\beta+\gamma+\delta+\epsilon+\zeta$$

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* Of course if a 4^{th} column were taken with the additional data

$$|a_1b_2c_4| = 0, \quad |a_1b_2d_4| = 0, \quad |a_1b_2e_4| = 0,$$

another 5-by-3 array could be proved evanescent: and still another with the data

$$|a_1b_2c_5| = 0, \quad |a_1b_2d_5| = 0, \quad |a_1b_2e_5| = 0.$$

We should then have reached a *square* array, whose 3-by-5 minor arrays we could treat in the same fashion without additional data, and so prove that if each of the 9 three-line minors be evanescent, of which the non-evanescent $|a_1b_2|$ is a common minor, then all the other three-line minors would vanish.

the three-line determinant

$$\begin{vmatrix} \alpha & -\epsilon & -\delta \\ -\epsilon & -\epsilon & \gamma \\ -\delta & \beta & -\delta \end{vmatrix},$$

and finding this to be equal to

$$\delta\epsilon\left(\alpha+\beta+\gamma+\delta+\epsilon-\frac{\alpha\beta\gamma}{\delta\epsilon}\right)$$

Dodgson concludes that *any expression of six terms such that the product of three of them differs only in sign from the product of the other three can be expressed as a three-line determinant.*

It deserves to be remarked in passing that although Dodgson considers only the cases of a three-line and of a four-line determinant, it is quite generally true that *any* n-*line determinant can be expressed by means of* $1+(n-1)^2$ *of its terms.* Thus, denoting by *pqrst* any term $a_p b_q c_r d_s e_t$ of the determinant $|\, a_1 b_2 c_3 d_4 e_5 \,|$ we have

$$|\, a_1 b_2 c_3 d_4 e_5 \,| = \frac{\begin{vmatrix} 12345 & 21345 & 32145 & 42315 & 52341 \\ 21345 & 21345 & 23145 & 24315 & 25341 \\ 32145 & 31245 & 32145 & 32415 & 32541 \\ 42315 & 41325 & 42135 & 42315 & 42351 \\ 52341 & 51342 & 52143 & 52314 & 52341 \end{vmatrix}}{21345 \cdot 32145 \cdot 42315 \cdot 52341},$$

or, if we denote by $\overline{rs}$ the result of performing on 12345 the interchange of r and s, and by $\overline{rs} \cdot \overline{uv}$ the result of performing on $\overline{rs}$ the interchange of u and v,

$$|\, a_1 b_2 c_3 d_4 e_5 \,| = \frac{\begin{vmatrix} \overline{11} & \overline{12} & \overline{13} & \overline{14} & \overline{15} \\ \overline{21} & \overline{21} \text{ or } \overline{12} & \overline{21}\cdot\overline{13} & \overline{21}\cdot\overline{14} & \overline{21}\cdot\overline{15} \\ \overline{31} & \overline{31}\cdot\overline{12} & \overline{31} \text{ or } \overline{13} & \overline{31}\cdot\overline{14} & \overline{31}\cdot\overline{15} \\ \overline{41} & \overline{41}\cdot\overline{12} & \overline{41}\cdot\overline{13} & \overline{41} \text{ or } \overline{14} & \overline{41}\cdot\overline{15} \\ \overline{51} & \overline{51}\cdot\overline{12} & \overline{51}\cdot\overline{13} & \overline{51}\cdot\overline{14} & \overline{51} \text{ or } \overline{15} \end{vmatrix}}{\overline{12}\cdot\overline{13}\cdot\overline{14}\cdot\overline{15}}$$

ECHEGARAY, J. (1868): DAVIDOV, A. J. (1868): "UN ABONNÉ" (1868).

[Memoria sobre la teoria de las determinantes. *Memorias y documentos* (Madrid), viii. pp. 244–444.]

[Algebra (in Russian). Moskva.]

[Exposé des principes élémentaires de la théorie des déterminants. *Nouv. Annales de Math.*, (2) vii. pp. 403–413, viii. pp. 97–113.]

The author of the first * of these is said to have considered his work little more than a free translation of Trudi's. The third is of no moment.

HESSE, O. (1868).

[Ein Determinantensatz. *Crelle's Journ.*, lxix. pp. 319–322; or *Werke*, pp. 557–560.]

The theorem in question is, in Hesse's notation, that if

$$\Delta = |a_{11}a_{22}\ldots a_{nn}|, \quad B = \partial\Delta/\partial a_{pq}, \quad P = \sum_{\kappa}\frac{\partial B}{\partial a_{\kappa\beta}}a_{\kappa q}\cdot\sum_{\lambda}\frac{\partial B}{\partial a_{\alpha\lambda}}a_{p\lambda},$$

then

$$\Delta\frac{\partial B}{\partial a_{\alpha\beta}} + P = B\frac{\partial B}{\partial a_{\alpha\beta}}a_{pq} - B\sum_{\kappa\lambda}\frac{\partial^2 B}{\partial a_{\alpha\beta}\partial a_{\kappa\lambda}}a_{\kappa q}a_{p\lambda}.$$

It is brought forward and established mainly for the sake of the case where $B = 0$, the deduction being that Δ is then expressible as the product of two factors of the form

$$(\lambda_1 a_{1q}+\lambda_2 a_{2q}+\ldots+\lambda_n a_{nq})(\mu_1 a_{p1}+\mu_2 a_{p2}+\ldots+\mu_n a_{pn});$$

or, taking without loss of generality $p = q = n$ and $\alpha = \beta = n-1$, that

$$-\Delta\frac{\partial B}{\partial a_{n-1,n-1}} = \left\{\frac{\partial B}{\partial a_{1,n-1}}a_{1n} + \frac{\partial B}{\partial a_{2,n-1}}a_{2n} + \ldots + \frac{\partial B}{\partial a_{n-1,n-1}}a_{n-1,n}\right\}$$
$$\cdot\left\{\frac{\partial B}{\partial a_{n-1,1}}a_{n1} + \frac{\partial B}{\partial a_{n-1,2}}a_{n,2} + \ldots + \frac{\partial B}{\partial a_{n-1,n-1}}a_{n,n-1}\right\}.$$

* Neither this nor the second have I been fortunate enough to see.

We are thus led up to the important fact that when in addition Δ is axisymmetric, the two factors on the right are identical. The theorem may consequently be viewed as a generalisation of Salmon's of the year 1859.

It may be noted, however, in passing that Hesse's notation rather obscures the true character of his result, which is nothing more nor less than the simplest case of Jacobi's theorem regarding a minor of the adjugate. For dispensing with differentiation by using A_{rs} for the cofactor of a_{rs} in Δ, we readily see that B is A_{pq} : that the first factor of P is the determinant got from B by changing the suffix β into q, that is to say, is $A_{p\beta}$: that similarly the second factor is $A_{\alpha\gamma}$: that $\partial B/\partial a_{\alpha\beta}$ is the cofactor of $|\, a_{\alpha\beta}a_{pq}\,|$ in Δ : that

$$\frac{\partial B}{\partial a_{\alpha\beta}}a_{pq} - \sum_{\kappa,\lambda}\frac{\partial^2 B}{\partial a_{\alpha\beta}\partial a_{\kappa\lambda}}a_{\kappa q}a_{p\lambda} \quad \text{is} \quad A_{\alpha\beta}:$$

and that therefore Hesse's result is

$$\begin{aligned} |\, a_{11}a_{22}\ldots a_{nn}\,| \cdot \text{cofactor of } |\, a_{\alpha\beta}a_{pq}\,| &= A_{pq}A_{\alpha\beta} - A_{p\beta}A_{\alpha q} \\ &= |\, A_{\alpha\beta}A_{pq}\,|. \end{aligned}$$

STUDNIČKA, F. J. (1869, 1870) : ZELEWSKI, A. v. (1870).

[O determinantech. 64 pp. V Praze.]

[Einleitung in die Theorie der Determinanten. Für Studirende an Mittelschulen und technischen Anstalten. vi+66 pp. Prag.]

[The Elements of the Theory of Determinants (in Russian). Prag.]

[Ein Beitrag zur Theorie der Determinanten. 31 pp. Breslau.]

The German title of Studnička's booklet accurately describes its character. It consists of an introduction followed by ten sections (§§ 1–10) under descriptive headings, each of the last four sections dealing shortly with a special determinant by name. The edition in Russian, published the same year, I have not seen.

Zelewski's so-called ' Beitrag ' is a very simple but very formally arranged introductory sketch.

TRZASKA, W. (1870).

[Krótkie wiadomości o wyznacznikach. Appendix to Folkierski's *Zasady ruchunku różniczkowego i calkowego* . . . . Tom i. xlvi+1087 pp. Paris.]

Trzaska's 'short account of determinants' extends to forty-nine pages (pp. 1031–1079), and, as is appropriate enough in a book dealing with the Differential Calculus, a considerable share of space (pp. 1059–1075) is devoted to 'functional determinants.' It is followed by a collection of seventeen exercises (pp. 1080–1087).

BALTZER, R. (1870).

[Theorie und Anwendung der Determinanten. 3te verbesserte Auflage. vi+241 pp. Leipzig.]

In this edition the 'Theorie' is enlarged to the extent of four or five pages. One interesting addition (p. 31) is a simple deduction from the identity

$$\begin{vmatrix} a_1-x & a_2 & a_3 \\ b_1 & b_2-y & b_3 \\ c_1 & c_2 & c_3-z \end{vmatrix}$$

$$= -xyz+a_1yz+b_2zx+c_3xy - |\,b_2c_3\,|\,x - |\,c_3a_1\,|\,y - |\,a_1b_2\,|\,z + |\,a_1b_2c_3\,|.$$

Putting x, y, z equal to the sums of the a's, b's, c's respectively, we have

$$0 = -\Sigma a\Sigma b\Sigma c+a_1\Sigma b\Sigma c+b_2\Sigma c\Sigma a+c_3\Sigma a\Sigma b - |\,a_1b_2\,|\,\Sigma c - |\,b_2c_3\,|\,\Sigma a - |\,c_3a_1\,|\,\Sigma b + |\,a_1b_2c_3\,|,$$

a result which, notwithstanding Baltzer's different standpoint, we may view as a generalisable relation between the coaxial minors of the determinant $|\,a_1b_2c_3\,|$ and the sums of its rows.

Another addition (§ 5. 8, pp. 46–49) concerns the result of the substitution which is possible when we have given

$x_1, x_2, \ldots, x_n$ linear functions of $x_1', x_2', \ldots, x_n'$,

$x_1', x_2', \ldots, x_n'$ linear functions of $x'', x_2'', \ldots, x_n''$,

and so on. The subject, however, belongs more naturally to the theory of matrices (see § 11 of Cayley's memoir of 1857).

UNTERHUBER, A. (1870): HOUEL, J. (1871): MAURER, F. (1872): UNTERHUBER, A. (1872).

[Einleitung in die Theorie der Determinanten. Leoben.]

[Notions élémentaires sur les déterminants. Paris. (Lithographed.)]

[Grundzüge der Determinantenlehre. Budweis.]

[Ueber Determinanten. 14 pp. Leoben.]

The second of these accurately describes itself, and the others are school-programs of the simplest type.

CAYLEY, A. (1871).

[Question 3329. *Educ. Times*, xxiii. p. 261, xxiv. p. 17; or *Math. from Educ. Times*, xv. pp. 66–67.]

The requirement here is to show that every permutation of 12345 can be obtained by means of the cyclical substitution (12345), and the interchange (12); and it is met by giving the equivalents of other interchanges in terms of the said two operations, K and α say. For example, the interchange (15) can be effected by performing in succession the six operations K, α, K, K, K, K, or, as this is written,

$$(15) = \mathrm{K}^4\alpha\mathrm{K}.$$

Similarly $$(13) = \alpha\mathrm{K}\alpha\mathrm{K}^4\alpha = \alpha\mathrm{K}^2\alpha\mathrm{K}\alpha\mathrm{K} = \ldots,$$

the variant forms being due to the facts

$$\mathrm{K}^5 = 1, \quad \alpha^2 = 1, \quad (\mathrm{K}\alpha)^4 = 1.$$

BRILL, A. (1871).

[Ueber diejenigen Curven eines Büschels, ... *Math. Annalen*, iii. pp. 459–468.]

Brill has incidentally to consider an identity which is the case of Schläfli's theorem of 1851, in which n, $r = 3, 2$, that is to say, the case which gives a six-line determinant, $\Delta_{3,2}$ say, as the equiva-

lent of $|a_1b_2c_3|^4$. The said determinant he multiplies row-wise by $2|b_1c_2|\cdot|b_2c_3|\cdot|b_3c_1|$ in the form

$$\begin{vmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ 2|b_2c_3| & . & . & . & |b_1c_2| & |b_3c_1| \\ . & 2|b_3c_1| & . & |b_1c_2| & . & |b_2c_3| \\ . & . & 2|b_1c_2| & |b_3c_1| & |b_2c_3| & \end{vmatrix}$$

obtaining a determinant equal to

$$2|a_1b_2c_3|^4\cdot\begin{vmatrix} b_1^2 & b_2^2 & b_3^2 \\ c_1^2 & c_2^2 & c_3^2 \\ b_1c_1 & b_2c_2 & b_3c_3 \end{vmatrix};$$

and, as the last three-line determinant is equal to

$$|b_1c_2|\cdot|b_2c_3|\cdot|b_3c_1|,$$

the desired object is obtained.

JAROŠENKO, S. (1871).

[Teorija opredělitelej i jeja priloženija. I. Teorija opredělitelej. xii+159 pp. Odessa.]

This text-book, which has almost the same title as Sperling's of 1858, contains a very full account of the subject, the fullness being possible by the allocation of so much space to the theory alone. It opens with a historical introduction of ten pages, and then follow three chapters (pp. 1–74) on determinants in general and other three on special forms, that on Compound Determinants (chap. iv.) being more than ordinarily full.

CALDARERA, F. (1871).

[Nota su talune proprietà dei determinanti, in ispecie di quelli a matrici composte con la serie dei numeri figurati. *Giornale di Mat.*, ix. pp. 223–232.]

Caldarera's fundamental theorem is Chio's of 1853, regarding the reduction of the order of a determinant, and is established in

essentially the same way. The only difference is that Caldarera takes $q = p$ from the outset, and insists on the importance of the theorem in evaluating axisymmetric determinants, and in reaching such a result as

$$\begin{vmatrix} a & a+\delta & a+2\delta & a+3\delta & \dots \\ a & 2a+\delta & 3a+3\delta & 4a+6\delta & \dots \\ a & 3a+\delta & 6a+4\delta & 10a+10\delta & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n = a^n,$$

which, of course, is easily obtained otherwise.

BOURGET, J. (1871).

[Des permutations. *Nouv. Annales de Math.*, (2) x. pp. 254–268.]

This resembles in subject the first part of Rothe's paper of 1800 (*Hist.*, i. pp. 55–60), the main difference arising from the adoption of a different principle for determining the order in which the $n!$ permutations of the first n integers have to be arranged. The order followed is Bezout's of 1764 (*Hist.*, i. p. 15), namely, when n is 3,

123, 132, 312, 213, 231, 321,

instead of

123, 132, 213, 231, 312, 321.

It has no advantage save that the r^{th} permutation from the end is obtainable from the r^{th} permutation from the beginning by reversing the order of the integers, a fact to which Bourget does not refer. A new definition of 'conjugate' permutations is introduced which entails the equivalence of *conjugate* and *reverse*, and which therefore, unlike the old, does not permit of the interesting idea of self-conjugateness. The fact, too, that *the total number of inverted-pairs in a permutation and its now so-called conjugate is* $\frac{1}{2}n(n-1)$ becomes self-evident when we change the word 'conjugate' into 'reverse.' The paper concludes with an investigation of the problem of finding the number of permutations which each have δ inverted-pairs, the result being practically successful although a general difference-equation is not obtained.

BECKER, J. C. (1871).

[Ueber einen Fundamentalsatz der Determinantentheorie. *Zeitschrift f. Math. u. Phys.*, xvi. pp. 526–530.]

This is a well-reasoned and clearly written criticism of Salmon and Baltzer for laxity of thought in their treatment of the proposition regarding the interchange of rows and columns. At the close the critic brings matters to a point by formulating for proof the theorem :

If the elements taken to form a term of a determinant be first arranged so that the row-indices are in their natural order, the unsigned result being

$$a_{1x}a_{2y}a_{3z}a_{4w} \ldots, \quad \textit{or } A, \textit{ say,}$$

and afterwards arranged so that the column-indices are in their natural order, the unsigned result being

$$a_{\xi 1}a_{\eta 2}a_{\zeta 3}a_{\omega 4} \ldots, \quad \textit{or } B, \textit{ say,}$$

then the number of inverted-pairs in x, y, z, w, ... *is the same as in* $\xi, \eta, \zeta, \omega, \ldots$.

As his own contribution he shows that the two numbers are at least of the same kind as regards influence on the sign. In effect he says that if one of the interchanges requisite for the transformation of A into B be the interchange of a_{2y} and a_{4w}, and this be performed, the double set of indices will be changed from

$$\begin{matrix} x & y & z & w \ldots \\ 1 & 2 & 3 & 4 \ldots \end{matrix} \quad \text{into} \quad \begin{matrix} x & w & z & y \ldots \\ 1 & 4 & 3 & 2 \ldots \end{matrix},$$

and the number of inverted-pairs being thus increased or diminished in each set by an odd number, the total number in the two sets will be increased or diminished by an even number. Further, the like being the case for every such interchange it follows that the number of inverted-pairs in the first double set

$$\begin{matrix} x & y & z & w \ldots \\ 1 & 2 & 3 & 4 \ldots \end{matrix}$$

will differ from the number in the last derived double set

$$\begin{matrix} 1 & 2 & 3 & 4 \ldots \\ \xi & \eta & \zeta & \omega \ldots \end{matrix}$$

by an even number, and consequently the sign determined from B will be the same as that determined from A.*

We note for ourselves in passing that $x, y, z, w, \ldots$ and $\xi, \eta, \zeta, \omega, \ldots$ are 'conjugate' permutations according to Rothe's definition of 1800. (*Hist.*, i. pp. 59–60.)

HESSE, O. (1871, 1871, 1872): HATTENDORFF, K. (1872).

[Die Determinanten, elementar behandelt. ii+46 pp. Leipzig.]

[De Determinanten, elementair behandeld door Dr. O. Hesse. In het Hollandsch overgebracht door Dr. F. van Wageningen. iv+48 pp. S'Gravenhage.]

[Die Determinanten, elementar behandelt. 2te Auflage. ii+48 pp. Leipzig.]

[Einleitung in die Lehre von den Determinanten. xii+60 pp. Hannover.]

In October 1870 the theory of determinants was included among the subjects prescribed to be taught in the Bavarian real-gymnasia, and Hesse sought to supply the want which thus arose for an elementary text-book. In his *Analytische Geometrie des Raumes* he had already devoted a lecture to the subject. (1st ed. of 1861, pp. 72–83: 2nd ed. of 1869, pp. 79–99.) As might have been expected, his new exposition is in its way excellent, twenty pages being devoted to linear equations and alternating functions, and the rest to sixteen carefully formulated propositions regarding determinants. It is more suitable, however, for teachers than for gymnasium pupils.

Hattendorff, on the other hand, aims at keeping the wants of the young student constantly in view, and therefore assumes less at the outset, advances more slowly, illustrates more fully by examples, and does not neglect to give numerous exercises for practice.

BRILL, A. (1871).

[Ueber zwei Berührungsprobleme. *Math. Annalen*, iv. pp. 527–549.]

Brill, in the discussion of his geometrical problems requires the aid of two theorems regarding determinants, and to these at the

* A variant of this proof was afterwards given in the second edition of Dölp's class-book (p. 18, 1877).

outset he consequently devotes a separate section (§ 2, pp. 530–533).

The first theorem may be formulated as follows: *If two arrays* C *and* D, *each consisting of* n−1 *rows and* n *columns, be so related to one another that in every case the product of the* r^{th} *and every succeeding row of* C *by the* r^{th} *row of* D *vanishes, then the principal minors of the last* k *rows of* C *are proportional to the principal minors of the first* n−k *rows of* D, *the series of minors in the latter case being arranged in reverse order.* Taking for shortness' sake the case where n is 5 and k is 2, the arrays being

$$\begin{array}{ccccc} a_1 & a_2 & \dots & a_5 \\ b_1 & b_2 & \dots & b_5 \\ c_1 & c_2 & \dots & c_5 \\ d_1 & d_2 & \dots & d_5 \end{array} \qquad \begin{array}{ccccc} x_1 & x_2 & \dots & x_5 \\ y_1 & y_2 & \dots & y_5 \\ z_1 & z_2 & \dots & z_5 \\ w_1 & w_2 & \dots & w_5, \end{array}$$

and the given relations therefore representable by

$$\begin{array}{llll} \Sigma ax = 0, & & & \\ \Sigma bx = 0, & \Sigma by = 0, & & \\ \Sigma cx = 0, & \Sigma cy = 0, & \Sigma cz = 0, & \\ \Sigma dx = 0, & \Sigma dy = 0, & \Sigma dz = 0, & \Sigma dw = 0, \end{array}$$

we append 1 row to C and prefix k rows to the first $n-k$ rows of D, thus obtaining the square arrays whose determinants are

$$|\, a_1 b_2 c_3 d_4 e_5 \,|, \qquad |\, m_1 n_2 x_3 y_4 z_5 \,|.$$

By the multiplication-theorem the product of these

$$= \begin{vmatrix} \Sigma am & \Sigma an & . & \Sigma ay & \Sigma az \\ \Sigma bm & \Sigma bn & . & . & \Sigma bz \\ \Sigma cm & \Sigma cn & . & . & . \\ \Sigma dm & \Sigma dn & . & . & . \\ \Sigma em & \Sigma en & \Sigma ex & \Sigma ey & \Sigma ez \end{vmatrix}$$

$$= \Sigma ex \cdot \Sigma ay \cdot \Sigma bz \cdot \begin{vmatrix} \Sigma cm & \Sigma cn \\ \Sigma dm & \Sigma dn \end{vmatrix}$$

$$= \Sigma ex \cdot \Sigma ay \cdot \Sigma bz \cdot \begin{Vmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{Vmatrix} \cdot \begin{Vmatrix} m_1 & m_2 & m_3 & m_4 & m_5 \\ n_1 & n_2 & n_3 & n_4 & n_5 \end{Vmatrix},$$

from which, by equating the cofactor of $|\, m_h n_k \,|$ on the one side to its cofactor on the other, we obtain the desired result.

The second theorem concerns what may be called in imitation of Dodgson (1867) the 'coevanescence of arrays.' We may enunciate it for ourselves thus: C *and* D *being a pair of arrays conditioned as in the previous theorem, and* Γ, Δ *a similarly related pair, if an array composed of the last* $n-k$ *rows of* C *and the last* $k+1$ *rows of* Γ *be evanescent, so also is the array composed of the first* k *rows of* D *and the first* $n-k-1$ *rows of* Δ: *and conversely.* The proof turns on postulating a set of linear homogeneous equations in $u_1, u_2, \ldots, u_n$, whose coefficients form the given evanescent array.

In connection with the proof of the first theorem it is important to note that when ascertaining on the left-hand the cofactor of $|\, m_h n_k \,|$ there is a sign-factor to be attended to, namely $(-1)^{1+2+h+k}$ It would therefore be an improvement to append to the enunciation of the theorem the words "and the members of one of the two series being signed." Thus, what is actually established above is that if the product of each row of

$$\begin{matrix} c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{matrix}$$

by each row of

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{matrix}$$

be zero, then

$$\frac{|\, c_1 d_2 \,|}{|\, x_3 y_4 z_5 \,|} = -\frac{|\, c_1 d_3 \,|}{|\, x_2 y_4 z_5 \,|} = \ldots = \frac{|\, c_4 d_5 \,|}{|\, x_1 y_2 z_3 \,|}.$$

RASCHI, L. (1872).

[Complementi d' Algebra, contenenti fra l' altre materie il metodo dei determinanti e frequenti applicazioni dello stesso. x+456 pp. Parma.]

The contents do not belie the subtitle. At the outset (pp. 16–81) an exposition of determinants is given, and throughout the rest of the volume frequent use is made of them. The subjects to which they are applied and the manner of application recall Trudi (1862).

COTTERILL, T. (1872).

[On an algebraical form, and the geometry of its dual connection with the polygon, plane or spherical. *Proceed. London Math. Soc.*, iv. pp. 139–143.]

The mainstay of the paper is a proposition regarding certain principal minors of the array,

$$\begin{array}{cccccccccc} t_1 & x_1 & y_1 & a_1 & b_1 & c_1 & \ldots & m_1 & n_1 \\ t_2 & x_2 & y_2 & a_2 & b_2 & c_2 & \ldots & m_2 & n_2 \\ t_3 & x_3 & y_3 & a_3 & b_3 & c_3 & \ldots & m_3 & n_3, \end{array}$$

namely, that the expression

$$\frac{1}{|t_1x_2y_3|}\left\{\frac{|t_1a_2b_3|}{|a_1x_2y_3|\,|b_1x_2y_3|}+\ldots+\frac{|t_1m_2n_3|}{|m_1x_2y_3|\,|n_1x_2y_3|}+\frac{|t_1n_2a_3|}{|n_1x_2y_3|\,|a_1x_2y_3|}\right\}$$

is independent of the t's, and that, by substituting in it the fourth or any succeeding column for the first column, various forms of the t-free equivalent are readily obtained. No proof is given.

Denoting any principal minor of the array by the numbers of the columns which it contains, we may write the simplest case of the expression in the form

$$\frac{1}{123}\left\{\frac{145}{423\cdot523}+\frac{156}{523\cdot623}+\frac{164}{623\cdot423}\right\};$$

and this being equal to

$$(145\cdot623+156\cdot423+164\cdot523)/(123\cdot423\cdot523\cdot623),$$

we can substitute for the dividend Sylvester's equivalent of 1851, namely, $123\cdot456$, with the result

$$\frac{456}{423\cdot523\cdot623},$$

which, besides being free of 1, is also exactly what is obtained by simply putting 4 for 1 in the expression as given.

In the next case of the expression the cofactor of 1/123 being

$$\frac{145}{423\cdot523}+\frac{156}{523\cdot623}+\frac{167}{623\cdot723}+\frac{174}{723\cdot423},$$

we add to and subtract from it $164/(623 \cdot 423)$, thus obtaining

$$\left(\frac{145}{423 \cdot 523}+\frac{156}{523 \cdot 623}+\frac{164}{623 \cdot 423}\right)+\left(\frac{167}{623 \cdot 723}+\frac{174}{723 \cdot 423}+\frac{146}{423 \cdot 623}\right)$$

the division of which by 123 we know from the previous case to give

$$\frac{456}{423 \cdot 523 \cdot 623}+\frac{674}{623 \cdot 723 \cdot 423}.$$

In similar manner we find

$$\frac{1}{123}\left\{\frac{145}{423 \cdot 523}+\frac{156}{523 \cdot 623}+\frac{167}{623 \cdot 723}+\frac{178}{723 \cdot 823}+\frac{184}{823 \cdot 423}\right\}$$
$$=\frac{456}{423 \cdot 523 \cdot 623}+\frac{467}{423 \cdot 623 \cdot 723}+\frac{478}{423 \cdot 723 \cdot 823}.$$

The right-hand member in both cases is got from the left by the substitution of 4 for 1.

STUDNIČKA, F. J. (1872).

[Ein Beitrag zur Theorie der Determinanten. *Sitzungsb. . . . böhm. Ges. d. Wiss.* Jahrg. 1872, pp. 78–80.]

[Nove poučky o determinantech. *Časopis pro pěstování math. a fys.*, i. pp. 203–206.]

Studnicka performs on any determinant Δ the two sets of operations which Hankel (1861) performed on the persymmetric determinant. For ourselves we may put the result in the simple form

$$\Delta = U \Delta U,$$

where

$$U = \begin{vmatrix} 1 & . & . & . & \ldots \\ -1 & 1 & . & . & \ldots \\ 1 & -2 & 1 & . & \ldots \\ -1 & 3 & -3 & 1 & \ldots \\ . & . & . & . & \ldots \end{vmatrix}.$$

Specialization is then made so as to obtain Hankel's result,* and to show that $| a^0 b^1 c^2 \ldots |$ is equal to the difference-product of $a, b, c, \ldots$.

* See also *Nouv. Corresp. Math.*, ii. p. 401, vi. pp. 231–232.

SIACCI, F. (1872).

[Teorema sui determinanti, ed alcune sue applicazioni. *Atti . . . Accad. . . . Torino*, vii. pp. 772-783.]

[Intorno ad alcune trasformazioni di determinanti. *Annali di Mat.*, (2) v. pp. 296–304.]

In reality these two papers deal with the same theorem, the second paper being an improvement on the first and containing two or three fresh deductions.

The theorem concerns one of that class of determinants whose every element is a constant function of the corresponding elements of two given general determinants, the exact connection in this case being that the new element is the sum of constant multiples of the parent elements. This is the same as to say that if

$$|a_{11} \ldots a_{nn}|, \qquad |b_{11} \ldots b_{nn}|$$

be the basic or originating determinants, the new composite determinant is

$$|(\lambda a_{11}+\mu b_{11}) \ldots . (\lambda a_{nn}+\mu b_{nn})|.$$

In his first paper Siacci partitions the new determinant into a series of determinants with monomial elements, and thence obtains an expression for it arranged according to descending powers of λ, the coefficient of λ^n being $|a_{1n}|$ or A, say, the coefficient of μ^n being $|b_{1n}|$ or B, say, and the coefficient of $\lambda^{n-r}\mu^r$ being a sum of determinants each composed of $n-r$ columns from A and r columns from B. Proceeding further with the coefficients of the latter type he substitutes for each determinant in them a sum of binary products, one factor being a minor from A and the other a minor from B. In this way the coefficient of $\lambda^{n-r}\mu^r$ is found to be obtainable by taking each $(n-r)$-line minor of A, multiplying this by the complementary of the corresponding minor of B, and adding the products. There being one product for every r-line minor of B, the total number would be $\{C_{n,r}\}^2$. For example, when n is 3, the coefficient of $\lambda^2\mu$ is

$$\begin{aligned}
&(\ \ |a_{11}a_{22}|, \ -|a_{11}a_{32}|, \ \ |a_{21}a_{32}| \,\}\, b_{33}, \ b_{23}, \ b_{13})\\
&+(-|a_{11}a_{23}|, \ \ |a_{11}a_{33}|, \ -|a_{21}a_{33}| \,\}\, b_{32}, \ b_{22}, \ b_{12})\\
&+(\ \ |a_{12}a_{23}|, \ -|a_{12}a_{13}|, \ \ |a_{22}a_{33}| \,\}\, b_{31}, \ b_{21}, \ b_{11}).
\end{aligned}$$

Siacci's mode of writing the general expansion in symbols is

$$\lambda^n A_n + \lambda^{n-1}\mu \cdot \sum(A_{n-1} \cdot \text{comp } B_{n-1}) + \lambda^{n-2}\mu^2 \cdot \sum(A_{n-2} \cdot \text{comp } B_{n-2}) + \ldots + \mu^n B_n,$$

where A_r means an r-line minor of A and where therefore A_n means A.

With this auxiliary result in his possession he has no difficulty in proving his main theorem, namely, the identity

$$|(\lambda a_{11}+\mu b_{11}) \ldots (\lambda a_{nn}+\mu b_{nn})| = AB \cdot \left|\left(\mu\frac{A_{11}}{A}+\lambda\frac{B_{11}}{B}\right) \ldots \left(\mu\frac{A_{nn}}{A}+\lambda\frac{B_{nn}}{B}\right)\right|,$$

or, as we may formulate it for ourselves, *If* P *be Siacci's composite determinant formed from the matrices of* A *and* B, *and* Q *be that similarly formed from the reciprocals of the matrices of* B *and* A, *then*

$$P = AB \cdot Q.$$

All he has to do is to show that the coefficient of $\lambda^{n-r}\mu^r$ in P is AB times the coefficient of $\lambda^{n-r}\mu^r$ in Q, and this he obtains by using Jacobi's theorem regarding a minor of the adjugate determinant.

In his second paper he views P as the eliminant of the set of equations

$$(\lambda a_{r1}+\mu b_{r1})x_1 + \ldots + (\lambda a_{rn}+\mu b_{rn}) = 0 \Big\}_{r=1}^{r=n}$$

arising from the two sets

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = \mu u_r \Big\}_{r=1}^{r=n},$$

$$b_{r1}x_1 + b_{r2}x_2 + \ldots + b_{rn}x_n = -\lambda u_r \Big\}_{r=1}^{r=n},$$

and he succeeds in deriving from the same two sets, by equating values of x_t, the set

$$\left(\mu\frac{A_{1t}}{A}+\lambda\frac{B_{1t}}{B}\right)u_1 + \ldots + \left(\mu\frac{A_{nt}}{A}+\lambda\frac{B_{nt}}{B}\right)u_n = 0 \Big\}_{t=1}^{t=n},$$

of which Q is the eliminant. As the resulting equations $P = 0$, $Q = 0$ must have the same roots, he concludes that P and Q are connected by a constant multiplier, and taking the special case $\lambda = 0$ he finds the multiplier to be AB.

Incidentally to this proof it is shown that

$$P = A\begin{vmatrix} \mu h_{11}+\lambda & h_{12} & \dots \\ h_{21} & \mu h_{22}+\lambda & \dots \\ \dots & \dots & \dots \end{vmatrix}_n, \quad Q = B\begin{vmatrix} \lambda k_{11}+\mu & k_{12} & \dots \\ k_{21} & \lambda k_{22}+\mu & \dots \\ \dots & \dots & \dots \end{vmatrix}_n,$$

where

$$h_{rs} = \frac{A_{r1}}{A} b_{s1} + \frac{A_{r2}}{A} b_{s2} + \dots + \frac{A_{rn}}{A} b_{sn},$$

and

$$k_{rs} = \frac{B_{r1}}{B} a_{s1} + \frac{B_{r2}}{B} a_{s2} + \dots + \frac{B_{rn}}{B} a_{sn},$$

and where therefore

$$|h_{1n}| = \frac{B}{A}, \qquad |k_{1n}| = \frac{A}{B}$$

Putting $f(\mu)$ for P when $\lambda = 1$, and multiplying $f(\mu)$ by the expression which the theorem gives for $f(-\mu)$, Siacci readily obtains

$$f(\mu)\cdot f(-\mu) = AB\begin{vmatrix} h_{11}-k_{11}\mu^2 & \dots & h_{1n}-k_{1n}\mu^2 \\ h_{21}-k_{21}\mu^2 & \dots & h_{2n}-k_{2n}\mu^2 \\ \dots & \dots & \dots \\ h_{n1}-k_{n1}\mu^2 & \dots & h_{nn}-k_{nn}\mu^2 \end{vmatrix},$$

where the h's and k's are such that

$$h_{r1}k_{1r} + h_{r2}k_{2r} + \dots + h_{rn}k_{nr} = 1,$$
$$h_{r1}k_{1s} + h_{r2}k_{2s} + \dots + h_{rn}k_{ns} = 0.$$

The remaining portions of the two papers are occupied with deductions, some of which fall under Orthogonants and one under Recurrents.

STUDNIČKA, F. J. (1872): WALKER, J. J. (1872).

[Nový dúkaz poučky o pomerech mezi původními a předruženými determinanty a subdeterminanty. *Časopis pro pěstování math. a phys.*, i. pp. 6–10.]

[Question 3690. *Educ. Times*, xxiv. p. 296; or *Math. from Educ. Times*, xvii. pp. 110–111.]

Studnička's so-called proof of Jacobi's theorem regarding a minor of the adjugate is only suggestive of the truth: and Walker's identity is that which began to appear with Monge in 1809, and which came later to be known as the 'extensional' of

$$|a_1b_2||c_1d_2| - |a_1c_2||b_1d_2| + |a_1d_2||b_1c_2| = 0.$$

MUIR, T.: WHITWORTH, W. A. (1872).

[Extension of a law of determinants. *Messenger of Math.*, ii. pp. 60–61, 80.]

Whitworth's further extension is that *if in an* n*-line determinant* Δ *there be a* p*-by-*q *array, each of whose elements contains the factor* a, *then* Δ *must contain the factor* a^{p+q-n}: and the reason is that if we multiply by a each of the n-p rows outside the array we obtain a determinant in which q columns contain a. The theorem should be compared with Dodgson's of 1867.

ALBEGGIANI, M. (1872).

[Sviluppo di un determinante ad elementi binomii, ed applicazione . . . . *Giornale di Mat.*, x. pp. 279–293.]

The determinant dealt with is

$$|(c_{11}a_{11}+d_{11}b_{11})(c_{22}a_{22}+d_{22}b_{22}) \ldots . (c_{nn}a_{nn}+d_{nn}b_{nn})|,$$

and the development is written in the form

$$\Delta_1^{(n)} + \sum\Delta_1^{(n-1)}\Delta_2^{(1)} + \sum\Delta_1^{(n-2)}\Delta_2^{(2)} + \ldots . + \Delta_2^{(n)},$$

where $\Delta_1^{(p)}$, $\Delta_2^{(p)}$ are p-line minors formed from

$$|c_{11}a_{11}\cdot c_{22}a_{22} \ldots c_{nn}a_{nn}| \quad \text{and} \quad |d_{11}b_{11}\cdot d_{22}b_{22} \ldots d_{nn}b_{nn}|$$

respectively, subject to the condition that in such a product as

$$\Delta_1^{(p)}\Delta_2^{(q)}, \quad \text{where } p+q = n,$$

the two minors are understood to be complementary as regards the rows and columns from which they are taken. The result is thus closely like that reached by Siacci earlier in the same year.

The remainder of the paper (pp. 285–293) is occupied with the deduction of four results of Siacci's set as 'questions' on p. 188 of the same volume (see under *Orthogonants*).

FONTEBASSO, D. (1872).

[I primi Elementi della Teoria dei Determinanti, e loro applicazioni all' algebra ed alla geometria, proposti agli alunni degli istituti tecnici. viii + 134 pp. Treviso.]

Fontebasso's aim was a text-book simpler still in style than Trudi's; but though he reached it there is nothing very praiseworthy

in the result of his effort. It is strange, for example, to find in the short opening chapter valuable space given to two unnecessary though interesting theorems regarding derangements while skew determinants and other important special forms are nowhere mentioned. The theorems in question are

1. *The total number of derangements in the* n! *permutations of the first* n *integers is* $\frac{1}{4}(n-1)n \cdot n!$.

2. *The total number of derangements in the* n *permutations got by permuting the first* n *integers cyclically is* $\frac{1}{6}(n-1)\,n(n+1)$.

Save for a very special persymmetric determinant there is nothing else worth noting.

GUNDELFINGER, S. (1873).

[Ueber einen Satz aus der Determinantentheorie. *Zeitschrift f. Math. u. Phys.*, xviii. pp. 312–315.]

The theorem in question may be said to concern the 'increment' produced on the product of two oblong arrays by prefixing to each a column of units. It being understood that the given array consists of m rows and n columns, the said increment

$$= \begin{Vmatrix} 1 & a_1 & a_2 & \dots & a_n \\ 1 & b_1 & b_2 & \dots & b_n \\ 1 & c_1 & c_2 & \dots & c_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix} \cdot \begin{Vmatrix} 1 & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ 1 & \beta_1 & \beta_2 & \dots & \beta_n \\ 1 & \gamma_1 & \gamma_2 & \dots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{Vmatrix} - \begin{Vmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \\ c_1 & c_2 & \dots & c_n \\ \cdot & \cdot & \cdot & \cdot \end{Vmatrix} \cdot \begin{Vmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \beta_1 & \beta_2 & \dots & \beta_n \\ \gamma_1 & \gamma_2 & \dots & \gamma_n \\ \cdot & \cdot & \cdot & \cdot \end{Vmatrix}$$

$$= \begin{vmatrix} 1+\Sigma a\alpha & 1+\Sigma a\beta & 1+\Sigma a\gamma & \dots \\ 1+\Sigma b\alpha & 1+\Sigma b\beta & 1+\Sigma b\gamma & \dots \\ 1+\Sigma c\alpha & 1+\Sigma c\beta & 1+\Sigma c\gamma & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_m - \begin{vmatrix} \Sigma a\alpha & \Sigma a\beta & \Sigma a\gamma & \dots \\ \Sigma b\alpha & \Sigma b\beta & \Sigma b\gamma & \dots \\ \Sigma c\alpha & \Sigma c\beta & \Sigma c\gamma & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n$$

$$= -\begin{vmatrix} \cdot & 1 & 1 & 1 & \dots \\ 1 & \Sigma a\alpha & \Sigma a\beta & \Sigma a\gamma & \dots \\ 1 & \Sigma b\alpha & \Sigma b\beta & \Sigma b\gamma & \dots \\ 1 & \Sigma c\alpha & \Sigma c\beta & \Sigma c\gamma & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_{m+1},$$

where $\Sigma a\alpha$ stands for $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$. This agrees with the result obtained in a different way by Sylvester in 1852 where $n = m-1$, and where, therefore, the product of the given arrays

vanishes and the enlarged arrays are square (see *Hist.*, ii. p. 122). All that is further done is to change it into

$$\frac{(-1)^m}{2^{m-1}} \begin{vmatrix} . & 1 & 1 & 1 & \dots \\ 1 & \Sigma(a-\alpha)^2 & \Sigma(a-\beta)^2 & \Sigma(a-\gamma)^2 & \dots \\ 1 & \Sigma(b-\alpha)^2 & \Sigma(b-\beta)^2 & \Sigma(b-\gamma)^2 & \dots \\ 1 & \Sigma(c-\alpha)^2 & \Sigma(c-\beta)^2 & \Sigma(c-\gamma)^2 & \dots \\ . & . & . & . & . \end{vmatrix}_{m+1}$$

exactly in Sylvester's fashion.

It is noted that when $n < m-1$ the product of the enlarged arrays also vanishes, so that the left-hand member is 0. Attention, however, is not drawn to the fact that when $n = m-2$, we are led to Cayley's results of 1841. For example, when $m = 3$ and $n = 1$, we have

$$0 = \begin{vmatrix} . & 1 & 1 & 1 \\ 1 & (a-\alpha)^2 & (a-\beta)^2 & (a-\gamma)^2 \\ 1 & (b-\alpha)^2 & (b-\beta)^2 & (b-\gamma)^2 \\ 1 & (c-\alpha)^2 & (c-\beta)^2 & (c-\gamma)^2 \end{vmatrix},$$

which becomes Cayley's first result on putting $\alpha, \beta, \gamma = a, b, c$.

Gundelfinger claims that his theorem and corollaries embrace most of the results contained in Baltzer's § 16, § 17. 1-7.

CATALAN, E. (1873): CAYLEY, A. (1873).

[Sur un déterminant. *Mélanges Math.*, ii. p. 141; or *Mém. Soc. R. des Sci.* (Liège), (2) xiii. p. 141.]

[Two Smith's prize dissertations. *Messenger of Math.*, ii. pp. 147–149.]

Both unimportant.

HAMBURGER, M. (1873).

[Bemerkung über die Form der Integrale der linearen Differentialgleichungen mit veränderlichen Coefficienten. *Crelle's Journ.*, lxxvi. pp. 113–125.]

On pp. 115–117 Hamburger deals with a lemma regarding four sets of quantities,

$$y_1, y_2, \dots, y_n \qquad x_1, x_2, \dots, x_n$$
$$\eta_1, \eta_2, \dots, \eta_n \qquad \xi_1, \xi_2, \dots, \xi_n,$$

in which any member of any set is a linear homogeneous function of all the members of any other set. Of course, it is impossible that more than three connecting sets of equations can be independent. For example, if it be given that

$$(y_1, y_2, y_3) = \left(\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \right) (x_1, x_2, x_3), \quad (\eta_1, \eta_2, \eta_3) = \left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right) (y_1, y_2, y_3),$$

$$(\eta_1, \eta_2, \eta_3) = \left(\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} \right) (\xi_1, \xi_2, \xi_3), \quad (\xi_1, \xi_2, \xi_3) = \left(\begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix} \right) (x_1, x_2, x_3),$$

it is evident that there must exist between the coefficients the matrical relation

$$\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \middle\| \begin{matrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{matrix} \right) = \left(\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} \middle\| \begin{matrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{matrix} \right).$$

Now, what Hamburger in effect says—he does not use Cayley's notation *—is that, if the third matrix in this last identity be the same as the second, then

$$\begin{vmatrix} a_1-\theta & a_2 & a_3 \\ b_1 & b_2-\theta & b_3 \\ c_1 & c_2 & c_3-\theta \end{vmatrix} = \begin{vmatrix} f_1-\theta & f_2 & f_3 \\ g_1 & g_2-\theta & g_3 \\ h_1 & h_2 & h_3-\theta \end{vmatrix}$$

for all values of θ. By way of proof it is pointed out that in this special case the matrical relation becomes

$$\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ l_1 & m_1 & n_1 \\ b_1 & b_2 & b_3 \\ l_1 & m_1 & n_1 \\ c_1 & c_2 & c_3 \\ l_1 & m_1 & n_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ l_2 & m_2 & n_2 \\ b_1 & b_2 & b_3 \\ l_2 & m_2 & n_2 \\ c_1 & c_2 & c_3 \\ l_2 & m_2 & n_2 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ l_3 & m_3 & n_3 \\ b_1 & b_2 & b_3 \\ l_3 & m_3 & n_3 \\ c_1 & c_2 & c_3 \\ l_3 & m_3 & n_3 \end{vmatrix} \right) = \left(\begin{vmatrix} l_1 & l_2 & l_3 \\ f_1 & g_1 & h_1 \\ m_1 & m_2 & m_3 \\ f_1 & g_1 & h_1 \\ n_1 & n_2 & n_3 \\ f_1 & g_1 & h_1 \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 \\ f_2 & g_2 & h_2 \\ m_1 & m_2 & m_3 \\ f_2 & g_2 & h_2 \\ n_1 & n_2 & n_3 \\ f_2 & g_2 & h_2 \end{vmatrix} \begin{vmatrix} l_1 & l_2 & l_3 \\ f_3 & g_3 & h_3 \\ m_1 & m_2 & m_3 \\ f_3 & g_3 & h_3 \\ n_1 & n_2 & n_3 \\ f_3 & g_3 & h_3 \end{vmatrix} \right),$$

* See *Hist.*, i. pp. 85–87.

this being Cayley's way of stating nine equalities of the type

$$\frac{a_1\ a_2\ a_3}{l_1\ m_1\ n_1} = \frac{l_1\ l_2\ l_3}{f_1\ g_1\ h_1}, \quad i.e. \quad a_1l_1 + a_2m_1 + a_3n_1 = l_1f_1 + l_2g_1 + l_3h_1.$$

On account of these equalities it is next seen that

$$\begin{vmatrix} a_1-\theta & a_2 & a_3 \\ b_1 & b_2-\theta & b_3 \\ c_1 & c_2 & c_3-\theta \end{vmatrix} \cdot \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \cdot \begin{vmatrix} f_1-\theta & f_2 & f_3 \\ g_1 & g_2-\theta & g_3 \\ h_1 & h_2 & h_3-\theta \end{vmatrix},$$

so that, if $|l_1m_2n_3|$ be not equal to 0, it follows that

$$\begin{vmatrix} a_1-\theta & a_2 & a_3 \\ b_1 & b_2-\theta & b_3 \\ c_1 & c_2 & c_3-\theta \end{vmatrix} = \begin{vmatrix} f_1-\theta & f_2 & f_3 \\ g_1 & g_2-\theta & g_3 \\ h_1 & h_2 & h_3-\theta \end{vmatrix},$$

as desired. Another way of stating the property is that the sum of the r-line coaxial minors of the one determinant is equal to the corresponding sum in the other; and a third way, only possible ten years later, would be: *If* m_1, m_2, μ *be matrices such that*

$$\mathrm{m}_1\mu = \mu\mathrm{m}_2,$$

then m_1, m_2 *have the same latent roots.*

A second property is also enunciated and proved, namely, that if in the case of the one determinant all the r-line minors vanish for a particular value of θ while all the $(r-1)$-line minors do not vanish, the same holds in the case of the other determinant;—in later phraseology, the 'rank' or 'characteristic' of the two determinants is the same.

RUBINI, R. (1873): HESSE, O. (1873).

[Trattato d'Algebra. Parte seconda. viii+364 pp. Napoli.]

[I Determinanti, elementarmente esposti. Traduzione del Valeriani Valeriano. *Giornale di Mat.*, x. pp. 217–229, 325–342. Also published separately.]

The chapters in Rubini regarding determinants (i.-xii., etc.) contain essentially the same matter as his text-books of 1864–1867 above referred to.

JANNI, V. (1873).

[Sul prodotto di due matrici. *Giornale di Mat.*, xi. pp. 357–358.]

The mode of proof which Bellavitis (1857) applied to the multiplication of two three-line determinants Janni applies to the multiplication of two ν-by-n arrays,—that is to say, he begins with the product-determinant and investigates the result of the partitionment of it into a sum of determinants with monomial elements.

GORDAN, P. (1873).

[Ueber den grössten gemeinsamen Factor. *Math. Annalen*, vii. pp. 443–448.]

If we return to Brill's second paper of 1871, we shall find that in proving his first theorem for the case where $k = 2$, we did not use the datum

$$\sum dw = 0,$$

and we used unnecessarily the data

$$\sum ax = 0, \quad \sum bx = 0, \quad \sum by = 0.$$

A little examination will also show that for each case a different selection of data must be made, and that in fact the theorem as stated gives the aggregate of the data requisite for all the values of k. Manifestly there would be considerable advantage in recasting the theorem so as to have indicated the exact data requisite in each case; and this is what Gordan here does. In effect, what he says is that *if an* r-*by*-n *array* P *and an* s-*by*-n *array* Q *be such that* r+s = n, *and the product of each row of the one by each row of the other be zero, then the* r-*line minors of* P *are proportional to the* s-*line minors of* Q, *the series of minors in the latter case being arranged in reverse order, and the members of one of the two series being signed.* The concluding condition of this does not appear in the original, but it is necessary, as we have already shown. No proof is given, but it can of course be readily effected in the same manner as before, namely, by annexing any s rows to P and any r rows to Q, and then multiplying together the two resulting square arrays.

In illustration of one of Gordan's applications we may note that if α, β, γ be the roots of the equation $x^3+c_1x^2+c_2x+c_3=0$, then

$$1 \quad c_1 \quad c_2 \quad c_3 \qquad \text{and} \qquad \begin{matrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{matrix}$$

are two arrays conditioned as in the theorem, and therefore that

$$\frac{1}{|\alpha^2\beta^1\gamma^0|} = \frac{-c_1}{|\alpha^3\beta^1\gamma^0|} = \frac{c_2}{|\alpha^3\beta^2\gamma^0|} = \frac{-c_3}{|\alpha^3\beta^2\gamma^1|},$$

—one of the oldest properties of alternants. This suggests, too, that the solution of a set of simultaneous linear equations is involved in the theorem; for example, the satisfying arrays being

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \qquad \begin{matrix} x & y & z & w \end{matrix}$$

we obtain

$$\frac{x}{|a_2b_3c_4|} = \frac{-y}{|a_1b_3c_4|} = \frac{z}{|a_1b_2c_4|} = \frac{-w}{|a_1b_2c_3|}.$$

WISSELINK, D. B. (1873).*

[Iets over Determinanten. 38 pp. Deventer.]

A simply and clearly written introduction well-printed on a large broad-margined page. It is the first instance—not a translation—of the subject being dealt with in the language of Holland.

DÖLP, H. (1873): REIDT, F. (1874).

[Die Determinanten, nebst Anwendung auf die Lösung algebraischer und analytisch-geometrischer Aufgaben. Elementar behandelt. iv+94 pp. Darmstadt.]

[Vorschule der Theorie der Determinanten für Gymnasien und Realschulen. vi+66 pp. Leipzig.]

These are workmanlike elementary class-books. The first is a particularly clear exposition, higher-pitched in its style than Hattendorff's: the second is a trifle more elementary even than the latter,

* No date on title-page.

giving much space to easy unworked exercises, of which there are over 200. As regards the arrangement of matter the second is superior.

STUDNIČKA, F. J. (1873).

[Příspěvek k theorii determinantů. *Časopis pro pěstování math. a fys.*, ii. pp. 282–285.]

[Ein neuer Determinantensatz. *Sitzungsb. . . . . Ges. d. Wiss.* (Prag), Jahrg. 1873, pp. 342–344.]

These two papers are practically the same, although the one is not an exact translation of the other. The theorem referred to in the title of the second is to the effect that *if the difference of any two rows be a multiple of the difference of any other two, the determinant vanishes*; and the generalization derived from it is of still less moment. It may perhaps be usefully compared with Dodgson's sixth proposition regarding the evanescence of arrays.

BALTZER, R. (1873).

[Mathematische Bemerkungen. *Berichte . . . Ges. d. Wiss.* (Leipzig), xxv. pp. 523–537.]

The seventh of Baltzer's notes derives Bezout's result

$$|a_1b_5c_6| \cdot |a_2b_3c_4| \quad - \quad |a_2b_5c_6| \cdot |a_1b_3c_4| \quad + \quad . \quad . \quad . \quad . \quad = \quad 0$$

from the more readily perceived identity

$$\begin{vmatrix} |a_1b_5c_6| & a_1 & b_1 & c_1 \\ |a_2b_5c_6| & a_2 & b_2 & c_2 \\ |a_3b_5c_6| & a_3 & b_3 & c_3 \\ |a_4b_5c_6| & a_4 & b_4 & c_4 \end{vmatrix} = 0.$$

The ninth note, while seeming to concern a different and less simple thing, is really occupied with an analogous identity, namely,

$$\begin{aligned} |a_1b_2c_3d_4e_5| \cdot |u_1v_2c_3d_4e_5| \quad &= \quad |u_1b_2c_3d_4e_5| \cdot |a_1v_2c_3d_4e_5| \\ &\quad - |a_1u_2c_3d_4e_5| \cdot |v_1b_2c_3d_4e_5|, \end{aligned}$$

first given by Desnanot in 1819 (*Hist.*, i. p. 141, G). The eighth note concerns Gundelfinger's paper of the same year (1873), the chief object apparently being to register a counter claim in favour of his own, § 16, 12.

JANNI, V. (1874).

[Dimostrazione di alcune teoremi sui determinanti. *Giornale di Mat.*, xii. pp. 142–145.]

Janni's theorems, though all simple, are treated with freshness. His first is that dealt with very shortly by Bellavitis in § 15 of his *Sposizione* of 1857 (*Hist.*, ii. pp. 94–95), his proof resting on the fact that, for example, when $|\, a_1b_2c_3\,| = 0$, we not only have

$$b_1A_1+b_2A_2+b_3A_3 = 0,$$

and

$$c_1A_1+c_2A_2+c_3A_3 = 0,$$

but also

$$a_1A_1+a_2A_2+a_3A_3 = 0.$$

Following on this are other properties of a null determinant established in his own way. He then proves Hermite's condensation-theorem (*Hist.*, ii. p. 46), deducing thence the corollary that *if all the two-line minors of a determinant be divisible by the same quantity, any* m*-line minor will be divisible by the* $(m-1)^{th}$ *power of the quantity*, and deriving therefrom the two main properties of the adjugate. His last result is that *a determinant is not altered by changing the sign of every element whose place-indices have an odd sum*, this change being equivalent to changing the signs of the even-numbered rows and thereafter the signs of the even-numbered columns.

KRONECKER, L. (1874).

[Ueber Schaaren von quadratischen und bilinearen Formen. *Monatsb. . . . Akad. d. Wiss.* (Berlin), Jahrg. 1874, pp. . . . , 214–215, . . . ; or *Werke*, i. pp. 165-174.]

Using the simplest case of Jacobi's theorem regarding a minor of the adjugate Kronecker makes a deduction, of which it will suffice to give merely an example, namely,

$$\frac{|a_1b_2c_3d_4|}{|b_2c_3d_4|} + \frac{|a_2c_3d_4|\cdot|b_1c_3d_4|}{|b_2c_3d_4|\cdot|c_3d_4|} + \frac{|a_3d_4|\cdot|c_1d_4|}{|c_3d_4|\cdot d_4} + \frac{a_4d_1}{d_4} = a_1,$$

the reader being referred to *Hist.*, ii. p. 40, and to a result given under Axisymmetric Determinants (1874).

SĄGAJŁO, A. (1874).

[Wykład Zupełny Algebry . . . . Tom ii. Teorya Wyznaczników i ich przedniejsze zastowania. . . . xii+400 pp. Paryz.]

The volume, although like others entitled 'The Theory of Determinants and its principal applications,' is practically a Polish edition of Salmon's *Modern Higher Algebra*, with a few additions and amendments.

ALBEGGIANI, M. (1874).

[Sviluppo di un determinante ad elementi polinomi. *Giornale di Mat.*, xiii. pp. 1–32.]

Albeggiani begins by restating his development of 1872 for the case of *binomial* elements, and then he proceeds to establish it for the case of trinomial elements which he states in the form

$$
\begin{aligned}
{}_{(3)}\Delta^{(n)} = {} & \Delta_1^{(n)} + \Delta_2^{(n)} + \Delta_3^{(n)} \\
& + \Sigma\Delta_1^{(n-1)}\Delta_2^{1} + \Sigma\Delta_1^{(n-1)}\Delta_3^{1} + \ldots . \\
& \quad \ldots \ldots \ldots \ldots \\
& + \Sigma\Delta_1^{(n-h)}\Delta_2^{(h)} + \ldots . + \Sigma\Delta_1^{(n-k)}\Delta_2^{(k)} + \ldots . \\
& + \Sigma\Delta_1^{(n-2)}\Delta_2^{(1)}\Delta_3^{(1)} + \ldots . \\
& + \Sigma\Delta_1^{(n-m)}\Delta_2^{(m-\nu)}\Delta_3^{\nu} + \ldots . ,
\end{aligned}
$$

where the minors following any Σ are quasi-complementary,—that is to say, no two of them have a row-number or a column-number in common, and yet all the row-numbers are used. This naturally suggests the statement which is likely to hold for all cases; and on the hypothesis that it is true for the case of p-termed elements he sets about trying to prove it true for the next higher case. With exemplary patience he gives seven pages to the attempt and is successful, the final result thus being in double-Σ notation

$$
{}_{(\rho)}\Delta^{(n)} = \sum\left(\Sigma\Delta_1^{(\alpha)}\Delta_2^{(\beta)}\Delta_3^{(\gamma)} \ldots \Delta_p^{(\pi)}\right),
$$

where $\alpha+\beta+\ldots+\pi=n$.

To make the form of development more familiar to the reader, Albeggiani points out its partial resemblance to that of

$$(a_1+a_2+\ldots+a_p)^n;$$

and on account of the latter development being better known he uses it as a mnemonic to recall the former. He thus writes his own theorem in the ultra-symbolic form

$${}_{(p)}\Delta^{(n)} = (\Delta_1+\Delta_2+\ldots+\Delta_p)^{(n)},$$

the implied instruction being that the n^{th} power of

$$\Delta_1+\Delta_2+\ldots+\Delta_p$$

is to be found in the usual way, and that then every term of it is to be suitably changed; for example,

$$\frac{n!}{\alpha!\beta!\ldots\pi!}\Delta_1^{\alpha}\Delta_2^{\beta}\ldots\Delta_p^{\pi} \quad \text{into} \quad \Sigma\Delta_1^{(\alpha)}\Delta_2^{(\beta)}\ldots\Delta_p^{(\pi)},$$

where it has to be noted as another point of analogy that the number of terms under the Σ is represented by the coefficient $n!/\alpha!\beta!\ldots\pi!$ which the Σ displaces.

Consideration is next given to another ultra-symbolic form and to certain rather fanciful applications (pp. 13–24); and finally a re-examination is made (pp. 24–32) of Siacci's theorems and deductions, some of which had been the original cause of Albeggiani's investigations.

GARBIERI, G. (1874).

[I Determinanti, con numerosi applicazioni. Parte Prima. Utile agli studiosi di matematica nei primi corsi universitari. xiv+ 267 pp. Bologna.]

In bulk this exceeds all previous Italian text-books except Trudi's (1862); and, as appears from the title, it was intended to be much larger; the Second Part, however, has not been published. Notwithstanding its greater fullness and general soundness, it cannot be said to show a marked advance in any particular. Of its twenty-two chapters the number strictly devoted to the theory is less than half, much space being given to geometrical * and alge-

* Those interested in such applications will find a valuable paper of this year by Frobenius in *Crelle's Journ.*, lxxix. pp. 185-247.

braical illustrations. The selection of matter is also somewhat arbitrary, some forms of special determinants being treated with considerable fullness and others not even mentioned.

Under the head of general determinants very little falls to be noted. The expansion of $|a_1b_2c_3|$ in terms of zero-axial determinants is reached by viewing the first row as being

$$a_1+0 \qquad 0+a_2 \qquad 0+a_3,$$

and partitioning the determinant into two; then treating the second of these two in similar manner; and so on. The procedure, however, is not so methodical as to suggest generalization.* There is also a fresh proof of the multiplication-theorem (pp. 155–158), but the importance attached to it by Garbieri is scarcely warranted.

GÜNTHER, S. (1875).

[LEHRBUCH DER DETERMINANTEN-THEORIE, für Studirende. viii+236 pp. Erlangen.]

Günther's aim was, like Trudi's, to produce a fairly comprehensive text-book less condensed than Baltzer's and in general more suitable for learners. Like Trudi, therefore, he ended by writing a lengthy book.

The first distinguishing feature of it is a fluently written and interesting chapter (i.) of thirty pages on the early history of the subject. This deservedly brings Rothe into notice, but in the desire for comprehensiveness does the same somewhat unnecessarily for Euler. A less justifiable peculiarity is a chapter (iv.) on *cubic*

* There exists an intermediate expansion of $|a_{1n}|$ which might well find a place in elementary text-books, namely, that in which each diagonal element, a_{rr}, is accompanied by the determinant evolved from A_{rr} by putting therein zeros in place of $a_{11}, a_{22}, \ldots, a_{r-1,\,r-1}$; for example:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 c_3 d_4 \end{vmatrix} + b_2 \begin{vmatrix} . & a_3 & a_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + c_3 \begin{vmatrix} . & a_2 & a_4 \\ b_1 & . & b_4 \\ d_1 & d_2 & d_4 \end{vmatrix} + d_4 \begin{vmatrix} . & a_2 & a_3 \\ b_1 & . & b_3 \\ c_1 & c_2 & . \end{vmatrix} + \begin{vmatrix} . & a_2 & a_3 & a_4 \\ b_1 & . & b_3 & b_4 \\ c_1 & c_2 & . & c_4 \\ d_1 & d_2 & d_3 & . \end{vmatrix}.$$

determinants; this, however, in the second edition he was well advised in relegating to a less prominent position. A third noteworthy characteristic is the assigning of a whole chapter (vi.) of thirty pages to *Kettenbruchdeterminanten*. The geometrical applications extend to only twelve pages. And, lastly, we may mention that, there being no footnotes, the bibliographical references are collected at the end of each chapter, where they form rather formidable-looking lists.

So far as general determinants are concerned, there is nothing fresh to note.

MANSION, P. (1874, 1875, 1876).

[Principes de la théorie des déterminants, d'après Baltzer et Salmon. *Nouv. Corresp. Math.*, i. pp. 114–128, 170–178, 206–222.]

[Éléments de la Théorie des Déterminants, d'après Baltzer et Salmon. 44 pp. Mons.]

[Introduction à la Théorie des Déterminants. À l'usage des établissements d'instruction moyenne. 24 pp. Gand.]

The second of these is a separate publication of the first; it is a singularly clear though short exposition with a sufficiency of worked and unworked examples to make it very suitable for use in schools. Nothing so good for beginners had as yet appeared in French. The third, as may be guessed, is still more elementary, having its origin in notelets contributed to the *Revue de l'Instruction Publique en Belgique* (1875, 1876).

GÜNTHER, S. (1875).

[Didaktische Bemerkungen zur Determinantentheorie. *Zeitschrift f. math. u. naturw. Unterricht*, vi. pp. 138–150.]

The remarks concern (1) the cofactor of a single element, or of the product of two or more elements, (2) special cases arising in the solution of simultaneous linear equations, (3) zero-axial skew determinants of even order. Under the last head the attempted proof of Cayley's property is based on a surprisingly questionable lemma.

HAMBURGER, M. (1875).

[Zur Theorie der Integration eines Systems von n linearen partiellen Differentialgleichungen . . . *Crelle's Journ.*, lxxxi. pp. 243–280.]

Incidentally (pp. 265–266) Hamburger states that the number of independent relations of the type

$$|fg|\cdot|hk| \;-\; |fh|\cdot|gk| \;+\; |fk|\cdot|gh| = 0$$
$$\text{or} \quad (f, g, h, k) = 0, \text{ say,}$$

between 2-line minors of a 2-by-n array is $\frac{1}{2}(n-2)(n-3)$, that is to say, the number of different sets of four columns in which two fixed columns always appear; and what he claims to prove is that if $|12|$ be not 0 and for all values of g from 3 to n $|1g|$ and $|2g|$ be not simultaneously 0, then all the relations in question are deducible from those of the type $(1, 2, h, k) = 0$. The simple facts are, that from the equations

$$\left.\begin{aligned} |12|\cdot|hk| \;-\; |1h|\cdot|2k| \;+\; |1k|\cdot|2h| &= 0 \\ |12|\cdot|kg| \;-\; |1k|\cdot|2g| \;+\; |1g|\cdot|2k| &= 0 \\ |12|\cdot|gh| \;-\; |1g|\cdot|2h| \;+\; |1h|\cdot|2g| &= 0 \end{aligned}\right\}$$

by using the multipliers $|1g|$, $|1h|$, $|1k|$, and adding, there is obtained

$$|12|\cdot(1ghk) = 0;$$

and having thus got

$$(1ghk) = 0, \quad (1gfh) = 0, \quad (1ghk) = 0,$$

there is similarly deducible

$$|1g|\cdot(fghk) = 0.$$

Again, instead of $(1ghk) = 0$, we might equally easily have got

$$(2ghk) = 0,$$

and thence

$$|2g|\cdot(fghk) = 0.$$

Provided therefore $|1g|$ and $|2g|$ be not simultaneously 0, it follows that

$$(fghk) = 0.$$

BELLAVITIS, G. (1875).

[Determinanti e loro uso nell' eliminazione. *Atti del R. Istituto veneto*, (5) i. pp. 1180–1182.]

This is merely a portion of a useful subject-index of Bellavitis' writings. Besides those to which we have in the foregoing directed attention he makes mention of his fifth, sixth, seventh and eighth 'riviste di giornali,' as bearing on determinants. These appeared in the *Atti* for the years 1862–1866.

BALTZER, R. (1875): SALMON, G. (1876).

[Theorie und Anwendung der Determinanten. . . . Vierte verbesserte Auflage. viii+247 pp. Leipzig.]

[Lessons Introductory to the Modern Higher Algebra. . . . Third edition. xx+318 pp. Dublin.]

Though the space devoted by Baltzer to the 'Theorie' is the same as in the third edition, quite a large number of changes, condensations and additions have been made. The most noteworthy change appears at the outset, where, probably as a consequence of Becker's criticisms, the sign of a term is settled by considering the inversions of order both in the series of row-numbers and in the series of column-numbers, + or − being taken according as the two series belong to the same or different classes of permutations.* Two questionable alterations in the choice of technical terms are made, namely, 'sub-determinant' for 'partial determinant,' and 'adjugate' for 'complementary.' Generally, however, the changes are for the better. The matter is as condensed as ever.

In the third edition of Salmon the first six chapters are increased, but as in Baltzer the added matter had already appeared elsewhere.

LEGGE, A. DI (1874).

[Teoria dei Determinanti. pp. (?) Roma.]

* This change first appeared in 1872 in his *Elemente d. Math.*, i. p. 144.

STUDNIČKA, F. J. (1875, 1876): MELLBERG, E. J. (1876).

[O původu a rozvoji nauky o determinantech. *Časopis pro pěstování math. a fys.*, v. pp. 1–9, 88–91, 193–199, 279–281.]

[Augustin Cauchy als formaler Begründer der Determinanten-Theorie. Eine literarisch-historische Studie. *Abhandl. . . . Ges. d. Wiss.* (Prag), (6) viii. 40 pp.]

[Teorin för Determinant-Kalkylen. 121 pp. Helsingfors.]

The first of these is either a preliminary sketch or an epitome of the second, and the interest of all of them is in connection with the *history* of the subject.

Mellberg's dissertation, it is true, devotes about seventy pages to an exposition of the theory and its applications, but these are of little moment compared with the fifty pages which precede. His plan is to enable the reader to judge for himself as to the exact nature and amount of the contributions of the early writers by giving the actual words of the relevant passages in their works. In this way Leibnitz, Cramer, Bezout, Vandermonde, Laplace, Lagrange, Gauss, Binet and Cauchy are dealt with, after which he makes short references to Jacobi, Spottiswoode, Brioschi, Baltzer, etc.

Studnička's memoir, although avowedly seeking to establish a special thesis, is constructed on the same model as Mellberg's. His list of authors is the same, save for the inclusion of Rothe and the omission of Binet. For the sake of additional clearness he appends to the account of each one's work a concise statement of what was essentially new in it, and before entering on the subject of Cauchy's memoir he gives a summing up of these epitomes. The result is a very readable and fair-minded 'study.'

The two publications, and especially Studnicka's, must have had considerable influence in spreading clearer and truer views of the early history. Their independent and practically simultaneous appearance,—Mellberg's dated the 1st of March and the other the 24th—show that it had begun to be felt that curt historical footnotes, which necessarily appear in the order fixed by the writer of the text-book for the purposes of exposition, have a tendency to mislead.

DIEKMANN, J. (1875, 1876): FALK, M. (1876): GULDBERG, A. S. (1876).

[Die Determinanten als Unterricht auf Gymnasien und Realschulen. *Zeitschrift f. math. u. naturw. Unterricht*, vi. pp. 1–21, 124–137, 193–211.]

[Einleitung in die Lehre von den Determinanten, und ihre Anwendung auf dem Gebiete der niederen Mathematik. Zum Gebrauch an Gymnasien, Realschulen u. andern höhern Lehranstalten, sowie zum Selbstunterricht. viii + 88 pp. Essen.]

[Lärobok i Determinant-teoriens första Grunder, för högre läroanstalter och till sjelfstudium. iv+96 pp. Upsala.]

[Determinanternes Teori. Udgivet med bidrag af videnskabernes selskab i Trondhjem. viii+88 pp. Kristiania.]

In his first publication Diekmann pressed on teachers his views as to the part that ought to be taken by determinants in a school course; he then almost immediately after brought out a booklet planned and written in accordance therewith. Less than a third part of the exposition is devoted to determinants pure and simple, but the little elementary knowledge thus acquired is utilized to the utmost in the remaining pages. In some respects it is not an advance on similar booklets already in use. The general question which he raised, however, was both timely and appropriate, and was certain sooner or later to be amply discussed by intemperate advocates of a novelty and by natural opponents of change.

Falk's publication, though meant for the same kind of readers as Diekmann's, is of a quite different character, about two-thirds of it being occupied with the establishment and illustration of twelve carefully formulated theorems, including three on Jacobians, and the remainder with forty exercises, worked and unworked, culled from well-known sources.

Guldberg's is still less of an elementary introduction, being indeed a reversion to an earlier type and intended apparently for such readers as Brioschi and Baltzer had in view. It makes no pretensions to freshness of manner or matter.

WRIGHT, W. J. (1875): KEMPER, D. (1876): STUDNIČKA, F. J. (1876): MÜLLER, H. (1876).

[Tracts relating to Modern Higher Mathematics. Tract No. 1, DETERMINANTS. viii+72 pp. London.]

[Determinants. *The Analyst*, iii. pp. 17–24.]

[Počátkove nauky o determinantech. *Časopis pro pěstování math. a fys.*, vi. pp. 49–58, 97–105, 201–211.]

[KURZE UND SCHULGEMÄSSE BEHANDLUNG DER DETERMINANTEN. 19 pp. Metz.]

Everywhere except on the title-page Wright's tract is styled 'Elementary.' It is of course more commonplace in its character than its immediate predecessor in English, Dodgson's (1867). If Todhunter's sketch in his 'Theory of Equations' had been published separately, it would have been much preferable to Wright's; the latter, however, is somewhat more extended in range.

The others are very elementary.

GÜNTHER, S. (1876): HOZA, F. (1876).

[Das allgemeine Zerlegungsproblem der Determinanten. *Archiv d. Math. u. Phys.*, lix. pp. 130–146.]

[Příspévek k theorii podřízených determinantu. *Časopis pro pěstování math. a phys.*, vi. pp. 21–34; or, in German, *Archiv d. Math. u. Phys.*, lix. pp. 387–400.]

Both these papers concern Laplace's expansion-theorem, their claim to attention being based on fullness of exposition and illustration. Taken together they occupy *thirty* pages of one and the same volume of the *Archiv*. In the former, which is the more ambitious, the question of the sign-factor is made a rather serious matter.*

* In this connection I take the opportunity of noting an unseen school-program published at Danzig in 1868: NEUMANN, Ueber Vorzeichensbestimmung in Formeln der Determinantentheorie.

JANNI, V. (1876, 1879).

[Lezioni di Algebra Complementare : Teoria dei Determinanti. (v)+50 pp. Napoli.]*

[Lezioni di Algebra Complementare. (viii)+228 pp. Napoli.]*

For its extent (50 pp.) this is an unusually pleasing exposition. Most of the points worth noting, however, have already been attended to in dealing with the same author's paper of 1874.

Trudi (1862) is followed in making the sign of a term of a determinant dependent on the combined number of inverted-pairs in the two sets of suffixes ; but Janni shows in addition that the interchange of two double-suffixed elements will make no alteration in the sign as thus determined ; that therefore the order in which the elements may happen to be taken in the formation of the term is of no real consequence ; and likewise that this order may be made such as to necessitate the counting of inverted-pairs in only one of the two sets of suffixes.

Trudi's theorem regarding a partitioned permutation is established only for the case in which the permutation

$$a_1 a_2 \ . \ . \ . \ a_m \mid a_{m+1} \ . \ . \ . \ a_n$$

has no inversions in either of its two parts taken separately, and where, therefore, the *total* number of inverted-pairs is

$$a_1 + a_2 + \ . \ . \ . \ + a_m - \tfrac{1}{2} m (m+1);$$

but Janni proceeds to take another permutation

$$\beta_1 \beta_2 \ . \ . \ . \ \beta_m \mid \beta_{m+1} \ . \ . \ . \ \beta_n$$

similarly partitioned and similarly in part arranged, and, asserting that the number of its inverted-pairs is

$$\beta_1 + \beta_2 + \ . \ . \ . \ + \beta_m - \tfrac{1}{2} m (m+1),$$

is thus able, by reason of $m(m+1)$ being even, to conclude that the combined number of inverted-pairs in the two permutations

$$a_1 a_2 \ . \ . \ . \ a_m \mid a_{m+1} \ . \ . \ . \ a_n$$
$$\beta_1 \beta_2 \ . \ . \ . \ \beta_m \mid \beta_{m+1} \ . \ . \ . \ \beta_n$$

* The first of these books seems a sort of trial edition of cap. i., ii., iii. (pp. 1-64) of the second ; but in neither is reference made to the other, and the publishers are different.

is even or odd according as

$$\alpha_1+\alpha_2+\ldots+\alpha_m+\beta_1+\beta_2+\ldots+\beta_m$$

is even or odd. The application which he afterwards makes of this in connection with Laplace's expansion-theorem will be readily guessed.

LAGUERRE, E. (1876).

[Sur la méthode de Monge pour l'intégration des équations linéaires aux différences partielles du second ordre. *Nouv. Annales de Math*, (2) xv. pp. 49–58.]

The first section of the paper (pp. 49–53) is occupied with a pure problem of determinants, namely, the finding of values for a, b, c, d, α, β, γ, δ so that

$$\begin{vmatrix} 1 & . & r & s \\ . & 1 & s & t \\ a & b & c & d \\ \alpha & \beta & \delta & \gamma \end{vmatrix}$$

may represent

$$\mathrm{H}r+2\mathrm{K}s+\mathrm{L}t-\mathrm{M}+\mathrm{N}(rt-s^2).$$

The problem is solved, and the relation between two solutions discussed.

FROBENIUS, G. (1876).

[Ueber das Pfaffsche Problem. *Crelle's Journ.*, lxxxii. pp. 230–315.]

Frobenius having to deal with the vanishing of the minors of a *skew* determinant takes a step backward (§ 4, pp. 239–241) to the consideration of determinants in general.

He first gives an interesting proof of the fundamental theorem regarding the evanescence of an array. For better means of comparison between it and Dodgson's proof, let us apply it to the same simple case, namely, where the array to be proved evanescent is

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \end{matrix}$$

with the data

$$|\, a_1 b_2 \,| \neq 0,$$
$$|\, a_1 b_2 c_3 \,| = 0, \qquad |\, a_1 b_2 d_3 \,| = 0, \qquad |\, a_1 b_2 e_3 \,| = 0.$$

In effect Frobenius says that by reason of data (2), (3), (4) we know that for *any* values of x, y, z

$$\begin{vmatrix} a_1 & a_2 & a_1x+a_2y+a_3z \\ b_1 & b_2 & b_1x+b_2y+b_3z \\ c_1 & c_2 & c_1x+c_2y+c_3z \end{vmatrix} = 0,$$

$$\begin{vmatrix} a_1 & a_2 & a_1x+a_2y+a_3z \\ b_1 & b_2 & b_1x+b_2y+b_3z \\ d_1 & d_2 & d_1x+d_2y+d_3z \end{vmatrix} = 0.$$

$$\begin{vmatrix} a_1 & a_2 & a_1x+a_2y+a_3z \\ b_1 & b_2 & b_1x+b_2y+b_3z \\ e_1 & e_2 & e_1x+e_2y+e_3z \end{vmatrix} = 0,$$

But, then, by reason of datum (1), values of x, y, z can be determined so as to make simultaneously

$$\left.\begin{aligned} a_1x+a_2y+a_3z &= 0 \\ b_1x+b_2y+b_3z &= 0 \end{aligned}\right\};$$

and, therefore, from the preceding three equations, with the help of (1), the same values of x, y, z will make

$$\left.\begin{aligned} c_1x+c_2y+c_3z &= 0 \\ d_1x+d_2y+d_3z &= 0 \\ e_1x+e_2y+e_3z &= 0 \end{aligned}\right\}.$$

The existence of these five equalities, exactly as in Dodgson's proof, involves the evanescence of the given array.

Frobenius then proceeds to establish a fresh theorem on the subject, namely, that *if in an oblong array all the* (m+1)*-line minors vanish, the* m*-line minors formed from any set of* m *rows are proportional to the corresponding minors formed from any other set.* We may note, however, that this may be best viewed as an extension of the theorem that *if a determinant vanishes, any two rows of the adjugate are proportional*, and that a proof on the same lines is readily devisable. A second theorem which he gives is an easy deduction from the first, namely, that *if in an oblong array all the* (m+1)*-line minors vanish, and it be possible to choose a set of* m *rows*

whose m-*line minors do not all vanish and a set of* m *columns with the same property, then the* m-*line minor common to the two sets cannot possibly vanish.* What follows on this mainly concerns skew determinants, and is therefore dealt with elsewhere.

HOZA, F. (1876).

[Ueber das Multiplicationstheorem zweier Determinanten n-ten Grades. *Archiv d. Math. u. Phys.*, lix. pp. 403–406; or, in Czech, *Casopis pro pěstování math. a fys.*, vi. pp. 87–89.]

The procedure is that adopted by Janni (1873), and is not more effectively carried out.

HOZA, F. (1876).

[Ueber Unterdeterminanten einer adjungirten Determinante. *Archiv d. Math. u. Phys.*, lix. pp. 401–403.]

This is simply the ordinary proof (Cayley's, 1843) with fuller details and explanations for the benefit of beginners.

JOHNSON, W. W. (1876): DICK, G. R. (1878): TANNER, H. W. L. (1879).

[On the determination of the sign of any determinant. *Messenger of Math.*, vii. p. 59.]

[On the sign of any term of a determinant. *Educ. Times*, xxxi. p. 161; or *Math. from Educ. Times*, xxix. pp. 99–100.]

[On the sign of any term of a determinant. *Messenger of Math.*, ix. pp. 51–52.]

Johnson's rule and Dick's are both to be found in Cauchy (1812), the one being that in which circular substitutions are counted, and the other that connected with the difference-product.

Tanner's is of considerably more interest. If the determinant be $|\, a_{15} \,|$, and the sign of $a_{13}a_{21}a_{32}a_{45}a_{54}$ be wanted, he writes the row-subscripts of the determinant and under them the column-subscripts, both in their natural order, thus

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{matrix};$$

then the first element in the given term being a_{13}, he draws a straight line connecting 1 in the first group with 3 in the second, similarly 2 in the first group with 1 in the second, and so on : lastly he counts the number of intersections of the connecting lines, and makes the sign + or − according as the number is even or odd.

No justification of Tanner's rule is given. It is not difficult, however, to see that it is an old friend (Cramer) with a new face, what is counted being the number of inverted-pairs in 31254.

CAYLEY, A. (1877).

[Note on a theorem in determinants. *Quart. Journ. of Math.*, xv. pp. 55–57 ; or *Collected Math. Papers*, x. pp. 265–266.]

Cayley here gives what he calls the ' proper ' proof of an identity first noted by Fontaine in 1748, namely,

$$0 = \begin{vmatrix} a_1 & . & . & . \\ b_1 & . & . & . \\ . & a_2 & a_3 & a_4 \\ . & b_2 & b_3 & b_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ . & a_2 & a_3 & a_4 \\ . & b_2 & b_3 & b_4 \end{vmatrix}$$

$$= |a_1b_2||a_3b_4| - |a_1b_3||a_2b_4| + |a_1b_4||a_2b_3|,$$

marking it like Bezout (1779) as the first of a series by proceeding to give

$$0 = \begin{vmatrix} a_1 & a_2 & . & . & . & . \\ b_1 & b_2 & . & . & . & . \\ c_1 & c_2 & . & . & . & . \\ . & . & a_3 & a_4 & a_5 & a_6 \\ . & . & b_3 & b_4 & b_5 & b_6 \\ . & . & c_3 & c_4 & c_5 & c_6 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ . & . & a_3 & a_4 & a_5 & a_6 \\ . & . & b_3 & b_4 & b_5 & b_6 \\ . & . & c_3 & c_4 & c_5 & c_6 \end{vmatrix}$$

$$= |a_1b_2c_3|\cdot|a_4b_5c_6| - |a_1b_2c_4|\cdot|a_3b_5c_6| + |a_1b_2c_5|\cdot|a_3b_4c_6| - |a_1b_2c_6|\cdot|a_3b_4c_5|,$$

or, as he writes it,

$$0 = 123\cdot456 - 124\cdot356 + 125\cdot346 - 126\cdot345.$$

Instead of following the matter up so as to include Sylvester's theorem of 1851, he ends his note by recording the fourteen other like identities connected with a 3-by-6 array, and remarking that only five such relations between the ten products are independent.

It is convenient to state here that in the following year, in the course of a short remark on Tanner's paper of 1878, Cayley asserts the existence of the very important theorem afterwards known as the *Law of Complementaries.* (See under Skew Determinants.)

GÜNTHER, S. (1877): SALMON, G. (1877): SERRET, J.-A. (1877).

[Lehrbuch der Determinanten . . . Zweite durchaus umgearbeitete, vermehrte und durch ein Aufgaben-Sammlung bereicherte Auflage. xii+209 pp. Erlangen.]

[Vorlesungen zur Einführung in die Algebra der linearen Transformationen. Deutsch bearbeitet von Dr. Wilhelm Fiedler. Zweite Auflage. xiv+477 pp. Leipzig.]

[Cours d'Algèbre Supérieure. 4ᵉ éd. Tome i. xiv+649 pp. Paris.]

Günther's second edition is all that the title-page proclaims it to be. The historical sketch is improved and lengthened by insertions in reference to Hindenburg, Reiss, Grassmann and others, and by giving deserved attention to Studnička's views of the preceding year. In the chapter on general determinants additional readable notelets on minor matters are introduced; for example, the paragraph (§ 5, p. 38) on the effect which is produced on a determinant by reversing the order of its rows or the order of its columns or the order of both. The bibliographical references at the end of this chapter are also largely increased; the same, indeed, is true of the whole. The Collection of Exercises (pp. 195–207) includes sixty-six items; and it is followed by a two-page List of Text-books in a variety of languages.

The second German edition of Salmon follows closely (pp. 1–66) the third English edition, the additional matter consisting almost entirely of illustrative examples.

Serret devotes a chapter (pp. 527–598) to determinants and certain of their algebraical applications. The latter are the more interesting, the subjects dealt with being the discriminant (still called 'l'invariant') of a quadric, the transformation of a quadric, Sturm's theorem and its application to Lagrange's determinantal equation.

BOURGET, J. (1877): DÖLP, H. (1877): TEIXEIRA, F. G. (1877): ŻELEWSKI, A. (1877).

[Principes élémentaires sur les déterminants, . . . *Journ. de Math. Élém.*, i. pp. 5–11, 33–37, 193–194.]

[Die Determinanten, . . . 2te Auflage, bearbeitet von W. Soldan. iv+94 pp. Darmstadt.]

[Noçoes elementares sobre a theoria dos determinantes. *Jorn. de Sci. Math. e Astron.*, i. pp. 138–141.]

[Theory of Determinants, with Applications. (In Polish.) x+191 pp. Kraków.]

In regard to the first three of these no remark is necessary save that the second edition of Dölp's booklet is an improvement on the first, the changes including (pp. 18–19) a variant of Becker's proof of 1871 regarding the interchange of rows and columns. The fourth I have not seen.

STUDNIČKA, F. J. (1877).

[Beitrag zur Determinanten-Theorie. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), pp. 120–125.]

The main object of the paper is to advocate the use of

$$(A_{11}A_{nn}-A_{1n}A_{n1}) \div |a_{22} \ldots a_{n-1,n-1}|$$

as a means of evaluating $|a_{11}a_{22} \ldots a_{nn}|$. The application of it to obtain a result in continuants and another in skew determinants had already been made (Brioschi, 1856: Trudi, 1862).

DOSTOR, G. (1877).

[Éléments de la Théorie des Déterminants, avec application à l'algébre, la trigonométrie et la géométrie analytique dans le plan et dans l'espace, à l'usage des classes de mathématiques spéciales. xxxi+352 pp. Paris.]

Up till the appearance of Dostor's work no text-book in French had been available for students save the translations of Brioschi

and Baltzer. It is consequently the more to be regretted that the new author does not show that great advance upon his predecessors' work which might reasonably have been expected. Although it is the lengthiest text-book as yet published, a disproportionately large part of it (195 pages) is assigned to so-called 'geometrical applications,' so that only about 155 pages are occupied with the treatment of the theory. Further, since much of this is taken up with rather prolonged explanations and with so-called theorems like that of 1874, some important parts of the theory have received insufficient attention or none at all. It is regrettable to note also that notwithstanding the pains which he takes to be helpful to the beginner he is not always a safe guide. For example, he quite irrelevantly brings in the subject of 'magic squares' (pp. 55–58): he is hopelessly wrong in evaluating a particular three-line determinant (p. 62); and he affirms (p. 75) that

$$\left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \right\| \text{ and } \left| \begin{matrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \right|$$

are equivalents.

D'OVIDIO, E. (1877).

[Le funzioni metriche fondamentali negli spazi di quante si vogliano dimensioni e di curvatura costante. *Atti . . . Accad. dei Lincei: Memorie*, (3) i. pp. 929–986. Abstract in *Math. Annalen*, xii. pp. 403–419.]

At an early stage in this extensive geometrical investigation the author has to deal with the determinants of an m-by-n array, and having shown that they involve $(n-m)m$ quantities that are absolutely independent, he concludes (p. 940) that they must be connected by

$$(n)_m - (n-m)m - 1$$

relations, from which all the other relations are deducible.

This formula, it should be noted, has never been associated with the name of d'Ovidio. Even in Italy of the twentieth century it is spoken of as Vahlen's formula of 1893.

MATZKA, W. (1877).

[Grundzüge der systematischen Einführung und Begründung der Lehre der Determinanten, vermittelst geigneter Auflösung der Gruppen allgemeiner linearer Gleichungen. *Abhandl. d. k. Böhm. Ges. d. Wiss.*, (6) ix. 61 pp.]

What is fresh in this interesting memoir is the mode in which the student is introduced to determinants and becomes acquainted with their fundamental properties. The set of equations

$$\left.\begin{aligned} a_1x+a_2y+a_3z+a_4u+a_5v+\ldots &= a_0,\\ b_1x+b_2y+b_3z+b_4u+b_5v+\ldots &= b_0,\\ c_1x+c_2y+c_3z+c_4u+c_5v+\ldots &= c_0,\\ \ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\}$$

is proposed for solution, and by multiplication and subtraction x is eliminated between every adjoining pair of them, the opportunity being taken to give a definition of a determinant of the second order and to use Laplace's notation for the same. From the resulting set of equations

$$\left.\begin{aligned} |a_1b_2|y+|a_1b_3|z+|a_1b_4|u+|a_1b_5|v+\ldots &= |a_1b_0|,\\ |b_1c_2|y+|b_1c_3|z+|b_1c_4|u+|b_1c_5|v+\ldots &= |b_1c_0|,\\ |c_1d_2|y+|c_1d_3|z+|c_1d_4|u+|c_1d_5|v+\ldots &= |c_1d_0|,\\ \ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\}$$

y is eliminated in like manner, the consequence being that something more is learned regarding two-line determinants, and an opportunity is given for using

$$|a_1b_2c_3| \quad \text{instead of} \quad a_3|b_1c_2| - b_3|a_1c_2| + c_3|a_1b_2|$$

and calling it a determinant of the third order. The second derived set of equations

$$\left.\begin{aligned} |a_1b_2c_3|z+|a_1b_2c_4|u+|a_1b_2c_5|v+\ldots &= |a_1b_2c_0|,\\ |b_1c_2d_3|z+|b_1c_2d_4|u+|b_1c_2d_5|v+\ldots &= |b_1c_2d_0|,\\ |c_1d_2e_3|z+|c_1d_2e_4|u+|c_1d_2e_5|v+\ldots &= |c_1d_2e_0|,\\ \ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\}$$

is next treated in similar fashion, and with the like gains to knowledge. A halt is made on reaching the fourth derived set at the

close of the 17th quarto page, and the opportunity is taken to ascertain, besides other things, what had incidentally been learned about the solution of two equations with two unknowns, three equations with three unknowns, and so forth.

After this introduction there comes a chapter (pp. 22–42) which establishes the basic properties of determinants; then one (pp. 42–46) which recurs to the work of the first, rounding it off by consideration of the $(n-1)^{th}$ derived set of equations; and finally a chapter (pp. 46–61) which treats of the solution of linear equations by the method of 'undetermined multipliers.'

The procedure throughout is not at all of the stereotyped kind.

PAIGE, C. LE (1877): JAMET, V. (1877).

[Sur la multiplication des déterminants. *Nouv. Corresp. Math.*, iii. pp. 141–144, 247, 275–276.]

The row-by-row product of D_1 by D_2 being Δ, Le Paige, founding on the fact that Δ vanishes if two columns of D_1 be identical, concludes that D_1 is a factor of Δ; then similarly that D_2 is a factor; and finally that the cofactor of D_1D_2 in Δ is 1.

To this Jamet naturally objects that the vanishing of D_1 does not necessarily entail the identity of two of its columns, and that therefore the demonstration is imperfect.

Le Paige returns to the subject, and from the vanishing of D_1 draws this time a less objectionable conclusion which leads him to the vanishing of Δ, after which he proceeds with this proof as before.

FROBENIUS, G. (1877).

[Ueber homogene totale Differentialgleichungen. *Crelle's Journ.*, lxxxvi. pp. 1–19.]

At the bottom of the first page Frobenius introduces his use of the word 'Rang' as applied to a determinant, his definition being that *if in a determinant all the minors of the* $(m+1)^{th}$ *order vanish but not all those of the* m^{th} *order, the determinant is said to be of the* m^{th} '*Rang.*' It is seen to be the same as the order of Rouché's 'critical minor,' the latter, however, being from the first not confined to a determinant but used in connection with any oblong array.

SCHERING, E. (1877, 1878).

[Analytische Theorie der Determinanten. *Abhandl. d. k. Ges. d. Wiss.* (Göttingen), xxii. 41 pp.]

[Théorie analytique des déterminants. *Comptes Rendus . . . Acad. des Sci.* (Paris), lxxxvi. pp. 1387–1389.]

This is an imposing memoir founded on a supposedly fresh definition. In essence, however, it involves nothing new, the same fundamental idea having occurred to Scherk in 1825. All Scherk's tediousness is consequently repeated: indeed, by reason of the use of subscripts that are themselves 'subscribed,' Schering's expressions quite outclass Scherk's. The writing of a single term of an n-line determinant occupies the whole breadth of a quarto page, as witness the following result which is reached on page 10:

"Ein Glied der Determinante kann demnach immer in der Form

$$\eta_\nu \kappa_\nu \times \mathfrak{Z} \prod_{m,\mu} (\eta_m - \eta_\mu)(\kappa_m - \kappa_\mu) \times \mathfrak{Z} \prod_{a,\alpha} (h_a - h_\alpha)(a - \alpha) \times \mathfrak{Z} \prod_{b,\beta} (k_b - k_\beta)(b - \beta)$$

dargestellt werden,"—where E is any element denuded of its compound suffix, and $\mathfrak{Z}$ is a function-symbol corresponding to Scherk's ϕ, and being such that $\mathfrak{Z}(x)$ is equal to $+1$ or 0 or -1 according as x is greater than, equal to, or less than 0. The consequences of investigating in this notation Laplace's expansion-theorem, the multiplication-theorem, and the properties connected with Cayley's 'skew symmetry' may well be left undescribed.

VOSS, A. (1877).

[Zur Theorie der orthogonalen Substitutionen. *Math. Annalen*, xiii. pp. 320–374.]

In the course of his work Voss arrives at a result which he recognises as being somewhat less general than Frobenius' second theorem of the preceding year: and with a view to proof he takes the determinant $| b_{16} |$ and formulates the data

(1) $| b_{14} b_{25} b_{36} | \neq 0$,

(2) every four-line minor $= 0$,

(3) $| b_{11} b_{22} b_{33} | = 0$.

He then says that on account of (3) it is possible to determine x, y, z so as to have

$$\left.\begin{aligned} b_{11}x+b_{12}y+b_{13}z &= 0 \\ b_{21}x+b_{22}y+b_{23}z &= 0 \\ b_{31}x+b_{32}y+b_{33}z &= 0 \end{aligned}\right\};$$

next that on account of (2) he can affirm that

$$\begin{vmatrix} b_{11}x+b_{12}y+b_{13}z & b_{14} & b_{15} & b_{16} \\ b_{21}x+b_{22}y+b_{23}z & b_{24} & b_{25} & b_{26} \\ b_{31}x+b_{32}y+b_{33}z & b_{34} & b_{35} & b_{36} \\ b_{a1}x+b_{a2}y+b_{a3}z & b_{a4} & b_{a5} & b_{a6} \end{vmatrix} = 0,$$

where x, y, z have any values whatever and a is 4 or 5 or 6; and in the third place that by giving x, y, z the special values alluded to and using (1), he is entitled to conclude that

$$b_{a1}x+b_{a2}y+b_{a3}z = 0.$$

This means that multiples of the first three columns of the given six-line determinant can be taken whose sum is 0, and therefore that every three-line minor formable from the said columns vanishes.

Since in the special case to which Voss had been led by the subject of his paper the three-line minors form either an axisymmetric or a skew array, the result reached involves the vanishing of $|b_{41}b_{52}b_{63}|$, and therefore is at variance with (1). His view in regard to this is that, if on the supposition, along with others, that a certain magnitude does not vanish, you can prove that it does vanish, then vanish it must.

BARTL, E. (1878).

EINLEITUNG IN DIE THEORIE DER DETERMINANTEN. Zum Gebrauch an Mittelschulen sowie zum Selbstunterricht. iv+96 pp. Prag.]

The space which Dölp gave to geometrical applications is occupied by Bartl with additional explanations and illustrations. The latter writer, indeed, is effusively helpful to the beginner, six pages being devoted to determinants of the second order, and eighteen pages to those of the third and fourth orders, as a preparation for the study of the general properties

MANSION, P. (1878): SERSAWY, V. (1878): ZAHRADNIK, K. (1878): MOLLAME, V. (1878).

[Elemente der Theorie der Determinanten. Mit vielen Uebungsaufgaben. iv+50 pp. Leipzig.]

[Die Fundamente der Determinanten-Theorie. Zum Gebrauche für das erste Studium. vi+42 pp. Wien.]

[O Determinantih drugoga i trecega stupnja. Za porabu visih srednjih ucilista. viii+39 pp. Zagreb.]

[I Determinanti, e loro applicazioni all' algebra ed alla geometria analitica. Nuova esposizione elementare. xii+125 pp. Napoli.]

Of these class-books the one which most invites attention is Mollame's. Notwithstanding its bulk it does not go deeply into the theory, seventy-two pages being given over to algebraical and geometrical applications; but the remaining fifty-one pages give a very clear and attractive exposition well suited as an introduction for university students.

Sersawy, who apparently writes for the same class of readers, gives much less matter and is less concerned with explanation and illustration: his work shows no advance.

The German edition of Mansion's *Éléments* is an improvement on what we have already seen to be good.

Zahradnik's little introduction confines itself in the main to determinants of the second and third order.

GARBIERI, G. (1878).

[Intorno al trattato sui determinanti del Prof. Sigismondo Günther. *Bull. di Bibliogr. e di Storia delle Sci. mat. e fis.*, xi. pp. 257–318.]

In the sixty-two pages of this 'rivista bibliographica' exactly thirty-one are devoted to an account of Günther's first chapter, which, it will be recalled, deals with the history of the subject and which itself contains only thirty-one pages in the original. Garbieri's rôle is almost that of a mere reporter, his main additions being due to the editor's practice of pedantically giving the full title-page of every writing referred to. He also makes several corrections.

SCHOLTZ, A. (1877–8).

[Sechs Punkte eines Kegelschnittes. *Archiv d. Math. u. Phys.*, lxii. (1878), pp. 317–324; or *Müegyetemi Lapok*, ii. pp. 65 *et seqq.*]

Like Brill (1871), Scholtz requires for geometrical purposes the case of Schläfli's theorem of 1851, in which $n, r = 3, 2$. His proof (§ 2, pp. 319–321) is not essentially different from Brill's.

FALK, M. (1878).

[Elementary demonstration of the theorem of multiplication of determinants. *Report . . . British Assoc. . . .* xlviii. pp. 473–475.]

Falk's mode of proof is the gradational,—that is to say, he shows that, if the theorem holds for determinants of the $(n-1)^{\text{th}}$ order, it will also hold for determinants of the n^{th} order. It is put forward as an improvement of what he calls Brioschi's, but which is really Spottiswoode's. A sufficient idea of it will be obtained by considering the case where n is 3. Manifestly we have

$$\left.\begin{aligned} x_1X_1+y_1Y_1+z_1Z_1 &= |x_1y_2z_3| \\ x_2X_1+y_2Y_1+z_2Z_1 &= 0 \\ x_3X_1+y_3Y_1+z_3Z_1 &= 0 \end{aligned}\right\},$$

and thence by multiplication and addition

$$\begin{aligned} (a_1x_1+a_2x_2+a_3x_3)X_1+(a_1y_1+a_2y_2+a_3z_3)Y_1+(a_1z_1+a_2z_2+a_3z_3)Z_1 &= a_1|x_1y_2 \\ (b_1x_1+b_2x_2+b_3x_3)X_1+(b_1y_1+b_2y_2+b_3z_3)Y_1+(b_1z_1+b_2z_2+b_3z_3)Z_1 &= b_1|x_1y_2 \\ (c_1x_1+c_2x_2+c_3x_3)X_1+(c_1y_1+c_2y_2+c_3z_3)Y_1+(c_1z_1+c_2z_2+c_3z_3)Z_1 &= c_1|x_1y_2 \end{aligned}$$

Consequently, if the determinant of the coefficients of X_1, Y_1, Z_1 here be denoted by Δ, we obtain on solution

$$\begin{aligned} X_1\Delta &= |x_1y_2z_3| \cdot \begin{vmatrix} a_1 & a_1y_1+a_2y_2+a_3y_3 & a_1z_1+a_2z_2+a_3z_3 \\ b_1 & b_1y_1+b_2y_2+b_3y_3 & b_1z_1+b_2z_2+b_3z_3 \\ c_1 & c_1y_1+c_2y_2+c_3y_3 & c_1z_1+c_2z_2+c_3z_3 \end{vmatrix} \\ &= |x_1y_2z_3| \cdot \begin{vmatrix} a_1 & a_2y_2+a_3y_3 & a_2z_2+a_3z_3 \\ b_1 & b_2y_2+b_3y_3 & b_2z_2+b_3z_3 \\ c_1 & c_2y_2+c_3y_3 & c_2z_2+c_3z_3 \end{vmatrix}. \end{aligned}$$

But the last determinant here is by hypothesis equal to

$$a_1|b_2c_3||y_2z_3| \;-\; b_1|a_2c_3||y_2z_3| \;+\; c_1|a_2b_3||y_2z_3|,$$

i.e. $|a_1b_2c_3|\cdot|y_2z_3|$;

so that $$X_1\Delta = |x_1y_2z_3|\cdot|a_1b_2c_3|\cdot|y_2z_3|,$$

and $$\therefore\ \Delta = |x_1y_2z_3|\cdot|a_1b_2c_3|$$

as desired.

BARANIECKI, M. A. (1878).

[Teorya Wyznaczników (Determinantów). Kurs uniwersytecki. xxii+600 pp. Paris.]

In bulk Baraniecki's text-book greatly surpasses all its predecessors. It contains 250 pages more than Dostor's, and the disparity is enhanced by the fact that whereas the French writer gives 195 pages to geometry, the Pole gives only 48, relegating them, too, not unjustifiably to an appendix. This great increase is due to a variety of causes,—a broad-margined page, an occasional want of discrimination between the important and the unimportant, and a general tendency to fullness of exposition both in the text and in the working of examples.

The first seven chapters (pp. 1–280) concern determinants in general; the last three (pp. 388–530) deal respectively with axisymmetric, skew and functional determinants: and the remaining two (pp. 281–387) with linear equations and other so-called algebraical applications. Besides the appendix just mentioned there is also one of sixteen pages dealing with cubic determinants.

All the contents are to be found in earlier text-books, but the writer has fully assimilated his material, and the sources used are as a rule scrupulously specified. For Polish students unfamiliar with other languages the handsome volume must be a most useful repository.

In view of the author's general accuracy it is surprising to find him led away by Dostor in the definition of

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

VAŠČENKO-ZACHARČENKO, M. E. (1878).

[THEORY OF DETERMINANTS AND THEORY OF FORMS. (In Russian.) vi+501 pp. Kieff.]

Ostensibly there is here devoted to the theory of determinants 240 pages, but much of the space is occupied with algebraical matter that is not closely relevant. On the main subject there is nothing calling for attention. The tables of symmetric functions and eliminants at the end (pp. 220–240) are apparently reproduced from Salmon's *Modern Higher Algebra*.

WOLSTENHOLME, J. (1878).

[MATHEMATICAL PROBLEMS, . . . (pp. 275–280). 2nd edition. x+480 pp. London.]

Wolstenholme * gives the 5-by-6 array

$$\begin{Vmatrix} . & . & . & a & b & c \\ . & . & z & a & b & . \\ . & y & . & a & . & c \\ x & . & . & . & b & c \\ x & y & z & . & . & . \end{Vmatrix},$$

which conveniently illustrates Dodgson's test for 'evanescence' (see above, p. 25). The first two primary minors can be shown to vanish by diminishing in each case the last row by the sum of the three immediately preceding rows, and they are easily seen to have a common 4-line minor which is not equal to 0.

FALK, M. (1878).

[Sur une propriété des déterminants nuls. *Nouv. Corresp. Math.*, iv. pp. 373–376.]

The property in question is that *if a determinant be equal to* 0, *it is possible to transform it so as to have a row or column of zeros.* By way of proof it is pointed out that the determinant being $|\, a_{1n} \,|$ the number of sets of non-zero values of $x_1, x_2, \ldots, x_n$ which will make the expressions

$$\begin{aligned} &a_{11}x_1+a_{12}x_2+\ldots+a_{1n}x_n, \\ &a_{21}x_1+a_{22}x_2+\ldots+a_{2n}x_n, \\ &\quad\ldots\ldots\ldots\ldots\ldots \\ &a_{n1}x_1+a_{n2}x_2+\ldots+a_{nn}x_n, \end{aligned}$$

* But it appears in *Cambridge Univ. Exam. Papers*, iv. (1875), p. 224.

vanish is known to be infinite, and that it is possible to have a set in which the value of one of the x's, x_k say, is 1. This being accepted, it is easily seen what multiples of the other columns must be subtracted from the k^{th} in order to transform it into a column of zeros. The deduction is then readily made that *to ensure the vanishing of a determinant it is necessary and sufficient that two of its columns have their elements proportional.*

DOSTOR, G. (1879).

[Evaluation d'un certain déterminant. *Archiv d. Math. u. Phys.*, lxiv. pp. 57–59.]

The result is a theorem of the type of Catalan's of 1846 and Tait's of 1867, and is to the effect that *the determinant got from* $|\, a_{1n} \,|$ *by diminishing every row by the sum of all the other rows is equal to*

$$-(n-2)2^{n-1} \,|\, a_{1n} \,|.$$

KÖNIG, J. (1879, 1877).

[Ein Beweis des Multiplicationstheorems für Determinanten. *Math. Annalen*, xiv. (1879), pp. 507–509; also, in Magyar, *Müegyetemi Lapok* (Budapest), ii. pp. 271–274.]

The determinant for the product of $|\, x_1y_2z_3 \,|$ and $|\, a_1b_2c_3 \,|$ is got as the third of a series of results of which the first two are

$$|\, x_1y_2z_3 \,| \cdot a_1 = \begin{vmatrix} a_1x_1 & x_2 & x_3 \\ a_1y_1 & y_2 & y_3 \\ a_1z_1 & z_2 & z_3 \end{vmatrix},$$

and

$$|\, x_1y_2z_3 \,| \cdot |\, a_1b_2 \,| = \begin{vmatrix} a_1x_1+a_2x_2 & b_1x_1+b_2x_2 & x_3 \\ a_1y_1+a_2y_2 & b_1y_1+b_2y_2 & y_3 \\ a_1z_1+a_2z_2 & b_1z_1+b_2z_2 & z_3 \end{vmatrix}.$$

The first is manifestly true; the right-hand member of the second is got from that of the first by performing the operation

$$\mathrm{col}_1 + a_2\,\mathrm{col}_2, \qquad |\, a_1b_2 \,|\,\mathrm{col}_2 + b_1\,\mathrm{col}_1, \qquad \mathrm{col}_2 \div a_1;$$

and the right-hand member of the third is got from that of the second by performing the operations,

$$\mathrm{col}_1 + a_3\mathrm{col}_3, \qquad \mathrm{col}_2 + b_3\,\mathrm{col}_3;$$
$$|\, a_1b_2c_3 \,|\,\mathrm{col}_3 - |\, b_1c_2 \,|\,\mathrm{col}_1 + |\, a_1c_2 \,|\,\mathrm{col}_2;$$
$$\mathrm{col}_3 \div |\, a_1b_2 \,|.$$

BATTELLI, S. (1878–9): LIT, R. R. (1879).

[I primi Elementi della Teoria dei Determinanti, con alcune applicazioni all' algebra ed alla geometria. Sch. Progr. Rovereto.]

[Beginselen van de Leer der Determinanten, met voorbeelden en toepassingen. viii+38 pp. Amsterdam.]

Both of these are quite elementary, and there is nothing about them more than their titles would lead one to expect.

SIMONNET (1879).

[Sur les conditions de l'existence d'un nombre déterminé de racines communes à deux équations données. *Comptes Rendus . . . . Acad. des Sci.* (Paris), lxxxviii. pp. 223–224.]

The basis of the special investigation here is a carefully formulated theorem in general determinants which we may put in the form $|a_{1m}| \cdot \mathrm{B}_{kr} = \mathrm{A}_{1r} \cdot |b_{1n}|_{(k),1} + \mathrm{A}_{2r} \cdot |b_{1n}|_{(k),2} + \ldots + \mathrm{A}_{mr} \cdot |b_{1n}|_{(k),m}$, where $|a_{1m}|$, $|b_{1n}|$ are any two determinants; $|\mathrm{A}_{1m}|$, $|\mathrm{B}_{1n}|$ their adjugates; and $|b_{1n}|_{(k),s}$ what $|b_{1n}|$ becomes when its k^{th} row is supplanted by the first n elements of the s^{th} row of $|a_{1m}|$. No proof is adduced; but if, in order to gain breadth, we view the theorem as giving the product of two determinants of the p^{th} and q^{th} orders ($p > q$) in the form of a sum of products of two determinants of the $(p-1)^{\text{th}}$ and $(q+1)^{\text{th}}$ orders, there is no difficulty in establishing it, and its relation to previous work becomes more readily apparent. For example, when $p, q = 4, 2$ (or, when $m, n = 4, 3$), we have

$$|a_1b_2c_3d_4| \cdot |e_1f_2| = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & . & . \\ b_1 & b_2 & b_3 & b_4 & . & . \\ c_1 & c_2 & c_3 & c_4 & . & . \\ d_1 & d_2 & d_3 & d_4 & . & . \\ e_1 & e_2 & e_3 & . & e_1 & e_2 \\ f_1 & f_2 & f_3 & . & f_1 & f_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & -a_1 & -a_2 \\ b_1 & b_2 & b_3 & b_4 & -b_1 & -b_2 \\ c_1 & c_2 & c_3 & c_4 & -c_1 & -c_2 \\ d_1 & d_2 & d_3 & d_4 & -d_1 & -d_2 \\ e_1 & e_2 & e_3 & . & . & . \\ f_1 & f_2 & f_3 & . & . & . \end{vmatrix}$$

$$= -|a_1b_2c_4| \cdot |d_1e_2f_3| + |a_1b_2d_4| \cdot |c_1e_2f_3| - |a_1c_2d_4| \cdot |b_1e_2f_3| + |b_1c_2d_4| \cdot |a_1e_2f_3|.$$

In Sylvester's similar theorem of 1839 p and q are equal.

PAIGE, C. LE (1879): JAMET, V. (1879).

[Sur la multiplication des déterminants. *Nouv. Corresp. Math.*, v. pp. 76–79.]

[Sur la multiplication des déterminants. *Nouv. Corresp. Math.*, v. pp. 79–81.]

Le Paige and Jamet recall their work of 1877, and again try to show that if D_1 vanish, then Δ must vanish, using now the fact that the vanishing of D_1 entails the results formulated by Falk.

MUIR, T. (1879).

[General theorems on determinants. *Transac. R. Soc. Edinburgh*, xxix. pp. 47–54.]

The first theorem, though only now published, is one referred to by the same author in a paper of 1873–74 as his "extension of Laplace's theorem" (*Proceed. R. Soc. Edinburgh*, viii. p. 23). It stands thus: *In a determinant of the* n^{th} *degree, if the rows from the* 1^{st} *to the* q^{th} *inclusive and the rows from the* p^{th} *to the* n^{th} *inclusive be taken; and if a minor of the* $(q-p+1)^{th}$ *degree be chosen from the rows common to these two sets; and if from the first set each minor of the* q^{th} *degree containing the chosen minor be multiplied by the minor which contains both the complementary of the former and the chosen minor; and if the sign* $(-1)^s$ *be prefixed to the product,* s *being the sum of the numbers indicating the rows and columns from which the first factor is formed increased by* $q-p+1$ *for every such number greater than* q; *then the sum of the products thus obtained is equal to the product of the chosen minor and the original determinant.* The mode of proof will probably be sufficiently explained by taking the case where $n, p, q = 5, 3, 4$. First we form the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & . & . & a_5 \\ b_1 & b_2 & b_3 & b_4 & . & . & b_5 \\ c_1 & c_2 & c_3 & c_4 & . & . & c_5 \\ d_1 & d_2 & d_3 & d_4 & . & . & d_5 \\ c_1 & c_2 & . & . & c_3 & c_4 & c_5 \\ d_1 & d_2 & . & . & d_3 & d_4 & d_5 \\ e_1 & e_2 & . & . & e_3 & e_4 & e_5 \end{vmatrix},$$

which, by Laplace's expansion-theorem, is equal to

$$|a_1b_2c_3d_4| \cdot |c_3d_4e_5| \quad - \quad |a_1b_3c_4d_5| \cdot |c_2d_3e_4| \quad + \quad |a_2b_3c_4d_5| \cdot |c_1d_3e_4|.$$

Next we perform on the determinant the operations

$$\begin{array}{cc} \mathrm{col}_3+\mathrm{col}_5, & \mathrm{col}_4+\mathrm{col}_6 \\ \mathrm{row}_5-\mathrm{row}_3, & \mathrm{row}_6-\mathrm{row}_4, \end{array}$$

and so obtain the form

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & . & . & a_5 \\ b_1 & b_2 & b_3 & b_4 & . & . & b_5 \\ c_1 & c_2 & c_3 & c_4 & . & . & c_5 \\ d_1 & d_2 & d_3 & d_4 & . & . & d_5 \\ . & . & . & . & c_3 & c_4 & . \\ . & . & . & . & d_3 & d_4 & . \\ e_1 & e_2 & e_3 & e_4 & e_3 & e_4 & e_5 \end{vmatrix}$$

which, by again using Laplace's expansion-theorem, is seen to be equal to

$$|\, a_1b_2c_3d_4e_5 \,| \cdot |\, c_3d_4 \,|.$$

The required result is thus reached.

Of the second theorem the first case is identical with that given by Hermite in 1849, and the other cases result from the continuance of the condensation so begun. For example, n being 5, we have

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} = \begin{vmatrix} |a_1b_2| & |a_2b_3| & |a_3b_4| & |a_4b_5| \\ |a_1c_2| & |a_2c_3| & |a_3c_4| & |a_4c_5| \\ |a_1d_2| & |a_2d_3| & |a_3d_4| & |a_4d_5| \\ |a_1e_2| & |a_2e_3| & |a_3e_4| & |a_4e_5| \end{vmatrix} \div a_2a_3a_4$$

$$= \begin{vmatrix} |a_1b_2c_3| & |a_2b_3c_4| & |a_3b_4c_5| \\ |a_1b_2d_3| & |a_2b_3d_4| & |a_3b_4d_5| \\ |a_1b_2e_3| & |a_2b_3e_4| & |a_3b_4e_5| \end{vmatrix} \div |a_2b_3| \cdot |a_3b_4|$$

$$= \begin{vmatrix} |a_1b_2c_3d_4| & |a_2b_3c_4d_5| \\ |a_1b_2c_3e_4| & |a_2b_3c_4e_5| \end{vmatrix} \div |a_2b_3c_4|.$$

Coordinate with these, but in a sense subordinate to them, are given

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} = \begin{vmatrix} |a_1b_2| & |a_1b_3| & |a_1b_4| & |a_1b_5| \\ |a_1c_2| & |a_1c_3| & |a_1c_4| & |a_1c_5| \\ |a_1d_2| & |a_1d_3| & |a_1d_4| & |a_1d_5| \\ |a_1e_2| & |a_1e_3| & |a_1e_4| & |a_1e_5| \end{vmatrix} \div a_1^3$$

$$= \begin{vmatrix} |a_1b_2c_3| & |a_1b_2c_4| & |a_1b_2c_5| \\ |a_1b_2d_3| & |a_1b_2d_4| & |a_1b_2d_5| \\ |a_1b_2e_3| & |a_1b_2e_4| & |a_1b_2e_5| \end{vmatrix} \div |a_1b_2|^2$$

$$= \begin{vmatrix} |a_1b_2c_3d_4| & |a_1b_2c_3d_5| \\ |a_1b_2c_3e_4| & |a_1b_2c_3e_5| \end{vmatrix} \div |a_1b_2c_3|,$$

the first being Chio's of 1853, and the others obtained by means of it.

The third and last theorem concerns the multiplication of an n-line determinant by an n-termed expression. Its nature will be fully understood from merely stating the case where n is 3, namely,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} (\xi_1+\xi_2+\xi_3)$$

$$= \begin{vmatrix} a_1\xi_1 & a_2\xi_2 & a_3\xi_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1\xi_1 & b_2\xi_2 & b_3\xi_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1\xi_1 & c_2\xi_2 & c_3\xi_3 \end{vmatrix},$$

and from suggesting an examination of the cofactor of any one of the ξ's on the right-hand side.

It is well to note in passing that the identities appearing under the second theorem may be viewed as propositions regarding *compound* determinants. As such the first set had been foreshadowed and in part formulated by Spottiswoode in 1853 (*Hist.*, ii. pp. 202–204), and the specialized set fully enunciated by Sylvester in 1851.

FÜRSTENAU, E. (1879).

[Beiträge zur Theorie der Determinanten. *Crelle's Journ.*, lxxxix. pp. 86–88.]

The first of Fürstenau's theorems is identical with Sylvester's of 1839, of which an easy proof was suggested by Cayley in 1843 (*Hist.*, i. p. 233 ; ii. pp. 10–11).

The second is that *if the successive minor diagonals on the upper side of and parallel to the main diagonal be multiplied by* $x, x^2, x^3, \ldots$ *respectively, and those on the under side by* $x^{-1}, x^{-2}, x^{-3}, \ldots$ *respectively, the value of the determinant remains unaltered.* In proof of this, it is pointed out that the multiplications in question are equivalent to multiplying the element in the $(r, s)^{\text{th}}$ place by x^{s-r}, and therefore equivalent to multiplying any term of the determinant by

$$x^{\Sigma s - \Sigma r}, \text{ that is, } x^0.$$

When $x = -1$ we have Janni's case of 1874.

CARR, G. S. (1879).

[Question 5752. *Math. from Educ. Times*, xxxii. pp. 54–55.]

What is here proposed to be indicated is a method for changing a given algebraical expression into the form of a determinant. The process, which is illustrated by the expression

$$abcd+bfgi+fh^2+def+cghp+ahr+elpr \\ -fhpr-ablr-ach^2-fgh-bdf^2-efhl-cdep,$$

is commonplace when compared with Dodgson's of 1867.

STUDNIČKA, F. J. (1879, 1880).

[Ueber eine neue Determinantentransformation. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), Jahrg. 1879, pp. 487–494.]

[O nové poučce determinantní. *Časopis pro pěstování math. a fys.*, ix. pp. 97–103.]

The transformation referred to is the second of those spoken of as 'condensations,' and fully dealt with earlier in the same year by Muir (see above, pp. 80–81).

CHAPTER II.

DETERMINANTS AND LINEAR EQUATIONS, FROM 1861 TO 1878.

In the preceding volume the contributions to this subject were so unimportant as not to warrant the segregation of them, and of course in the first volume segregation, even if possible, would have been highly undesirable. For the period now reached, however, the circumstances are in every respect different, and facility of reference can receive due attention.

SMITH, H. J. S. (1861).

[On systems of linear indeterminate equations and congruences. *Philos. Transac. R. Soc. London*, cli. pp. 293–326 ; or *Collected Math. Papers*, i. pp. 367–409.]

There here appears the theorem that *if every determinant of the augmented array of a redundant system of linear equations be equal to zero, while those of the unaugmented array are not all zero, the system admits of one solution and of one only.* Thus, the redundant set of equations

$$\left.\begin{aligned} a_1x+a_2y &= f \\ b_1x+b_2y &= g \\ c_1x+c_2y &= h \\ d_1x+d_2y &= k \end{aligned}\right\}$$

has one and only one solution if

$$\begin{Vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ f & g & h & k \end{Vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{vmatrix} \neq 0.$$

It will be noticed that two useful terms are introduced incidentally in the statement, 'augmented array' and 'unaugmented array' representing important entities connected with the study of simultaneous linear equations.

TRUDI, N. (1862).

[Teoria de' Determinanti, . . . xii+268 pp. Napoli.]

The section devoted to the Solution of Linear Equations (pp. 113–121) is fuller than in any previous exposition, and a great improvement on Baltzer's of 1857. The fifth paragraph of it deals with the case of

$$a_{r1}x_1+a_{r2}x_2+\ \ldots\ +a_{rn}x_n = u_r \Big\}_{r=1}^{r=n}$$

when the common denominator of the unknowns vanishes and also one of the numerators. From this the first deduction of course is that *the other numerators must vanish also*; and the proof rests on the fact that the vanishing of the numerator of x_1 gives us

$$u_1\mathrm{A}_{11}+u_2\mathrm{A}_{21}+\ \ldots\ +u_n\mathrm{A}_{n1} = 0,$$

and the vanishing of the common denominator gives us

$$\frac{\mathrm{A}_{11}}{\mathrm{A}_{1s}} = \frac{\mathrm{A}_{21}}{\mathrm{A}_{2s}} = \ \ldots\ = \frac{\mathrm{A}_{n1}}{\mathrm{A}_{ns}}.$$

The second deduction is that *one of the given equations*, say the first, *is derivable from the others*; and the proof consists in obtaining from these others by multiplication and addition the equality

$$\begin{aligned}&(a_{21}\mathrm{A}_{2s}+a_{31}\mathrm{A}_{3s}+\ \ldots\ +a_{n1}\mathrm{A}_{ns})x_1\\ &+\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\\ &+\ a_{2n}\mathrm{A}_{2s}+a_{3n}\mathrm{A}_{3s}+\ \ldots\ +a_{nn}\mathrm{A}_{ns})x_n\\ &=\ u_2\,\mathrm{A}_{2s}+u_3\,\mathrm{A}_{3s}+\ \ldots\ +u_n\mathrm{A}_{ns},\end{aligned}$$

and thence the equality

$$-a_{11}\mathrm{A}_{1s}x_1-a_{12}\mathrm{A}_{1s}x_2-\ \ldots\ -a_{1n}\mathrm{A}_{1s}x_n = -u_1\mathrm{A}_{1s},$$

from which the first equation of the set at once follows *if* A_{1s} *is not zero*. The appearance of this extra condition at the close deserves note, as also the fact that it is unmentioned in connection with the

first deduction. A proof is also given of the converse of one of the theorems regarding the eliminant, namely, that *if*

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & u_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & u_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & u_{n+1} \end{vmatrix} = 0,$$

then any one, say the last, *of the equations*

$$a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n = u_r \Big\}_{r=1}^{r=n+1}$$

is a consequence of the others; the procedure being to perform on the given vanishing determinant the operation,

$$\mathrm{col}_{n+1} - (x_1\mathrm{col}_1 + x_2\mathrm{col}_2 + \cdots + x_n\,\mathrm{col}_n),$$

with the result

$$\{u_{n+1} - (a_{n+1,1}x_1 + a_{n+1,2}x_2 + \cdots + a_{n+1,n}x_n)\}\,|\,a_{1n}\,| = 0,$$

after which we have only to assume $|\,a_{1n}\,| \neq 0$.

Lastly, attention is drawn to the simple fact that if we have n linear equations

$$\left.\begin{aligned} a_1x + a_2y + a_3z + a_4 &= 0 \\ b_1x + b_2y + b_3z + b_4 &= 0 \\ c_1x + c_2y + c_3z + c_4 &= 0 \end{aligned}\right\}$$

determining n unknowns x, y, z, then any other linear function of the same unknowns, $D_1x + D_2y + D_3z + D_4$, is readily expressible in terms of the coefficients, namely, is equal to

$$|\,a_1b_2c_3D_4\,| \div |\,a_1b_2c_3\,|.$$

BALTZER R. (1864).

[Theorie und Anwendungen der Determinanten. 2te verm. Aufl. vi+224 pp. Leipzig.]

Baltzer now notes (§ 8. 2) how the solution of n non-homogeneous equations in n unknowns is deducible from the solution of n homogeneous equations in $n+1$ unknowns, without, however, referring to the possibility of the reverse procedure. He also gives the second half of Kronecker's contribution already mentioned (p. 14), namely (§ 8. 3, iii.), that *if the minor* $|\,a_{11} \ldots a_{kk}\,|$ *of* $|\,a_{11} \ldots a_{nn}\,|$

do not vanish and all the minors of higher order do vanish, the set of equations

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = 0 \Big\}_{r=1}^{r=n}$$

is consistent and (n−k)*-fold indeterminate.*

For example, taking $n = 5$ and $k = 3$, we can, because of the datum regarding $|\, a_1b_2c_3 \,|$, obtain from the first three equations satisfying values of x_1, x_2, x_3 in terms of x_4, x_5; and we can then prove that these values will also satisfy the remaining equations, because the requisite substitution in the left-hand members of the said equations leads to the expressions

$$|\, a_1b_2c_3d_4 \,|\, x_4 \quad + \quad |\, a_1b_2c_3d_5 \,|\, x_5,$$
$$|\, a_1b_2c_3e_4 \,|\, x_4 \quad + \quad |\, a_1b_2c_3e_5 \,|\, x_5,$$

which, by reason of the second datum, are each equal to 0 for any finite values of x_4, x_5.

PADULA, F. (1864).

[Ricerche di geometria analitica. *Atti del R. Ist. d' Incorrig.* . . . (Napoli), (2) i. pp. 37–76.]

In a footnote extending over parts of pp. 42–46 a proof is attempted of the theorem that *if any two primary minors of an* m*-by-*(m+1) *array vanish, all the others vanish also.* This, which is the case of Cayley's theorem of 1843, where $n = q = 2$, is also identified with the theorem already formulated by Trudi, that *if one of the unknowns in the set of equations*

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = u_r \Big\}_{r=1}^{r=n}$$

take the form 0/0, *so also must all the others.*

DODGSON, C. L. (1867).

[An Elementary Treatise on Determinants, ... viii+143 pp. London.]

Chapters iii. and iv. establish conditions under which equations of a set are consistent, inconsistent, interdependent, and so forth: also, consequences flowing from consistency: and, finally, necessary and sufficient tests for consistency. They may thus be expected to

contain extensions and improvements of previous incidental work on the subject. The results are helpfully collected in tabular form (pp. 134–138); and by way of illustration four different sets of equations with arithmetical coefficients are investigated in full detail (pp. 111–116). To a certain extent tabulation is facilitated and conciseness furthered by using the term 'V-block' to stand for the array composed of the coefficients of the unknowns, and 'B-block' for the augmented array, thus making possible such short sentences as

$$\begin{array}{ll} \mathrm{V} = 0, \text{ or } \neq 0; & \mathrm{B} = 0, \text{ or } \neq 0; \\ |\mathrm{V}| = 0, \text{ or } \neq 0; & |\mathrm{B}| = 0, \text{ or } \neq 0; \\ \|\mathrm{V}\| = 0, \text{ or } \neq 0; & \|\mathrm{B}\| = 0, \text{ or } \neq 0. \end{array}$$

This has its advantages, but there is also much against it, and it has certainly not commended itself to subsequent writers. As hitherto, therefore, we shall state the propositions at full length, merely agreeing with Dodgson to speak of an array as being 'evanescent' when all its primary minors vanish.

Taking chapter iii. (pp. 32–59) and noting first those which are not specially concerned with equations that are homogeneous, we have:

(Ia_1). *If there be* n *linear equations containing* n+r *unknowns, and if a primary minor of the unaugmented array be not zero, the equations are consistent; further, any values being arbitrarily assigned to the* r *unknowns unconnected with the said minor, one satisfying value, and only one, can be found for each of the other unknowns.* (Prop. II.)

(Ia_2). *If there be* n *linear equations containing* n+r *unknowns (where* r *may be* 0), *and if the unaugmented array be evanescent but the augmented array be not, the equations are inconsistent: and, on the other hand, if they be consistent, and have their unaugmented array evanescent, their augmented array is also evanescent.* (Prop. V., XV.)

(Ia_3). *If there be* n *linear equations containing* n+r *unknowns (where* r *may be* 0): *and if there be* n−k *of the equations having their unaugmented array not evanescent: and if when these* n−k *equations are taken along with each of the remaining equations in succession, each so formed set of* n−k+1 *equations has its augmented array evanescent; then* (1) *the equations are consistent:* (2) *if any non-evanescent primary*

minor of the unaugmented array of these n−k *equations be selected, the* k+r *unknowns whose coefficients are not contained in it may have arbitrary values assigned to them, and for each such set of arbitrary values there is one set of satisfying values for the other unknowns*: *and* (3) *the remaining equations are dependent on the said* n−k *equations.* (Prop. X., Cor.)

Next we have propositions which, like the foregoing, do not specially concern homogeneous equations, but which bear on 'redundant' systems:

(Ib_1). *If there be* n *linear equations containing* n−r *unknowns, and if the augmented array be not evanescent, the equations are inconsistent*: *and, conversely, if the equations be consistent, the augmented array is evanescent.* (Prop. III., XIII.)

(Ib_2). *If there be* n *linear equations containing* n−r *unknowns*: *and if there be* n−r *of the equations whose unaugmented array is not evanescent*: *and if when these* n−r *equations are taken along with each of the remaining equations in succession, each so formed set of* n−r+1 *equations has its augmented array evanescent*; *then* (1) *the* n *equations are consistent*: (2) *there is only one set of satisfying values for the unknowns*: *and* (3) *the remaining equations are dependent on the said* n−r *equations.* (Prop. VIII.)

In the third place we have propositions which refer specially to homogeneous equations, the first being Jacobi's of 1841 (*Hist.*, ii. p. 324):

(IIa_1). *If there be* n *linear homogeneous equations containing* n+1 *unknowns, then in every set of satisfying values for the unknowns the values bear to one another one and the same set of ratios.* (Prop. II., Cor.)

(IIa_2). *In every set of* n *linear homogeneous equations containing* n+r *unknowns, there is for the unknowns a set of satisfying values of which two at least are not zero.* (Prop. XII.)

(IIa_3). *If* n *linear homogeneous equations containing* n+r *unknowns have for the unknowns a set of satisfying values of which one is not zero, and if on deleting the coefficients of the unknown which takes this non-zero value the remaining array is evanescent, then the whole array is evanescent.* (Prop. XV., Cor.)

Lastly, we have propositions which still refer specially to homogeneous equations, but now concern 'redundant' systems mainly:

(IIb_1). *If* n *linear homogeneous equations containing* n−r *unknowns* (*where* r *may be* 0) *have their unaugmented array evanescent, there is for the unknowns a set of satisfying values of which two at least are not zero, and of the* n *equations* r+1 *are dependent on the rest: and, conversely, if the equations are satisfied by a set of values which are not all zero, the unaugmented array is evanescent.* (Prop. XI., Cor. xix.)

As already implied, the proofs of all these propositions are most carefully set forth by Dodgson: we have considered it sufficient to indicate, by giving their numbers, the order which he found to be logically suitable for them.

The main proposition of chap. iv., the chapter which deals with tests for consistency, is (p. 61) as follows: *The necessary and sufficient test for a set of linear equations, which are not all homogeneous, being consistent is that either* (1) *there is one of them such that when it is taken along with every one of the remaining equations separately, each pair of equations so formed has its augmented array evanescent: or* (2) *there are* m *of them,* m > 1, *which contain at least* m *unknowns and have their unaugmented array not evanescent and are such that, when they are taken along with one of the remaining equations, each so formed set of equations has its augmented array evanescent.*

Of five Appendices the first (pp. 111–116) gives a very thorough exposition of how to proceed in the investigation of a given set of linear equations. As examples, sets of equations that are not all homogeneous, the two

$$\left.\begin{aligned} u+\ v-2x+y-\ z &= 6\\ 2u+2v-4x-y+\ z &= 9\\ u+\ v-2x\qquad\quad &= 5\\ u-\ v+\ x+y-2z &= 0 \end{aligned}\right\},\qquad \left.\begin{aligned} 3x-\ y &= -7\\ 6x-2y &= -14\\ x+\ y &= -1\\ x+5y &= 3\\ 5x+\ y &= -9 \end{aligned}\right\},$$

are dealt with in succession, being tested (1) for consistency, (2) for interdependence, and (3) for the existence of unknowns to which arbitrary values may be simultaneously assigned. Similarly, the set of homogeneous equations

$$\left.\begin{aligned} 2u+\ v+2x+\ y+\ 3z &= 0\\ 5u+3v-4x+3y-\ 6z &= 0\\ u+\ v-8x+\ y-12z &= 0 \end{aligned}\right\}$$

is examined, the test for consistency being now in regard to satisfying values that are not all zero.

VERSLUYS, J. (1870): VALERIANO, V. (1871): DONNINI, P. (1875).

[Discussion complète d'un système d'équations linéaires. *Archiv d. Math. u. Phys.*, lii. pp. 257–277.]

[Sistema generale di n equazioni lineari fra n incognite. *Giornale di Mat.*, ix. pp. 371–376.]

[Un sistema particolare d'equazioni lineari. pp. (?). Livorno.]

The theorem in determinants which is made the basis of Versluys' discussion is identical with Dodgson's of 1867 (chap. v. p. 67); and the body of the paper may be compared in subject with Dodgson's chapters iii. and iv. Versluys distinguishes eight different cases in the solution of

$$a_r x+b_r y+c_r z = d_r \Big\}_{r=1}^{r=3},$$

and twenty-five in the solution of

$$a_r x+b_r y+c_r z+d_r w = e_r \Big\}_{r=1}^{r=4}.$$

Valeriano confines himself to cases where the number of equations and the number of unknowns are the same, beginning with equations that are homogeneous.

FONTENÉ, G. (1875).

[Théorème pour la discussion d'un système de n équations du premier degré à n inconnues. *Nouv. Annales de Math.*, (2) xiv. pp. 481–487.]

In his subject Fontené restricts himself as Valeriano did (1871); but his aim is quite different, namely, to embody all his results in a single proposition. His success is more apparent than real, the enunciation of the proposition occupying almost a full page and not being a very compact whole. The purely determinant theorem which he uses (pp. 485–486) is in essence identical with Kronecker's of 1864.

ROUCHÉ, E. (1875, 1880): MÉRAY, C. (1875).

[Sur la discussion des équations du premier degré. *Comptes Rendus . . . Acad. des Sci.* (Paris), lxxxi. pp. 1050–1052.]

[Note sur les équations linéaires. *Journ. de l'Éc. Polyt.*, cah. xlviii. pp. 221–228.]

[Sur la discussion d'un système d'équations linéaires simultanées. *Comptes Rendus . . . . Acad. des Sci.* (Paris), lxxxi. pp. 1203–1204.]

Rouché is still more ambitious than Fontené, his aim being to condense into one proposition the means of completely investigating a set of linear equations. Simplicity in enunciation is sought to be attained by introducing the term 'déterminant principal' for *the non-zero minor of highest order in the unaugmented array*, and the term 'déterminants caractéristiques' for *minors of the augmented array but not of the unaugmented array which have the 'déterminant principal' for a primary minor.* Unfortunately, the terms, like Dodgson's, are not happily chosen. After explaining them at considerable length (pp. 221–223) Rouché says: We can now formulate "la proposition qui renferme *toute la théorie des équations linéaires.*" In English the proposition is:

In order that n *linear equations containing* m *unknowns may be consistent it is necessary and sufficient that all the 'characteristic determinants' of the set shall vanish; and, if this condition be satisfied, the set has only one solution or is indeterminate according as the order of the said determinants is greater or less than* m.

The logical foundation for the proposition is well set forth (pp. 223–227), but will readily be guessed by any one familiar with earlier work on the subject. In the actual application of the test to a given set of equations the first and fundamental requirement is of course the finding of a 'déterminant principal,' and unfortunately Rouché gives no indication of the procedure which he considered best for attaining this end. Nor are any examples like those of Dodgson's first appendix worked out. It is thus impossible to say whether in the actual practice of solution the condensation and simplification apparent in the statement of the proposition be realities of any appreciable value. The concept, however, of a 'déterminant

principal' is one not to be lost sight of: probably the most suitable English equivalent for the name would be 'critical minor.'

Méray's contribution is embodied in three theorems, and is less original in both form and substance.

DARBOUX, G. (1876).

[Sur la théorie de l'élimination entre deux équations à une inconnue. *Bull. des Sci. Math.*, x. (1), pp. 56–64.]

As a lemma requisite for his main purpose Darboux formulates in his own way the theorem regarding the solution of n linear homogeneous equations in n unknowns whose determinant $|\, a_{1n} \,|$ and all its minors down to $|\, a_{1,n-p} \,|$ are equal to 0. This and a converse he takes pains to establish anew (pp. 61–63).

VENTÉJOLS (1877): D'OVIDIO, E. (1877): BIEHLER, C. (1878).

[Sur un problème comprenant la théorie de l'élimination. *Comptes Rendus . . . . Acad. des Sci.* (Paris), lxxxiv. pp. 546–549.]

[Ricerche sui sistemi indeterminati di equazioni lineari. *Atti . . . . Accad. delle Sci.* (Torino), xii. pp. 334–349.]

[Sur la théorie des équations. Dissert. 60 pp. Paris.]

Ventéjols' fundamental propositions are but variants of already known results. (See Dodgson's text-book, p. 48 cor., p. 64 prop., p. 50 cor.)

D'Ovidio confines himself to a full and thorough investigation of the case where there are n homogeneous linear equations in $n+r$ unknowns, his proposed object being to effect a generalisation of the theorem known from Jacobi (1841) to hold for the case where r is 1.* He notes at the outset that the generalisation had in effect been reached by Clebsch in 1872 (*Abhandl. . . . . Ges. d. Wiss. zu Gottingen*, xvii. 60 pp.; or *Math. Annalen*, iv. pp. 427–434). As a matter of fact, however, the theorem which he attributes to Clebsch is nothing else than Brill's theorem of 1871, which Gordan had restated in 1873. Viewed as a property of an indeterminate

* See above under Dodgson, IIa_1.

set of linear homogeneous equations it takes with D'Ovidio the following form : *If*

$$(x_1^{(1)}, \ldots, x_n^{(1)}), \quad \ldots\ldots, \quad (x_1^{(r)}, \ldots, x_n^{(r)})$$

be r *independent solutions of the set of equations*

$$\left.\begin{aligned} a_1^{(1)}x_1 \quad &+ \ldots\ldots + a_n^{(1)}x_n \quad = 0 \\ &\cdots\cdots\cdots\cdots \\ a_1^{(n-r)}x_1 &+ \ldots\ldots + a_n^{(n-r)}x_n = 0 \end{aligned}\right\},$$

then the primary minors of

$$\left\| \begin{matrix} x_1^{(1)}, \ldots\ldots, x_n^{(1)} \\ \cdots\cdots \\ x_1^{(r)}, \ldots\ldots, x_n^{(r)} \end{matrix} \right\|$$

are proportional to those of

$$\left\| \begin{matrix} a_1^{(1)}, \quad \ldots\ldots, a_n^{(1)} \\ \cdots\cdots \\ a_1^{(n-r)}, \ldots\ldots, a_n^{(n-r)} \end{matrix} \right\|,$$

the ratio

$$(-1)^{\omega} \left| x_a^{(1)} \ldots x_b^{(r)} \right| : \left| a_c^{(1)} \ldots a_d^{(n-r)} \right|$$

being constant if a, . . . , b, c, . . . , d *be a permutation of* 1, 2, . . . , n, *and have* ω *inverted-pairs.*

Biehler opens his thesis (pp. 5–10) with the solution of n linear equations in n unknowns ; he does not, however, give anything really fresh.

CHAPTER III.

AXISYMMETRIC DETERMINANTS, FROM 1846 TO 1879.

NOTE has to be taken that four of the writings here to be dealt with, namely, those of Cayley (1846, 1856), Brioschi (1854), and d'Arrest (1857) belong to the preceding period. In the case of Brioschi the account is supplementary to that already given.

CAYLEY, A. (1846, 1856).

[On some analytical formulae, and their application to the theory of spherical coordinates. *Cambridge and Dub. Math. Journ.*, i. pp. 22–33 ; or *Collected Math. Papers*, i. pp. 213–223.]

[Note upon a result of elimination. *Philos. Magazine*, xi. pp. 378–379 ; or *Collected Math. Papers*, ii. pp. 214–215.]

Save for the notation used Cayley's fundamental property of 1846 is

$$\begin{vmatrix} \begin{array}{ccc|c|ccc|c} \xi_2 & \eta_2 & \zeta_2 & & \xi_3 & \eta_3 & \zeta_3 & \\ \hline a & f & e & \xi_1 & a & f & e & \xi_1 \\ f & b & d & \eta_1 & f & b & d & \eta_1 \\ e & d & c & \zeta_1 & e & d & c & \zeta_1 \end{array} \\ \begin{array}{ccc|c|ccc|c} \xi_2 & \eta_2 & \zeta_2 & & \xi_3 & \eta_3 & \zeta_3 & \\ \hline a & f & e & \xi_4 & a & f & e & \xi_4 \\ f & b & d & \eta_4 & f & b & d & \eta_4 \\ e & d & c & \zeta_4 & e & d & c & \zeta_4 \end{array} \end{vmatrix} = \begin{array}{ccc|c} |\eta_2\zeta_3| & |\zeta_2\xi_3| & |\xi_2\eta_3| & \\ \hline A & F & E & |\eta_1\zeta_4| \\ F & B & D & |\zeta_1\xi_4| \\ E & D & C & |\xi_1\eta_4| \end{array},$$

an extension of Cauchy's of 1844 into which it degenerates (*Hist.*, ii. p. 26) when the suffixes 1, 4 are changed into 2, 3.

Similarly the result of 1856 is that if A, B, ... be negative elements of the adjugate of

$$\begin{vmatrix} a & h & g & \xi \\ h & b & f & \eta \\ g & f & c & \zeta \\ \xi & \eta & \zeta & . \end{vmatrix}$$

and α, β, γ be quite arbitrary, then

$$\begin{array}{ccc|c} \beta\zeta-\gamma\eta & \gamma\xi-\alpha\zeta & \alpha\eta-\beta\xi & \\ \hline a & h & g & \beta\zeta-\gamma\eta \\ h & b & f & \gamma\xi-\alpha\zeta \\ g & f & c & \alpha\eta-\beta\xi \end{array} = \begin{array}{ccc|c} \alpha & \beta & \gamma & \\ \hline \mathrm{A} & \mathrm{H} & \mathrm{G} & \alpha \\ \mathrm{H} & \mathrm{B} & \mathrm{F} & \beta \\ \mathrm{G} & \mathrm{F} & \mathrm{C} & \gamma \end{array}.$$

BRIOSCHI, F. (1854).

[La Teorica dei Determinanti, ... viii+116 pp. Pavia.]

Brioschi (§ 6), in defining after Gauss (*Hist.*, i. pp. 64–65) the form *adjunct* or *adjugate* to

$$\sum a_{rs}x_rx_s \qquad \begin{pmatrix} r=1,\ 2,\ \ldots,\ n \\ s=1,\ 2,\ \ldots,\ n \end{pmatrix}$$

to be the form

$$\sum \mathrm{A}_{rs}\xi_r\xi_s \qquad \begin{pmatrix} r=1,\ 2,\ \ldots,\ n \\ s=1,\ 2,\ \ldots,\ n \end{pmatrix},$$

where A_{rs} is the cofactor of a_{rs} in $|\,a_{1n}\,|$, but where there is no connection between the x's and ξ's, writes the two as determinants of the $(n+1)^{\text{th}}$ order, namely,

$$\frac{-1}{|\,a_{1n}\,|^{n-2}} \begin{vmatrix} . & x_1 & x_2 & \ldots & x_n \\ x_1 & \mathrm{A}_{11} & \mathrm{A}_{12} & \ldots & \mathrm{A}_{1n} \\ x_2 & \mathrm{A}_{21} & \mathrm{A}_{22} & \ldots & \mathrm{A}_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_n & \mathrm{A}_{n1} & \mathrm{A}_{n2} & \ldots & \mathrm{A}_{nn} \end{vmatrix}, \quad - \begin{vmatrix} . & \xi_1 & \xi_2 & \ldots & \xi_n \\ \xi_1 & a_{11} & a_{12} & \ldots & a_{1n} \\ \xi_2 & a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \xi_n & a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

the justification for which is seen on developing the determinants in terms of binary products of the x's and the ξ's respectively, and using in the case of the first a result of Cauchy's regarding the adjugate of the adjugate (*Hist.*, i. p. 110). The first of the two identities, for the case of n equal to 3, was given by Sylvester in 1850 (*Hist.*, ii. pp. 117–118). The second we should have drawn attention to when reporting (*Hist.*, ii. p. 153) Salmon's remark

regarding the form taken by the evectant of the discriminant of a quadric when the discriminant vanishes. We see now that the property then stated is equivalent to saying that *when the discriminant of a quadric vanishes, the adjunct quadric is expressible as a square.*

Brioschi also notes that if, in addition to the above-mentioned relation between the a's and A's, there be the following connection between the x's and ξ's, namely,

$$\begin{aligned}
\xi_1 &\equiv a_{11}x_1+a_{12}x_2+ \ . \ . \ . \ +a_{1n}x_n, \\
\xi_2 &\equiv a_{21}x_1+a_{22}x_2+ \ . \ . \ . \ +a_{2n}x_n, \\
&\ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\
\xi_n &\equiv a_{n1}x_1+a_{n2}x_2+ \ . \ . \ . \ +a_{nn}x_n,
\end{aligned}$$

so that each ξ is the differential-coefficient of the given quadric with respect to the corresponding x, and consequently so that

$$x_1\xi_1+x_2\xi_2+ \ . \ . \ . \ +x_n\xi_n$$

is equal to the quadric itself, then

$$\sum A_{rs}\xi_r\xi_s = |\, a_{11}a_{22} \ . \ . \ . \ a_{nn}\,| \,.\sum a_{rs}x_rx_s.$$

It must not be forgotten, however, that the case of this for n equal to 3 is included in a result noted by Cayley in 1848 (*Hist.*, ii. pp. 116–117), and that the proof there given by us is generally applicable. As alternative proofs we may now add that

$$-\begin{vmatrix} . & \xi & \eta & \zeta \\ \xi & a & h & g \\ \eta & h & b & f \\ \zeta & g & f & c \end{vmatrix} = - \begin{vmatrix} . & x & y & z \\ \xi & 1 & . & . \\ \eta & . & 1 & . \\ \zeta & . & . & 1 \end{vmatrix} . \begin{vmatrix} 1 & . & . & . \\ . & a & h & g \\ . & h & b & f \\ . & g & f & c \end{vmatrix}$$

$$= (x\xi+y\eta+z\zeta)\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \ . \ . \ . ,$$

and that

$$-\begin{vmatrix} . & x & y & z \\ x & A & H & G \\ y & H & B & F \\ z & G & F & C \end{vmatrix} . \begin{vmatrix} 1 & . & . & . \\ . & a & h & g \\ . & h & b & f \\ . & g & f & c \end{vmatrix} = - \begin{vmatrix} . & \xi & \eta & \zeta \\ x & \Delta & . & . \\ y & . & \Delta & . \\ z & . & . & \Delta \end{vmatrix}$$

$$= (x\xi+y\eta+z\zeta)\Delta^2,$$

where Δ stands for the multiplier on the left.

Another fact to be supplied is that Brioschi, when dealing (§ 7) with the subject of linear transformation, gives a property of pure

determinants which we may formulate for ourselves thus: *If the square of any determinant formed by row-multiplication be* D, *and the square formed by column-multiplication be* D′, *then the sum of the* m-*line coaxial minors in* D *is equal to the corresponding sum in* D′.

D'ARREST, H. (1857).

[Beobachtungen des Cometen iii. 1857. *Astron. Nachrichten*, xlvii. col. 17–19.]

From consideration of three special cases d'Arrest ventures on the generalisation that if in the application of the Method of Least Squares the so-called 'normal' equations be

$$\left.\begin{aligned}(aa)x+(ab)y+(ac)z+\ \ldots\ &=0\\ (ab)x+(bb)y+(bc)z+\ \ldots\ &=0\\ (ac)x+(bc)y+(cc)z+\ \ldots\ &=0\\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ &\end{aligned}\right\},$$

then the weights of x, y, z, . . . are

$$\frac{\Delta}{[aa]},\ \frac{\Delta}{[bb]},\ \frac{\Delta}{[cc]},\ \ldots\ ,$$

where Δ is the determinant of the set of equations and $[aa]$ is the cofactor of (aa) in Δ. In the same serial in 1866 (vol. lxvii. pp. 174–175) a proof of the proposition is given but unaccompanied by any author's name.

It is perhaps worth adding that the array of coefficients in the 'normal' set of equations is got by multiplying the array of the original set columnwise by itself.

FERRERS, N. M. (1861).

[An Elementary Treatise on Trilinear Co-ordinates (chap. iii. pp. 58–71). xii+154 pp. London.]

One of the results, unfortunately inaccurate, given at the close of the short chapter on determinants is

$$\begin{vmatrix} . & 1 & 1 & 1 & \ldots \\ 1 & . & a+b & a+c & \ldots \\ 1 & b+a & . & b+c & \ldots \\ 1 & c+a & c+b & . & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = abc\ldots\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ \ldots\right).$$

It is readily seen that the left-hand member

$$= \begin{vmatrix} . & 1 & 1 & 1 & . & . & . & . \\ 1 & -2a & . & . & . & . & . & . \\ 1 & . & -2b & . & . & . & . & . \\ 1 & . & . & -2c & . & . & . & . \\ . & . & . & . & . & . & . & . \end{vmatrix},$$

and that therefore the right-hand member ought to be

$$(-1)^n 2^{n-1} abc \ldots \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+ \ldots\right),$$

where n is the number of variables.*

BOOLE, G. (1862).

[On the theory of probabilities. *Philos. Transac. R. Soc. London*, clii. pp. 225–252.]

An important part of Boole's investigation turns upon the solution of a peculiar set of linear equations, and he consequently devotes considerable space (pp. 235–240) to an examination of the determinant of the set.

As the determinant is axisymmetric he begins by establishing the proposition which results, in regard to such determinants, from subtracting λ times the j^{th} row from the i^{th} row, and thereafter λ times the j^{th} column from the i^{th} column. This operation does not do away with the axisymmetry, and the diagonal element a_{ii} is thereby changed into

$$a_{ii} - 2a_{ij} + \lambda^2 a_{jj}.$$

*There is an interesting connection between this determinant, Φ say, and Ferrers' second determinant of 1855, F say (*Hist.*, ii. p. 142). For, putting the first variable here equal to 1, we have

$$\Phi(1, a_2, a_3, \ldots a_n) = (-1)^n 2^{n-1} a_2 a_3 \ldots a_n \left(1+\frac{1}{a_2}+\frac{1}{a_3}+ \ldots +\frac{1}{a_n}\right),$$

and therefore

$$\Phi(1, a_2, a_3, \ldots a_n) = (-1)^n 2^{n-1} \mathrm{F}(a_2, a_3, \ldots a_n).$$

In the examination papers of 1862 for the Ferguson scholarship, a case of this, with incorrect sign, is set for independent proof, namely,

$$\begin{vmatrix} . & 1 & 1 & 1 & 1 \\ 1 & . & 1+a & 1+b & 1+c \\ 1 & a+1 & . & a+b & a+c \\ 1 & b+1 & b+a & . & b+c \\ 1 & c+1 & c+a & c+b & . \end{vmatrix} = 2^3 \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}.$$

Following this lemma and made in part dependent on it comes the rather notable proposition that *if an axisymmetric determinant have all its elements of the form*

$$\lambda a+\mu b+\nu c+ \ldots ,$$

and the coefficients λ, μ, ν, . . . *in all the diagonal elements be positive, and generally be such that all those joined to any one of the variables in any row be in order proportional to the coefficients of the same variable in any other row, then the final development of the determinant will contain only positive terms.* The proof is disappointingly lengthy, occupying very nearly three pages (pp. 226–238).

The other proposition, which is a deduction from this, is to the effect that *if* F *be a rational integral function of* n *variables* $x_1, x_2, \ldots x_n$, *having no powers of them above the first and having all its terms positive, then the final development of*

$$\begin{vmatrix} F & x_1F_1 & x_2F_2 & \ldots & x_nF_n \\ x_1F_1 & x_1F_1 & x_1x_2F_{12} & \ldots & x_1x_nF_{1n} \\ x_2F_2 & x_2x_1F_{21} & x_2F_2 & \ldots & x_2x_nF_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_nF_n & x_nx_1F_{n1} & x_nx_2F_{n2} & \ldots & x_nF_n \end{vmatrix},$$

where F_r, F_{rs} *stand for* $\partial F/\partial x_r$, $\partial^2 F/\partial x_r \partial x_s$ *respectively, will contain nothing but positive terms.* For the case of two variables the result is

$$\begin{vmatrix} axy+bx+cy+d & axy+bx & axy+cy \\ axy+bx & axy+bx & axy \\ axy+cy & axy & axy+cy \end{vmatrix} = abcx^2y^2+abdx^2y+acdxy^2 + bcdxy ;$$

and for the development in the case of three variables we are referred to a similar memoir * of the year 1857, where we find the expression

$$\begin{aligned} &(d+h+e+f)(abc+acg+abg+bcg) \\ &+(a+b+c+g)(dhe+dhf+def+hef) \\ &+(ac+bg)(df+dh+ef+eh) \\ &+(ag+bc)(df+eh+de+fh) \\ &+(ab+cg)(de+dh+fe+fh) \\ &+4agfe+4bcdh, \end{aligned}$$

* Boole, G., "On the Application of the Theory of Probabilities to the Question of the Combination of Testimonies or Judgments," *Transac. R. Soc. Edin.*, xxi. pp. 597–652 (see p. 648).

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and infer that it is the equivalent of

$$\begin{vmatrix} a+b+c+\ldots+h & a+c+d+e & a+b+d+f & a+b+c+g \\ a+c+d+e & a+c+d+e & a+d & a+c \\ a+b+d+f & a+d & a+b+b+f & a+b \\ a+b+c+g & a+c & a+b & a+b+c+g \end{vmatrix}.$$

In connection with unisignant determinants like these, it is desirable to recall Sylvester's paper of 1855 and Borchardt's of 1859 (*Hist.*, ii. pp. 456–9).

SIEBECK, F. H. (1862).

[Ueber die Determinante deren Elemente die Quadrate der sechzehn Verbindungslinien der Eckpunkte zweier beliebigen Tetraeder sind. *Crelle's Journ.*, lxii. pp. 151–159.]

The determinant in question is brought forward as a companion to Sylvester's of the year 1852, being, in fact, the complementary minor of the zero element in the latter. It is shown that the ratio of the one to the other is

$$2r\rho\cos\phi,$$

where r, ρ are the radii of the spheres circumscribing the tetraedra and ϕ the angle of intersection of the said spheres. The interest of the paper, like Sylvester's, is mainly geometrical.

FREUCHEN, P. (1863).

[To Determinanter af *n*te Grad. *Math. Tidsskrift*, v. p. 42.]

The two determinants are the cases of Ferrers' of the year 1855, in which $a_1=a_2=a_3=\ldots$.

ROBERTS, M. (1864).

[Question 694. *Nouv. Annales de Math.*, (2) iii. pp. 139–140. Solutions by L. Ferrara, G. Torelli, in *Giornale di Mat.*, ii. pp. 95–96, p. 191 : solutions by A. Smet-Jamar, M. Cornu, H. Picquet, in *Nouv. Annales de Math.*, (2) iii. pp. 395–399, and by "Mirza-Nizam," in (2) iv. pp. 500–504.]

Roberts' result is essentially the same as Ferrers' second, but is expressed more suggestively, namely,

$$\begin{vmatrix} a_1 & -1 & -1 & \dots & -1 \\ -1 & a_2 & -1 & \dots & -1 \\ -1 & -1 & a_3 & \dots & -1 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -1 & -1 & \dots & a_n \end{vmatrix} = f(1) - f'(1),$$

where $f(x) = (x+a_1) \dots (x+a_n)$. Ferrara and Smet-Jamar develop the determinant in Cayley's manner of 1847, obtaining $f(1)-f'(1)$ in the form

$$(1-n) + (2-n)\sum a_1 + (3-n)\sum a_1 a_2 + \dots + a_1 a_2 \dots a_n;$$

Cornu and Picquet subtract the last column from each of the others and obtain the development in the form

$$(1+a_1)(1+a_2) \dots (1+a_n) \cdot \left[1 - \left(\frac{a_1}{1+a_1} + \frac{a_2}{1+a_2} + \dots + \frac{a_n}{1+a_n}\right)\right].$$

Torelli, after generalising the determinant by putting $-x$ everywhere for -1, writes it in the form

$$\begin{vmatrix} (x+a_1)-x & 0-x & 0-x & \dots \\ 0-x & (x+a_2)-x & 0-x & \dots \\ 0-x & 0-x & (x+a_3)-x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

which, if expressed as a sum of determinants with monomial elements, gives

$$(x+a_1)(x+a_2) \dots (x+a_n) \quad - \quad x\sum (x+a_2)(x+a_3) \dots (x+a_n),$$

$$i.e. \quad f(x) - xf'(x):$$

and "Mirza Nizam" obtains this wider result of Torelli's by following Cornu and Picquet.

WALKER, J. J. (1865, 1868).

[Properties of the discriminant of the quaternary quadric form. *O. C. and D. Messenger of Math.*, iv. pp. 25–31, 189–190.]

[Question 2730. *Educ. Times*, xxi. p. 139; or *Math. from Educ. Times*, xi. pp. 107–108.]

The subject here is mainly the vanishing of the primary minors of any axisymmetric determinant of the fourth order, the final

result being that *all the primary minors will vanish when three of them vanish, provided that two, and not more than two, of the three be taken from the same rows of the adjugate.* The number *three* is Sylvester's $\frac{1}{2}(4-3+1)(4-3+2)$ of the year 1850.

In proving a simple case of the well-known theorem regarding a minor of the adjugate he incidentally rediscovers the equality

$$\begin{vmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & j \\ g & i & j & d \end{vmatrix} = \begin{vmatrix} e^2-ab & ef-ah & eg-ai \\ ef-ah & f^2-ac & fg-aj \\ eg-ai & fg-aj & g^2-ad \end{vmatrix} \div a^2,$$

and thence later shows that when $|\, a_1b_2c_3d_4 \,|$ is axisymmetric and vanishes, we have

$$|\, a_1b_2 \,| \cdot |\, a_1c_3d_4 \,| \;-\; |\, a_1b_3 \,| \cdot |\, a_1c_2d_4 \,| \;+\; |\, a_1b_4 \,|\, |\, a_1c_2d_3 \,| = 0.$$

Here, however, we must note that for this last identity axisymmetry is not necessary, and that the other condition may also be dispensed with if in place of 0 we put $a_1 \,|\, a_1b_2c_3d_4 \,|$.

CALDARERA, F. (1871).

[Nota su talune proprietà dei determinanti, in ispecie di quelli a matrici composte con la serie dei numeri figurati. *Giornale di Mat.*, ix. pp. 223–232.]

The main determinant dealt with is

$$\begin{vmatrix} a & a+\delta & a+2\delta & a+3\delta & \dots \\ a & 2a+\delta & 3a+3\delta & 4a+6\delta & \dots \\ a & 3a+\delta & 6a+4\delta & 10a+10\delta & \dots \\ a & 4a+\delta & 10a+5\delta & 20a+15\delta & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n \quad \text{or, say } \mathrm{D}(a,\delta),$$

it being first shown to be independent of δ, and consequently to be equal to

$$a^n \begin{vmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 3 & 4 & \dots \\ 1 & 3 & 6 & 10 & \dots \\ 1 & 4 & 10 & 20 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n .$$

The evaluation of the cofactor of a^n Caldarera effects with much trouble (pp. 226–231) by means of his (or, rather, Chio's) condensation-theorem, not observing that a series of similar steps can be made by simply diminishing each row by the row immediately preceding : for example, when $n = 4$, we have

$$\frac{\mathrm{D}(a,0)}{a^4} = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = 1,$$

the theorem repeatedly used in performing the subtractions being

$$\mathrm{C}_{n,r} - \mathrm{C}_{n-1,r} = \mathrm{C}_{n-1,r-1}.$$

The same process applied directly to $\mathrm{D}(a, \delta)$ gives the value a^n with equal ease, showing at one and the same time that $\mathrm{D}(a, \delta) = \mathrm{D}(a, 0)$, and that the axisymmetric cofactor of a^n in the latter is equal to 1.

SCHULTZE, E. (1871).

[Über die aus einer symmetrischen Determinante $\Delta_n = \Sigma \pm a_{11} \ldots a_{nn}$ gebildete Reihe Δ_n, Δ_{n-1}, . . . , Δ_0. Sch. Progr. 22 pp. Berlin.]

The title is misleading, the subject really being the linear transformation of a quadric, whose discriminant is Δ_n, into an aggregate of multiples of squares. As, however, one of the modes of transformation referred to is that resuscitated by Brioschi in 1856,* but originally due to Lagrange (1759 †), namely, that which changes

$$a_{11}x_1^2 + a_{22}x_2 + \ldots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + \ldots .$$

into

$$\Delta_1 y_1^2 + \frac{\Delta_2}{\Delta_1} y_2^2 + \frac{\Delta_3}{\Delta_2} y_3^2 + \ldots + \frac{\Delta_n}{\Delta_{n-1}} y_n^2$$

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* Brioschi, F., "Sur les séries qui donnent le nombre de racines réelles des équations algébriques à une ou à plusieurs inconnues," *Nouv. Annales de Math.*, xv. pp. 264–285 ; or *Zeitschrift f. Math. u. Phys.*, ii. pp. 209–222 ; or *Opere Mat.*, v. pp. 127–143.

† Lagrange, J. L., "Recherche sur la méthode de maximis et minimis," *Miscell... Taurinensis*, i. p. 18 ; or *Œuvres*, i. pp. 3–20. See also Gauss (1823), *Werke*, iv. pp. 27–53, and Jacobi (1847), *Crelle's Journ.*, liii. pp. 265–270.

by a transformation of the form

$$\left.\begin{aligned} x_1 &= y_1 + c_{12}y_2 + c_{13}y_3 + \ldots + c_{1n}y_n, \\ x_2 &= \qquad y_2 + c_{23}y_3 + \ldots + c_{2n}y_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n &= \qquad\qquad\qquad\qquad y_n, \end{aligned}\right\}$$

the series $\Delta_n, \Delta_{n-1} \ldots, \Delta_0$ does come up for consideration. The only property established (p. 7) is: *If* Δ_n *and all its coaxial minors of the* $(n-1)^{th}$ *order vanish, then every other coaxial minor of the said order will also vanish.* This is easily arrived at by considering any two-line coaxial minor of the adjugate.

We may note that in dealing with a second transformation—the orthogonal—Schultze formulates another proposition regarding the evanescence of minors, but that this is identical with the first of Dodgson's pair of 1867.

HUNYADY, E. (1872).

[Question 979 (proposée par H. Brocard). *Nouv. Annales de Math.*, (2) xi. pp. 39–44.]

In seeking to solve anew Lagrange's interpolation-problem, namely, to obtain the values of the A's in the set of equations

$$\left.A_1 \cos\frac{1r\cdot\pi}{n+1} + A_2 \cos\frac{2r\cdot\pi}{n+1} + \ldots + A_n \cos\frac{nr\cdot\pi}{n+1} = y_r\right\}_{r=1,}^{r=n}$$

Hunyady finds that

$$\left|\cos\frac{1\cdot1\cdot\pi}{n+1}\cos\frac{2\cdot2\cdot\pi}{n+1}\ldots\ldots\cos\frac{n\cdot n\cdot\pi}{n+1}\right|^2 = \begin{cases}\frac{1}{4}\left(\frac{n+1}{2}\right)^{n-2} & \text{for } n \text{ even}\\ 0 & \text{for } n \text{ odd.}\end{cases}$$

HESSE, O. (1872).

[Ein Cyclus von Determinanten-Gleichungen. *Crelle's Journ.*, lxxv. pp. 1–12; or *Werke*, pp. 585–598.]

As is explained under *Bordered Determinants*, Hesse uses $[\alpha, \beta]$ to stand for the result of bordering $[u_{1n}]$ vertically and horizontally by

$$\alpha_1, \alpha_2, \ldots, \alpha_n \quad \text{and} \quad \beta_1, \beta_2, \ldots, \beta_n$$

respectively, and similarly $[\alpha\gamma, \beta\delta]$ in the case where a two-line border has been affixed.

The connecting relation

$$|\, u_{1n} \,| \cdot [\alpha\beta, \gamma\delta] \;=\; [\alpha, \gamma]\cdot[\beta, \delta] - [\alpha, \delta]\cdot[\beta, \gamma]$$

comes to his hand at once, on account of the member on the right being viewable as a two-line minor of the adjugate of the second factor on the left. In like manner, or by the mere interchange of letters, he obtains

$$|\, u_{1n} \,| \cdot [\alpha\gamma, \beta\delta] \;=\; [\alpha, \beta][\gamma, \delta] - [\alpha, \delta][\gamma, \beta],$$

$$|\, u_{1n} \,| \cdot [\alpha\delta, \gamma\beta] \;=\; [\alpha, \gamma][\delta, \beta] - [\alpha, \beta][\delta, \gamma].$$

Then, positing the axisymmetry of $|\, u_{1n} \,|$, he notes that

$$[\beta, \delta] = [\delta, \beta], \quad \ldots\ldots,$$

and that therefore the first of the three right-hand members is exactly the sum of the two others. There thus follows the theorem

$$[\alpha\beta, \gamma\delta] \;=\; [\alpha\gamma, \beta\delta] + [\alpha\delta, \gamma\beta],$$

or, as we may prefer to write it,

$$[\alpha\beta, \gamma\delta] - [\alpha\gamma, \beta\delta] + [\alpha\delta, \beta\gamma] = 0.$$

A second theorem which he gives is easily remembered from this, namely, that *if we annex to the three terms on the left the outwardly resembling multipliers*

$$[\alpha, \beta]\cdot[\gamma, \delta], \qquad [\alpha, \gamma]\cdot[\beta, \delta], \qquad [\alpha, \delta]\cdot[\beta, \gamma]$$

respectively, the right-hand side need not be changed. This results from using the said multipliers on the three original equations, and then reasoning as before.

For ourselves, we may note that the existence of the two equations side by side is a consequence of the general fact that if we have

$$X = a - b, \qquad -Y = b - c, \qquad Z = c - a,$$

it follows that we have both

$$X - Y + Z = 0 \quad \text{and} \quad cX - aY + bZ = 0.$$

WILLIAMSON, B. (1872): BUCHWALD, E. (1872).

[Condition for a maximum or a minimum in a function of any number of variables. *Quart. Journ. of Math.*, xii. pp. 48–51; or his *Differential Calculus*, 1st edition, pp. 340–343, 2nd edition, pp. 363–367, etc.]

[Question 3683. *Educ. Times*, xxiv. p. 296, xxv. p. 18; or *Math. from Educ. Times*, xvii. pp. 66–68.]

[Betingelsen for at den algebraisk rationale homogene Function af anden Grad af n variable er positiv for alle Vœrdier af de variable eller negativ for dem alle. *Tidsskrift for Math.*, (3) ii. pp. 20–25.]

The real object here is to obtain the conditions that must be fulfilled in order that a quadric may remain positive for all real values of the variables.

The investigation, due in essence to Lagrange (1759),* is based on a simple transformation, three examples of which we may write for ourselves in the form

$$a_1 \cdot \begin{array}{cc|c} x & y & \\ \hline a_1 & a_2 & x \\ b_1 & b_2 & y \end{array} = (a_1x+a_2y)^2 + |a_1b_2|\,y^2,$$

$$a_1 \cdot \begin{array}{ccc|c} x & y & z & \\ \hline a_1 & a_2 & a_3 & x \\ b_1 & b_2 & b_3 & y \\ c_1 & c_2 & c_3 & z \end{array} = (a_1x+a_2y+a_3z)^2 + \begin{array}{cc|c} y & z & \\ \hline |a_1b_2| & |a_1b_3| & y \\ |a_1c_2| & |a_1c_3| & z, \end{array}$$

$$a_1 \cdot \begin{array}{cccc|c} x & y & z & w & \\ \hline a_1 & a_2 & a_3 & a_4 & x \\ b_1 & b_2 & b_3 & b_4 & y \\ c_1 & c_2 & c_3 & c_4 & z \\ d_1 & d_2 & d_3 & d_4 & w \end{array} = (a_1x+\ldots+a_4w)^2 + \begin{array}{ccc|c} y & z & w & \\ \hline |a_1b_2| & |a_1b_3| & |a_1b_4| & y \\ |a_1c_2| & |a_1c_3| & |a_1c_4| & z \\ |a_1d_2| & |a_1d_3| & |a_1d_4| & w \end{array}$$

where, it must be borne in mind, the square arrays are axisymmetric.†

* See immediately preceding footnote.

† This restriction may be done away with if we alter the squared expressions on the right hand into

$$(a_1x+a_2y)(a_1x+b_1y), \quad (a_1x+a_2y+a_3z)(a_1x+b_1y+c_1z), \quad \ldots.$$

By dividing both sides by a_1 it is seen that in the case of the binary quadric the conditions are

$$a_1 > 0, \qquad |\, a_1 b_2 \,| > 0.$$

In the next case it is equally evident that they are $a_1 > 0$ and the like conditions for the quadric in y, z : and as by the previous case the latter are

$$|\, a_1 b_2 \,| > 0, \qquad \begin{vmatrix} |\, a_1 b_2 \,| & |\, a_1 b_3 \,| \\ |\, a_1 c_2 \,| & |\, a_1 c_3 \,| \end{vmatrix} > 0,$$

we obtain in all for the ternary quadric

$$a_1 > 0, \qquad |\, a_1 b_2 \,| > 0, \qquad |\, a_1 b_2 c_3 \,| > 0.$$

Similarly for the quaternary quadric we must have a_1, $|\, a_1 b_2 \,|$, $|\, a_1 b_2 c_3 \,|$, $|\, a_1 b_2 c_3 d_4 \,|$ all positive : and so on, the last determinant in each case being the discriminant of the quadric. It is casually added that if the 1st, 3rd, 5th, ... of the series be negative and the others positive, the quadric will be negative for all real values of the variables.

As might be expected from the connection with Lagrange, it is also pointed out that, calling these determinants Δ_1, Δ_2, Δ_3, . . . , we can by repeated applications of the fundamental transformation change the n-ary quadric into the form

$$\Delta_1 U_n^2 + \frac{\Delta_2}{\Delta_1} U_{n-1}^2 + \frac{\Delta_3}{\Delta_2} U_{n-2}^2 + \ldots + \frac{\Delta_n}{\Delta_{n-1}} U_1^2,$$

that is to say, into an aggregate of multiples of squares of linear functions of the variables : for example,

$$ax^2 + by^2 + cz^2 + 2dyz + 2ezx + 2fxy$$

$$= a\left(x + \frac{f}{a}y + \frac{e}{a}z\right)^2 + \frac{ab - f^2}{a}\left(y + \frac{ad - ef}{ab - f^2}z\right)^2 + \frac{abc + 2def - \ldots}{ab - f^2} z^2.$$

Finally, as the quadric is invariant to a variety of sets of interchanges, there must be a corresponding variety of sets of conditions : and, as these latter sets must be all coexistent, there follows an interesting theorem which we may formulate for ourselves thus : *If the axisymmetric determinant* $|\, a_1 b_2 c_3 d_4 \,|$ *and its coaxial minors* $|\, a_1 b_2 c_3 \,|$, $|\, a_1 b_2 \,|$, a_1 *be positive, then all the other coaxial minors are positive also.*

Having had in the foregoing to transform

$$\left|\,|a_1b_2| \quad |a_1c_3|\,\right|, \qquad \left|\,|a_1b_2| \quad |a_1c_3| \quad |a_1d_4|\,\right|, \quad \ldots$$

into $\quad a_1\,|\,a_1b_2c_3\,|, \qquad a_1^2\,|\,a_1b_2c_3d_4\,|, \quad \ldots$

Williamson proposed the general problem in the *Educ. Times*, where solutions by Townsend, Laverty, and himself duly appeared. Chio's more general result of 1853 was apparently unknown.

Buchwald's paper is closely on the lines of Williamson's.

RITSERT, E. (1872).

[Die Herleitung der Determinante für den Inhalt des Dreiecks aus den drei Seiten. *Zeitschrift f. Math. u. Phys.*, xvii. pp. 518–520.]

This merely gives a variant on the usual way of deducing the four-line determinant in question from the determinant which involves the coordinates of the vertices.

BAUER, G. (1873).

[Bemerkungen über einige Determinanten geometrischer Bedeutung. *Sitzungsb. . . . . Akad. d. Wiss.* (München), ii. pp. 345–354.]

The determinants are of the type dealt with by Sylvester (1852) and Siebeck (1862), and though there is considerable generalisation it only concerns the geometrician.

Interesting on the same account and written about the same time is a paper of Gundelfinger's, which, as it starts with a theorem regarding general determinants, has already been dealt with elsewhere.

JANNI, V. (1874).

[Dimostrazione di alcune teoremi sui determinanti. *Giornale di Mat.*, xii. pp. 142-145.]

One of the theorems in question is Salmon's of 1859 (*Hist.*, ii. p. 153). The given determinant being

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & a & b & c & d \\ x_2 & b & e & f & g \\ x_3 & c & f & h & i \\ x_4 & d & g & i & j \end{vmatrix}, \quad \text{or } \Delta \text{ say,}$$

Janni puts D, G, I, J for the signed complementary minors of d, g, i, j in the vanishing cofactor of x_0, and, performing the operation

$$\mathrm{J}\cdot\mathrm{col}_5 \;+\; \mathrm{I}\cdot\mathrm{col}_4 \;+\; \mathrm{G}\cdot\mathrm{col}_3 \;+\; \mathrm{D}\cdot\mathrm{col}_2,$$

obtains

$$\mathrm{J}\cdot\Delta = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4\mathrm{J}+x_3\mathrm{I}+x_2\mathrm{G}+x_1\mathrm{D} \\ x_1 & a & b & c & . \\ x_2 & b & e & f & . \\ x_3 & c & f & h & . \\ x_4 & d & g & i & . \end{vmatrix}$$

$$= -(x_4\mathrm{J}+x_3\mathrm{I}+x_2\mathrm{G}+x_1\mathrm{D})^2,$$

as desired.

We recall, however, our remark under Hesse (1868), that if we take the determinant formed from the corner elements of the adjugate of Δ and apply Jacobi's theorem regarding a minor of the adjugate, we have at once

$$-\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ a & b & c & d \\ b & e & f & g \\ c & f & h & i \end{vmatrix}^2 = \Delta\cdot\begin{vmatrix} a & b & c \\ b & e & f \\ c & f & h \end{vmatrix}.$$

Further we note that it would have been equally appropriate to give

$$\mathrm{H}\Delta = -(\mathrm{C}x_1+\mathrm{F}x_2+\mathrm{H}x_3+\mathrm{I}x_4)^2,$$

or $$\mathrm{E}\Delta = -(\mathrm{B}x_1+\mathrm{E}x_2+\mathrm{F}x_3+\mathrm{G}x_4)^2,$$

or $$\mathrm{A}\Delta = -(\mathrm{A}x_1+\mathrm{B}x_2+\mathrm{C}x_3+\mathrm{D}x_4)^2;$$

and that each of the four leads to the original form

$$\Delta = -(x_1\mathrm{A}^{\frac{1}{2}}+x_2\mathrm{E}^{\frac{1}{2}}+x_3\mathrm{H}^{\frac{1}{2}}+x_4\mathrm{J}^{\frac{1}{2}})^2.$$

GARBIERI, G. (1874).

[Determinanti, con numerose applicazioni. xiii+267 pp. Bologna.]

Some of Garbieri's deductions (pp. 67–75) from the result

$$\begin{vmatrix} a_1 & x & x & \dots & x \\ x & a_2 & x & \dots & x \\ x & x & a_3 & \dots & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x & x & x & \dots & a_n \end{vmatrix} = (-1)^n\{f(x)-xf'(x)\},$$

where

$$f(x) \equiv (x-a_1)(x-a_2) \ldots (x-a_n),$$

are more worth noting than his proof of the result itself (pp. 168–169).

Thus, taking the case where n is 3, and where therefore the right-hand side is

$$(x-a_1)(x-a_2)(x-a_3)\left\{-1+\frac{x}{x-a_1}+\frac{x}{x-a_2}+\frac{x}{x-a_3}\right\},$$

he shows that it is effectively applicable to the determinant of the well-known equation

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

on account of the said determinant being transformable into

$$\begin{vmatrix} f^2(a-\lambda) & fgh & fgh \\ fgh & g^2(b-\lambda) & fgh \\ fgh & fgh & h^2(c-\lambda) \end{vmatrix} \div f^2g^2h^2,$$

and the equation thus taking the form

$$\frac{gh}{gh-af+f\lambda}+\frac{hf}{hf-bg+g\lambda}+\frac{fg}{fg-ch+h\lambda}=1$$

or

$$\frac{g^2h^2}{\lambda-\frac{\mathrm{F}}{f}}+\frac{h^2f^2}{\lambda-\frac{\mathrm{G}}{g}}+\frac{f^2g^2}{\lambda-\frac{\mathrm{H}}{h}}=fgh$$

for establishing the reality of the roots and determining their limits.

Again, putting x equal to one of the a's, say the first, in the general result, he obtains

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ a_1 & a_2 & a_1 & \ldots & a_1 \\ a_1 & a_1 & a_3 & \ldots & a_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_1 & a_1 & \ldots & a_n \end{vmatrix} = (-1)^{n+1}f'(a_1).$$

CAYLEY, A. (1874).

[On the number of distinct terms in a symmetrical or partially symmetrical determinant. *Monthly Notices R. Astron. Soc.*, xxxiv. pp. 303–307, 335; or *Collected Math. Papers*, ix. pp. 185–190.]

Denoting by $\phi(m, n)$ the number of distinct terms in a determinant of the $(m+n)^{\text{th}}$ order having an m-line coaxial minor which is axisymmetric, Cayley, by considering two different forms of development—Bezout's of 1764 and his own of 1847—arrives at the equations

$$\phi(m, n) = m\phi(m-1, n) + n\phi(m, n-1),$$

$$\phi(m, 0) = \phi(m-1, 0) + m\phi(m-2, 0) + \tfrac{1}{2}m(m-1)\phi(m-3, 1),$$

and thence obtains

$$\begin{matrix} & & \phi(0,0) & & \\ & \phi(1,0), & & \phi(0,1) & \\ \phi(2,0), & & \phi(1,1) & & \phi(0,2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \Bigg| = \begin{vmatrix} & & 1 & & \\ & 1 & & 1 & \\ 2 & & 2 & & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

Then, confining himself to $\phi(m, 0)$, he shows that the differential equation of the generating function of $\phi(m, 0)/m!$ is

$$2\frac{du}{dx} - u - xu = \frac{u}{1-x},$$

and consequently that

$$u = \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{(1-x)^{\frac{1}{2}}}.$$

This readily leads to similar results for $\phi(m, 1)$, $\phi(m, 2)$, ...

As an afterthought (p. 335) he puts the said differential equation in the form

$$2(1-x)\frac{du}{dx} = (2-x^2)u,$$

and thence deduces the equation of differences

$$u_n = nu_{n-1} - \tfrac{1}{2}(n-1)(n-2)u_{n-3}.$$

CUNNINGHAM, A. (1874): ROBERTS, S. (1874).

[An investigation of the number of constituents, elements, and minors of a determinant. *Quart. Journ. of Sci.*, (2) iv. pp. 212–228.]

[Question 4392. *Educ. Times*, xxvii. pp. 45, 66; or *Math. from Times*, xxi. pp. 81–83.]

Cunningham devotes §§ 8–10 (pp. 220–224) to the number of terms in devertebrate and vertebrate axisymmetric determinants. There is an oversight, however, in his reasoning, and his results are correct only as far as the fifth order. He concerns himself also (§ 14) with the number of k-line minors in an n-line axisymmetric determinant.

Roberts' difference-equation for the number of distinct terms is

$$u_n - u_{n-1} - (n-1)^2 u_{n-2} + \tfrac{1}{2}(n-1)(n-2)\{u_{n-3} + (n-3)u_{n-4}\} = 0,$$

and Cayley establishes it by taking his own, namely,

$$u_n - nu_{n-1} + \tfrac{1}{2}(n-1)(n-2)u_{n-3} = 0$$

or, say, $\quad \mathrm{E}_n = 0,$

and showing that the other is

$$\mathrm{E}_n + (n-1)\mathrm{E}_{n-1} = 0.$$

The first eight values of u_n he finds to be

$$1,\ 2,\ 5,\ 17,\ 73,\ 388,\ 2461,\ 18156,\ \ldots\ .$$

Roberts himself identifies the problem with that of finding "the number of distinct ways in which $2n$ things, two of a sort, can be made into parcels of 2."

PAINVIN, L. (1874).

[Note sur la méthode d'élimination de Bezout. *Nouv. Annales de Math.*, (2) xiii. pp. 278–285.]

A clear exposition of the mode of obtaining the axisymmetric eliminant of two equations in x.

SEELIGER, H. (1875).

[Bemerkungen über symmetrische Determinanten, und Anwendung auf eine Aufgabe der analytischen Geometrie. *Zeitschrift f. Math. u. Phys.*, xx. pp. 467–474.]

With the help of an unwieldy multiple-sigma representation of the elements of a power-determinant Seeliger arrives at Sylvester's unproved proposition of 1852 regarding the p^{th} power of an axisymmetric determinant. To this he adds the statement that the four modes of performing the multiplication lead to the same result. Another proposition is that *the* p^{th} *power of a vanishing two-line determinant is axisymmetric.*

He next investigates the consequences of the simultaneous vanishing of a determinant and one of its primary minors, say the determinant $|a_1b_2c_3d_4|$ and B_3. Since *all* the two-line minors of the adjugate must vanish, he has of course

$$\left.\begin{aligned} 0 &= |A_1B_3| = |A_2B_3| = |A_4B_3| \\ &= |C_1B_3| = |C_2B_3| = |C_4B_3| \\ &= |D_1B_3| = |D_2B_3| = |D_4B_3| \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \therefore\ 0 &= A_3B_1 = A_3B_2 = A_3B_4 \\ &= C_3B_1 = C_3B_2 = C_3B_4 \\ &= D_3B_1 = D_3B_2 = D_3B_4 \end{aligned}\right\},$$

from which it is easy to see that

either $$0 = A_3 = C_3 = D_3,$$

or $$0 = B_1 = B_2 = B_4.$$

The general result is that *if a determinant and one of its primary minors,* M, *vanish, then the other primary minors which are in the same row of the adjugate with* M *must also vanish, or those which are in the same column.* As a corollary it is added that *when in addition* Δ *is axisymmetric and* M *is coaxial, there is no alternative.*

Following on this is a proposition less readily acceptable, namely : *If an* n*-line axisymmetric determinant and* n-2 *of its primary coaxial minors simultaneously vanish, then all the other primary minors vanish also.* It is at once seen that after the application of the preceding corollary the only elements of the adjugate that require

consideration are those of the last two-line coaxial minor; in other words, that the adjugate must be of the form

$$\begin{vmatrix} 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & X & Y \\ 0 & 0 & \dots & 0 & Y & Z \end{vmatrix}.$$

Further, it may be agreed that the data give us

$$\left.\begin{aligned} xX+yY &= 0 \\ yY+zZ &= 0 \\ XZ-Y^2 &= 0 \end{aligned}\right\};$$

but the deduction made therefrom, namely, that

$$X = Y = Z = 0,$$

is clearly not the only one. A test case that we may put to ourselves is

$$\begin{vmatrix} a & b & b \\ b & c & c \\ b & c & c \end{vmatrix}.$$

Finally, Seeliger considers the primary minors of the product-determinant, showing that they must all vanish if the one factor and its primary minors vanish and the other factor does not vanish.

GRAVELAAR, N. L. W. A. (1875): PICQUET, H. (1875).

[Neuer Beweis für die Realität der Wurzeln einer wichtigen Gleichung. *Archiv d. Math. u. Phys.*, lviii. pp. 301–318.]

[Sur une propriété du discriminant des formes quadratiques. *Assoc. franç. pour l'avancem. des sci.* 4e sess. (Nantes), pp. 225–230.]

Both here, when dealing with Lagrange's determinantal equation, and in a similar paper of 1878, Gravelaar rehearses at the outset a few necessary properties of axisymmetric determinants. One of them is that *if an axisymmetric determinant with real elements vanishes, all the primary coaxial minors have the same sign*: it is an immediate consequence of an old result (Lebesgue, 1837).

Another is identical with Schultze's of 1871 : and a third is a slight variant of Janni's of 1874.

This last is also Picquet's main algebraical result.

GLAISHER, J. W. L. (1874, 1878).

[On the solution of the equations in the method of least squares. *Monthly Notices R. Astron. Soc.*, xxxiv. (1874), pp. 311–334.]

[Questions 4418, 5530. *Educ. Times*, xxvii. p. 69 ; xxxi. p. 21. Solution by E. J. Nanson in *Math. from Educ. Times*, (2) ii. pp. 95–96.]

What is given in the *Educ. Times* is an expression in the form of a series for d'Arrest's ratio of an axisymmetric determinant to one of its coaxial primary minors. A perusal of the paper on the Method of Least Squares will show how the expression originates * (see §§ 5, 6). When freed of awkward notations it is, in the case of the fourth order,

$$\frac{|a_1b_2c_3d_4|}{|a_1b_2c_3|} = d_4 - \frac{a_4^2}{a_1} - \frac{|a_1b_4|^2}{a_1|a_1b_2|} - \frac{|a_1b_2c_4|^2}{|a_1b_2|\cdot|a_1b_2c_3|}.$$

We may note for ourselves that the corresponding identity when axisymmetry is not insisted on is (*Hist.*, ii. p. 40 ; iii. p. 51)

$$\frac{|a_1b_2c_3d_4|}{|a_1b_2c_3|} = d_4 - \frac{a_4d_1}{a_1} - \frac{|a_1b_4|\,|a_1d_2|}{a_1|a_1b_2|} - \frac{|a_1b_2c_4|\,|a_1b_2d_3|}{|a_1b_2|\,|a_1b_2c_3|},$$

and that a very instructive way of establishing it is to combine the first two terms on the right into one, then in similar manner combine the result thus obtained with the third term, and so on, the successive 'approximations,' so to speak, being

$$d_4, \quad \frac{|a_1d_4|}{a_1}, \quad \frac{|a_1b_2d_4|}{|a_1b_2|}, \quad \frac{|a_1b_2c_3d_4|}{|a_1b_2c_3|}.$$

The series is thus seen to be one of those that close up telescopically.

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* Other papers on the same subject are—

GEER, P. VAN, "Over het gebruik van determinanten bij de methode der kleinste kwadraten," *Nieuw Archief voor Wiskunde*, i. (1875), pp. 179–188.

CATALAN, E., "Remarques sur la théorie des moindres carrés," *Mém. de l'Acad. . . . de Belgique*, xliii. (1878), pp. 24–33.

SMITH, H. J. S. (1876).

[On the value of a certain arithmetical determinant. *Proceed. London Math. Soc.*, vii. pp. 208–212 ; or *Collected Math. Papers*, ii. pp. 161–165.]

The determinant in question is that in which the element in the $(r, s)^{\text{th}}$ place is the greatest common measure of r and s, namely,

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 1 & 2 & \dots \\ 1 & 1 & 3 & 1 & \dots \\ 1 & 2 & 1 & 4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

and Smith's theorem is that the determinant of the n^{th} order, D_n say, is equal to

$$\psi(1) \cdot \psi(2) \cdot \psi(3) \dots \psi(n),$$

where $\psi(r)$ is the number of integers not greater than r and prime to r. Recalling the fact that

$$\psi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots$$

$$= n - \frac{n}{p_1} - \frac{n}{p_2} - \dots + \frac{n}{p_1 p_2} + \frac{n}{p_1 p_3} + \dots,$$

where the p's are the prime factors of n, he performs the operation of subtracting from the n^{th} row the rows whose order-numbers are

$$\frac{n}{p_1}, \frac{n}{p_2}, \dots,$$

adding those whose order-numbers are

$$\frac{n}{p_1 p_2}, \frac{n}{p_1 p_3}, \dots,$$

subtracting those whose order-numbers are

$$\frac{n}{p_1 p_2 p_3}, \dots,$$

and so on : and this being done he is able to show that all the elements of the n^{th} row vanish except the last, which becomes $\psi(n)$.

There is thus obtained

$$D_n = \psi(n) \cdot D_{n-1},$$

and thence at once the desired result.

A number of similar theorems are briefly referred to : (*a*) where the element in the $(r, s)^{\text{th}}$ place is a constant function of the greatest common measure of r and s ; (*b*) where it is the least common multiple of r and s ; (*c*) where the diagonal elements are certain conditioned numbers other than the first n integers.

GLAISHER, J. W. L. (1876).

[Theorem relating to the differentiation of a symmetrical determinant. *Quart. Journ. of Math.*, xiv. pp. 245–248.]

The theorem is to the effect that, *If* $u = 1/\Delta^{\frac{1}{2}}$, *where* Δ *stands for*

$$\begin{vmatrix} 2a_{11} & a_{12} & a_{13} & \dots \\ a_{21} & 2a_{22} & a_{23} & \dots \\ a_{31} & a_{32} & 2a_{33} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{a_{rs}=a_{sr}},$$

then any derivative of u *with respect to one or more of the* a*'s is equal to any other such derivative having the same number of occurrences of each suffix-number.* For example,

$$\frac{\partial^2}{\partial a_{12}^2}\frac{\partial}{\partial a_{23}}\frac{\partial}{\partial a_{34}}u = \frac{\partial}{\partial a_{11}}\frac{\partial}{\partial a_{22}}\frac{\partial}{\partial a_{24}}\frac{\partial}{\partial a_{33}}u = \frac{\partial^2}{\partial a_{13}^2}\frac{\partial}{\partial a_{22}}\frac{\partial}{\partial a_{24}}u = \dots .$$

It is an inference from a property of the multiple integral

$$\left(\int_{-\infty}^{\infty}\right)^n e^{-\frac{1}{2}Q}\partial x_1 \partial x_2 \dots \partial x_n,$$

where Q is the n-ary quadric

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & \dots & \\ \hline 2a_{11} & a_{12} & a_{13} & \dots & x_1 \\ a_{21} & 2a_{22} & a_{23} & \dots & x_2 \\ a_{31} & a_{32} & 2a_{33} & \dots & x_3 \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

whose discriminant is Δ.

TRZASKA, W. (1876): FERRERS, N. M. (1876).

[Question 201. *Nouv. Corresp. Math.*, ii. p. 401. Solution by Even, iii. pp. 91–92.]

[An Elementary Treatise on Trilinear Coordinates, 3rd ed. (pp. 59–73, 172, 179). xiv+184 pp. London.]

Exactly in Cayley's manner it is shown that higher-order determinants formed like his third of 1841 (*Hist.*, ii. p. 110) vanish also.

The result intended to be given by Ferrers (p. 72, ex. 8) is

$$\begin{vmatrix} (b+c)^2 & c^2 & b^2 \\ c^2 & (c+a)^2 & a^2 \\ b^2 & a^2 & (a+b)^2 \end{vmatrix} = 2(bc+ca+ab)^3.$$

JAMET, V. (1877): LONGCHAMPS, G. DE (1877).
PAIGE, C. LE (1879): WOLSTENHOLME, J. (1879).

[Sur une application des déterminants. *Nouv. Annales de Math.*, (2) xvi. pp. 372–373.]

[Des fractions étagées (p. 325). *Giornale di Mat.*, xv. pp. 299–328.]

[Question 514. *Nouv. Corresp. Math.*, v. p. 452. Solution by Jamet, vi. pp. 92–93.]

[Question 6038. *Educ. Times*, xxxii. pp. 243, 315; or *Math. from Educ. Times*, xxxii. p. 91.]

None of these is of any moment, the first and third being but instances of the square of an oblong array, the fourth a simple instance of the multiplication of two determinants, and the second a reproduction of Ferrers' first result of 1855.

MANSION, P. (1877).

[On an arithmetical theorem of Professor Smith's. *Messenger of Math.*, vii. pp. 81–82.]

[Généralisation d'un théorème de M. H.-J.-S. Smith. *Annales . . . Soc. Sci. . . . Bruxelles*, ii. (B), pp. 211–224; or as § iii. of pamphlet "Sur la théorie des nombres." 16 pp. Gand, 1878.]

Mansion, recognising not only that Smith's determinant is axisymmetric but that it also has the element in the place (r, s) equal

to the element in the place $(r-s, s)$, obtains the array of D_{k+1} from that of D_k by annexing to the diagonal of the latter the additional element $k+1$ and completing the new row and column by inserting in both cases the elements of the secondary diagonal of D_k: for example,* $D_5 = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 5 \end{vmatrix}$.

He next states that with a suitable alteration in the meaning of ψ the theorem holds when in all the places of D_n we put instead of 1, 2, 3, . . . any quantities x_1, x_2, x_3, . . . respectively; for example,

$$\begin{vmatrix} x_1 & x_1 & x_1 & x_1 & x_1 & x_1 \\ x_1 & x_2 & x_1 & x_2 & x_1 & x_2 \\ x_1 & x_1 & x_3 & x_1 & x_1 & x_3 \\ x_1 & x_2 & x_1 & x_4 & x_1 & x_2 \\ x_1 & x_1 & x_1 & x_1 & x_5 & x_1 \\ x_1 & x_2 & x_3 & x_2 & x_1 & x_6 \end{vmatrix} = x_1(x_2-x_1)(x_3-x_1)(x_4-x_2)(x_5-x_1)(x_6-x_3-x_2+x_1).$$

This is seen to be transformable by substitution so as to have all the factors on the right-hand side of the equality monomials, the result then being

$$\begin{vmatrix} y_1 & y_1 & y_1 & y_1 & y_1 & y_1 \\ y_1 & y_1+y_2 & y_1 & y_1+y_2 & y_1 & y_1+y_2 \\ y_1 & y_1 & y_1+y_3 & y_1 & y_1 & y_1+y_3 \\ y_1 & y_1+y_2 & y_1 & y_1+y_2+y_4 & y_1 & y_1+y_2 \\ y_1 & y_1 & y_1 & y_1 & y_1+y_5 & y_1 \\ y_1 & y_1+y_2 & y_1+y_3 & y_1+y_2 & y_1 & y_1+y_2+y_3+y_6 \end{vmatrix} = y_1y_2y_3y_4y_5y_6,$$

where the suffixes in any place (r, s) are the common divisors of r and s. The same result is also suggested if we take Smith's theorem and by a well-known property of integers express every element of the determinant as a sum of ψ's; for example,

$$3 = \psi(1)+\psi(3), \qquad 6 = \psi(1)+\psi(2)+\psi(3)+\psi(6).$$

* This means that instead of defining $a_{r,s}$ as being the G.C.M. of r and s, we may substitute $a_{r,r} = r$, $a_{r,s} = a_{r-s,s}$, $a_{r,s} = a_{s,r}$.

CATALAN, E. (1878).

[Théorèmes de MM. Smith et Mansion. *Nouv. Corresp. Math.*, iv. pp. 103–112.]

This is a fresh and clearly written exposition, the last of Mansion's theorems, however, being altered for the worse into "Tout produit est égal à un déterminant."

STODOLKIEWICZ, J. (1878).

[Proof of Cayley's formula for calculating the number of different kinds of terms in an axisymmetric determinant (In Polish). *Izwiestja Cesarskiego Uniwersytetu Warszawskiego*, 1878, No. 6.*]

The formula referred to is the difference-equation

$$u_n = nu_{n-1} - \tfrac{1}{2}(n-1)(n-2)u_{n-3},$$

which is now established by merely using elementary properties of determinants. Putting v_n for the number of different kinds of terms in the special form of n-line determinant which is only axisymmetric after a row and a column have been deleted,† we see, from familiar developments of the two determinants, that

$$\left.\begin{aligned} u_n &= u_{n-1}+(n-1)u_{n-2} + \tfrac{1}{2}(n-1)(n-2)v_{n-2}, \\ v_n &= u_{n-1}+(n-1)v_{n-1}; \end{aligned}\right\}$$

and between these the v's can readily be eliminated, with the desired result.

PAIGE, C. LE (1878).

[Sur un théorème de M. Mansion. *Nouv. Corresp. Math.*, iv. pp. 176–178.]

Taking Mansion's determinant for $y_1y_2 \ldots y_6$ Le Paige performs the operations

$$\begin{aligned} &\mathrm{col}_2-\mathrm{col}_1, \quad \mathrm{col}_3-\mathrm{col}_1, \quad \ldots\ldots, \quad \mathrm{col}_6-\mathrm{col}_1; \\ &\mathrm{col}_4-\mathrm{col}_2, \quad \mathrm{col}_6-\mathrm{col}_2; \\ &\mathrm{col}_6-\mathrm{col}_3; \end{aligned}$$

* Or Baraniecki's text-book, chap. x. § 94.

† *e.g.* an axisymmetric determinant unsymmetrically bordered.

obtaining thereby

$$\begin{vmatrix} y_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_1 & y_2 & \cdot & \cdot & \cdot & \cdot \\ y_1 & \cdot & y_3 & \cdot & \cdot & \cdot \\ y_1 & y_2 & \cdot & y_4 & \cdot & \cdot \\ y_1 & \cdot & \cdot & \cdot & y_5 & \cdot \\ y_1 & y_2 & y_3 & \cdot & \cdot & y_6 \end{vmatrix}.$$

He then follows up with the like operations on rows, showing, contrary to Catalan's opinion, that all the non-diagonal elements could be made zeros.

The intermediate form here reached by Le Paige is not unworthy of note.

MANSION, P. (1878).

[Démonstration d'un théorème relatif à un déterminant remarquable. *Bull.* . . . *Acad.* . . . *Belgique*, xlvi. pp. 892–899.]

If in Mansion's determinant for $y_1y_2y_3y_4y_5y_6$ we change y_r everywhere into y_rz_r, we must obtain a determinant equal to

$$y_1z_1 \cdot y_2z_2 \cdot y_3z_3 \cdot \cdot \cdot \cdot y_6z_6,$$

and therefore equal to

$$y_1y_2y_3y_4y_5y_6 \cdot z_1z_2z_3z_4z_5z_6,$$

thus indicating a very simple rule for the multiplication of two such determinants. This, however, is not the result which the ordinary multiplication-theorem of determinants gives: and what is now shown is that if the two factors be taken in Le Paige's intermediate form the said theorem gives the product in Mansion's form.*

The law of formation of Le Paige's form is given as

$$\begin{aligned} a_{r,s} &= y_r && \text{when } s = r \\ &= 0 && \text{when } s > r \\ &= a_{r-s,s} && \text{when } s < r. \end{aligned}$$

* Denoting the three forms in the order of simplicity by M, P, O, we may express the relation between them thus:

$$\begin{aligned} \mathrm{M}(x_1, x_2, \ldots) &= \mathrm{P}(x_1, x_2, \ldots) \times \mathrm{P}(1, 1, \ldots) \\ &= \mathrm{P}(1, 1, \ldots) \times \mathrm{O}(x_1, x_2, \ldots) \times \mathrm{P}(1, 1, \ldots). \end{aligned}$$

WOLSTENHOLME, J. (1878).

[MATHEMATICAL PROBLEMS ... 2nd edition. x+480 pp. London.]

Under No. 1627 (1), (2), Wolstenholme gives

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(b+c)(c+a)(a+b),$$

and Ferrers' result of 1876, the latter of which includes No. 1627 (4). No. 1630, so far as it is correct, is identical with a result of Caldarera's of 1871, and is included in one of Baltzer's of 1864. No. 1633 asserts that

$$\begin{vmatrix} 1 & \cos\alpha & \cos(\alpha+\beta) & \cos(\alpha+\beta+\gamma) \\ \cos\alpha & 1 & \cos\beta & \cos(\beta+\gamma) \\ \cos(\alpha+\beta) & \cos\beta & 1 & \cos\gamma \\ \cos(\alpha+\beta+\gamma) & \cos(\beta+\gamma) & \cos\gamma & 1 \end{vmatrix}$$

and all its primary minors vanish, a fact which we may verify for ourselves by showing that it equals

$$\sin^2\alpha \cdot \sin^2(\alpha+\beta) \cdot \sin^2(\alpha+\beta+\gamma) \cdot \begin{vmatrix} 1 & . & . & . \\ \cos\alpha & 1 & 1 & 1 \\ \cos(\alpha+\beta) & 1 & 1 & 1 \\ \cos(\alpha+\beta+\gamma) & 1 & 1 & 1 \end{vmatrix}.$$

SYLVESTER, J. J. (1879).

[Note on determinants and duadic disynthemes. *American Journ. of Math.*, ii. pp. 89–96, 214–222; or *Collected Math. Papers*, iii. pp. 264–280.]

This interesting paper, which is based unconsciously on Cauchy's expression of a substitution as a quasi-product of circular substitutions (*Hist.*, i. pp. 101, 304–305) deals with the number of terms in axisymmetric and skew determinants. In addition to previously known results he asserts (p. 93) that *if* u_n *be the number of distinct terms in an 'invertebrate'* (*i.e.* zero-axial) *axisymmetric determinant, then*

$$u_n = (n-1)(u_{n-1}+u_{n-2}) - \tfrac{1}{2}(n-1)(n-2)u_{n-3},$$

thus giving 0, 1, 1, 6, 22, 130, 822, ...

for the first seven values of u_n.

CHAPTER IV.

SYMMETRIC DETERMINANTS THAT ARE NOT AXISYMMETRIC, FROM 1862 TO 1879.

THE two kinds of determinants that are here considered are exemplified by

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ b_4 & b_3 & b_2 & b_1 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix}, \qquad \begin{vmatrix} x_1 & a_2 & a_3 & a_4 \\ a_1 & x_2 & a_3 & a_4 \\ a_1 & a_2 & x_3 & a_4 \\ a_1 & a_2 & a_3 & x_4 \end{vmatrix}.$$

In the first kind the r^{th} row from the end is, when reversed, the same as the r^{th} row from the beginning,—a peculiarity which, as a consequence, is found also in the columns: a fitting name for it is *centrosymmetric*. In the second kind each column has only two distinct elements, one in the intersection with the diagonal, and the other occupying all the remaining places. If each row be multiplied by the element which does not appear in the row, the result is an axisymmetric determinant; for example,

$$\begin{vmatrix} x_1 & a_2 & a_3 & a_4 \\ a_1 & x_2 & a_3 & a_4 \\ a_1 & a_2 & x_3 & a_4 \\ a_1 & a_2 & a_3 & x_4 \end{vmatrix} = \begin{vmatrix} a_1x_1 & a_1a_2 & a_1a_3 & a_1a_4 \\ a_2a_1 & a_2x_2 & a_2a_3 & a_2a_4 \\ a_3a_1 & a_3a_2 & a_3x_3 & a_3a_4 \\ a_4a_1 & a_4a_2 & a_4a_3 & a_4x_4 \end{vmatrix} \div a_1a_2a_3a_4.$$

An n-line determinant of the first of these kinds evidently involves $\frac{1}{2}n^2$ variables when n is even and $\frac{1}{2}(n^2+1)$ when n is odd; an n-line determinant of the second kind involves only $2n$.

ZEHFUSS, G. (1862).

[Zwei Sätze über Determinanten. *Zeitschrift f. Math. u. Phys.*, vii. pp. 436–439.]

Zehfuss' second theorem concerns a determinant whose elements are situated symmetrically with respect to the centre of the square. When the order-number is even, 6 say, it is to the effect that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_6 & c_5 & c_4 & c_3 & c_2 & c_1 \\ b_6 & b_5 & b_4 & b_3 & b_2 & b_1 \\ a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix} = \begin{vmatrix} a_1+a_6 & a_2+a_5 & a_3+a_4 \\ b_1+b_6 & b_2+b_5 & b_3+b_4 \\ c_1+c_6 & c_2+c_5 & c_3+c_4 \end{vmatrix} \cdot \begin{vmatrix} a_1-a_6 & a_2-a_5 & a_3-a_4 \\ b_1-b_6 & b_2-b_5 & b_3-b_4 \\ c_1-c_6 & c_2-c_5 & c_3-c_4 \end{vmatrix},$$

the result being reached by performing the operations

$$\text{row}_1+\text{row}_6, \qquad \text{row}_2+\text{row}_5, \qquad \text{row}_3+\text{row}_4$$

followed by

$$\text{col}_6-\text{col}_1, \qquad \text{col}_5-\text{col}_2, \qquad \text{col}_4-\text{col}_3.$$

When the order-number is odd, 5 say, it is similarly seen that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_2 & c_1 \\ b_5 & b_4 & b_3 & b_2 & b_1 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{vmatrix} = \begin{vmatrix} a_1+a_5 & a_2+a_4 & 2a_3 \\ b_1+b_5 & b_2+b_4 & 2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_1-a_5 & a_2-a_4 \\ b_1-b_5 & b_2-b_4 \end{vmatrix}.$$

The latter case, however, is incorrectly stated by Zehfuss, the 2's in the first determinant on the right being omitted and a power of 2 prefixed to the said determinant as a factor.

FERRERS, N. M. (1866).

[An Elementary Treatise on Trilinear Coordinates. (Chap. iii. pp. 59–72.) 2nd edition. xiv+182 pp. London.]

Ferrers' last exercise when correctly stated* depends on the result

* Another mode of correcting would make the underlying result

$$\begin{vmatrix} a^2 & (c+a)^2 & (a+b)^2 \\ (b+c)^2 & b^2 & (a+b)^2 \\ (b+c)^2 & (c+a)^2 & c^2 \end{vmatrix} = (a+b+c)^3(-\Sigma a^3-\Sigma a^2 b).$$

$$\begin{vmatrix} (b+c)^2 & b^2 & c^2 \\ a^2 & (c+a)^2 & c^2 \\ a^2 & b^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3,$$

given the same year in Salmon's *Modern Higher Algebra* (p. 14). The determinant has the simple axisymmetric form

$$\begin{vmatrix} (b+c)^2 & ab & ac \\ ab & (c+a)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix}.$$

SARDI, C. (1868).

[Un teorema su' determinanti. *Giornale di Mat.*, vi. pp. 357–360.]

Sardi viewed his determinant as a generalisation of Torelli's axisymmetric determinant of 1864. When of the fourth order it is

$$\begin{vmatrix} x_1 & a_2 & a_3 & a_4 \\ a_1 & x_2 & a_3 & a_4 \\ a_1 & a_2 & x_3 & a_4 \\ a_1 & a_2 & a_3 & x_4 \end{vmatrix},$$

all the elements of any column being identical except the diagonal element. Of his two ways of treating it the second is the better, namely, changing it into a determinant of the fifth order by appending a row of four zeros and then a column of five 1's. This five-line determinant is then seen to be equal to

$$\begin{vmatrix} x_1-a_1 & . & . & . & 1 \\ . & x_2-a_2 & . & . & 1 \\ . & . & x_3-a_3 & . & 1 \\ . & . & . & x_4-a_4 & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & 1 \end{vmatrix},$$

and therefore equal to

$$\begin{aligned} &(x_1-a_1)(x_2-a_2)(x_3-a_3)(x_4-a_4) \\ &+ (x_1-a_1)(x_2-a_2)(x_3-a_3)\,a_4 \\ &+ (x_1-a_1)(x_2-a_2)(x_4-a_4)\,a_3 \\ &+ (x_1-a_1)(x_3-a_3)(x_4-a_4)\,a_2 \\ &+ (x_2-a_2)(x_3-a_3)(x_4-a_4)\,a_1 \end{aligned}$$

or

$$(x_1-a_1)(x_2-a_2)(x_3-a_3)(x_4-a_4)\left\{1+\frac{a_1}{x_1-a_1}+\frac{a_2}{x_2-a_2}+\frac{a_3}{x_3-a_3}+\frac{a_4}{x_4-a_4}\right\}.$$

He might appropriately and with equal success have followed Torelli's method, namely, beginning by changing the diagonal elements into

$$x_1-a_1+a_1, \quad x_2-a_2+a_2, \quad x_3-a_3+a_3, \quad x_4-a_4+a_4.$$

He might also have remarked that when each of the x's is made equal to x and each of the a's equal to a, the value of the determinant is $(x-a)^{4-1}\{x+(4-1)a\}$.

LUCAS, F. (1870).

[Sur une formule d'analyse. *Comptes rendus . . . Acad. des. Sci.* (Paris), lxx. pp. 1167–1168; or *Journ. (de Liouville) de Math.*, (2) xv. p. 179.]

Lucas' determinant is the particular case of Sardi's, in which the diagonal elements are such that the sum of the elements of each row is x. The value of the n-line determinant is thus $x(x-\Sigma a)^{n-1}$.

WOLSTENHOLME, J. (1870).

[Question 3285. *Educ. Times*, xxiii. p. 209, p. 259; or *Math. from Educ. Times*, xv. pp. 40–42; solution by V. Mollame in *Giornale di Mat.*, ix. (1871), pp. 58–59.]

Wolstenholme's result is that, if s stand for $a+b+c+d$,

$$\begin{vmatrix} (s-a)^2 & a^2 & a^2 & a^2 \\ b^2 & (s-b)^2 & b^2 & b^2 \\ c^2 & c^2 & (s-c)^2 & c^2 \\ d^2 & d^2 & d^2 & (s-d)^2 \end{vmatrix} = 2s^5 . abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}-\frac{4}{s}\right).$$

It is seen to be the next higher case of the general theorem foreshadowed under Ferrers (1861). To establish it S. Roberts in effect transforms the determinant by subtracting the first column from each of the others and removing the factors s^2-2sa, s^2-2sb, s^2-2sc, s^2-2sd, the development being then found in the form

$$s^3\cdot(s-2a)(s-2b)(s-2c)(s-2d)\cdot\left(s+\frac{a^2}{s-2a}+\frac{b^2}{s-2b}+\frac{c^2}{s-2c}+\frac{d^2}{s-2d}\right),$$

from which Wolstenholme's form * is readily derived. The merit of Roberts' result is that it is true for any value of s and is easily generalizable, the corresponding expression for the three-line determinant being

$$s^2 \cdot (s-2a)(s-2b)(s-2c) \cdot \left(s + \frac{a^2}{s-2a} + \frac{b^2}{s-2b} + \frac{c^2}{s-2c}\right),$$

which, when s stands for $a+b+c$, is easily reducible to $2s^3 \cdot abc$.

J. J. Walker's procedure, on the other hand, is to start with the more general but self-evident result

$$\begin{vmatrix} A+\alpha & \alpha & \alpha & \alpha \\ \beta & B+\beta & \beta & \beta \\ \gamma & \gamma & C+\gamma & \gamma \\ \delta & \delta & \delta & D+\delta \end{vmatrix} = ABCD + \sum ABC\delta,$$

and then make the requisite substitution.

Mollame's process at the outset resembles that of Roberts, but is not an improvement.

GÜNTHER, S. (1873, 1876).

[Ueber einige Determinantensätze. *Sitzungsb. . . . Soc. zu Erlangen*, v. pp. 85–95.]

[Ueber aufsteigende Kettenbrüche. *Zeitschrift f. Math. u. Phys.*, xxi. pp. 178–191.]

What is here established, in two different ways, is Lucas' special result of 1870.

In the paper of 1876 Günther incidentally returns to the subject (§ 5, pp. 183–184), the penultimate form of his development, however, being the same as Albeggiani's of the year preceding.

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* Wolstenholme's form of development seems oddly chosen; for it manifestly

$$= 2s^5 \cdot \Sigma abc - 8s^4 \cdot abcd = 2s^4(s \cdot \Sigma abc - 4abcd)$$
$$= 2s^4 \cdot \Sigma a^2bc.$$

The corresponding determinant of the fifth order, when $s = a+b+c+d+e$, is equal to

$$2s^5(s \cdot \Sigma a^2bc + 12abcde),$$

from which the preceding result is got by putting $e = 0$.

ALBEGGIANI, M. (1875).

[Dimostrazione d'una formula d'analisi di F. Lucas. *Giornale di Mat.*, xiii. pp. 107–112.]

With considerable fullness of explanation the determinant is expanded in a series arranged according to descending powers of x, the result being

$$x^n - C_{n-1,1}\cdot(\Sigma a)\cdot x^{n-1} + C_{n-1,2}\cdot(\Sigma a)^2\cdot x^{n-2} - \ \ldots\ + (-1)^{n-1}(\Sigma a)^{n-1}x,$$
$$i.e. \qquad x(x-\Sigma a)^{n-1}.$$

Specialization is then needlessly used to obtain the otherwise self-evident result

$$\begin{vmatrix} x & a & a & \ldots & a & a \\ a & x & a & \ldots & a & a \\ a & a & x & \ldots & a & a \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a & a & a & \ldots & x & a \\ 1 & 1 & 1 & \ldots & 1 & 1 \end{vmatrix}_{n+1} = (x-a)^n,$$

and Sylvester's evaluation of the circulant

$$C(x, 1, 1, 1, \ldots, 1)_n.$$

JOHNSON, W. W. (1877).

[Problem 156. *Analyst*, iv. pp. 90–91.]

As the interchange of any two variables of the determinant

$$\begin{vmatrix} 1 & b & c & \ldots \\ a & 1 & c & \ldots \\ a & b & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

may be effected by an even number of interchanges of lines, it follows that the determinant must be a symmetrical function of the variables. In the next place it may be shown that the determinant must be of the form

$$1 - x\Sigma ab + y\Sigma abc - z\Sigma abcd + \ \ldots,$$

and lastly that

$$x, y, z, \ldots = 1, 2, 3, \ldots.$$

SCOTT, R. F. (1878).

[On some theorems on determinants. *Messenger of Math.*, viii. pp. 33–37.]

Scott's mode of dealing with Lucas' determinant is independent of any development, the factors being obtained by inference. That x is a factor is known from the sum of the columns : and, by subtracting a suitable multiple of a particular column from any one of the other columns and then putting $x = \Sigma a$, it is seen that $x - \Sigma a$ is a factor of the latter column as thus altered, and that therefore $(x-\Sigma a)^{n-1}$ is a factor of the determinant.

The same procedure, he adds, gives the related result

$$\begin{vmatrix} x-a_1 & \Sigma a-a_2 & \Sigma a-a_3 & \dots \\ \Sigma a-a_1 & x-a_2 & \Sigma a-a_3 & \dots \\ \Sigma a-a_1 & \Sigma a-a_2 & x-a_3 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n = \{x+(n-2)\Sigma a\}(x-\Sigma a)^{n-1}.$$

WOLSTENHOLME, J. (1878).

[Mathematical Problems. . . . 2nd edition. x+480 pp. London.]

Under no. 1627 (6), Wolstenholme gives

$$\begin{vmatrix} a-n(b+c) & (n+1)b & (n+1)c \\ (n+1)a & b-n(c+a) & (n+1)c \\ (n+1)a & (n+1)b & c-n(a+b) \end{vmatrix} = n^2(a+b+c)^3,$$

apparently not observing that it includes 1627 (3) and 1627 (7). The case of it where n is 1 is also a case of the first equality given by "H. 1" in 1846 (*Hist.*, ii. p. 114).

SCOTT, R. F. (1879).

[On some symmetrical forms of determinants. *Messenger of Math.*, viii. pp. 131–138.]

The latter half of this paper is occupied with very special forms of centrosymmetric determinants, beginning with the $2m$-line determinant whose primary diagonal consists of a's, secondary

diagonal of b's, and whose every place else is occupied by c's. Without using Zehfuss' theorem, this is found to be equal to

$$(a-b)^n(a+b-2c)^{n-1}(a+b+2c\cdot\overline{n-1}),$$

the factor $(a-b)^n$ being got by considering the effect of putting $a=b$ in the determinant, and so on.

The other determinants are still more fanciful in form: for example,

$$\begin{vmatrix} x & a & a & b & b & y \\ d & x & a & b & y & c \\ d & d & x & y & c & c \\ c & c & y & x & d & d \\ c & y & b & a & x & d \\ y & b & b & a & a & x \end{vmatrix} = \begin{cases} \dfrac{(a-b)(x-y+c-d)^3+(c-d)(x-y-a+b)^3}{a-b+c-d} \\ \cdot \dfrac{(a+b)(x+y-c-d)^3-(c+d)(x+y-a-b)^3}{a+b-c-d}, \end{cases}$$

the first step in the evaluation here being made with the help of Zehfuss' result.

SCOTT, R. F. (1879).

[On some symmetrical forms of determinants. *Messenger of Math.*, viii. pp. 145–150.]

In returning to determinants of Sardi's form, Scott returns also to the original method of Sardi, namely, substitution of a suitable determinant of higher order. He thus obtains for

$$\begin{vmatrix} (x-a)^2 & b^2 & c^2 & \dots \\ a^2 & (x-b)^2 & c^2 & \dots \\ a^2 & b^2 & (x-c)^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{n'} \quad \begin{vmatrix} (x-a)^2 & ab & ac & \dots \\ ab & (x-b)^2 & bc & \dots \\ ac & bc & (x-c)^2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{n}$$

the forms

$$\begin{vmatrix} -a^2 & -b^2 & -c^2 & \dots \\ x(x-2a) & . & . & \dots \\ . & x(x-2b) & . & \dots \\ . & . & x(x-2c) & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{n+1'} \quad \begin{vmatrix} 1 & -a & -b & -c & \dots \\ a & x(x-2a) & . & . & \dots \\ b & . & x(x-2b) & . & \dots \\ c & . & . & x(x-2c) & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_{n+}$$

respectively, whence comes the common development

$$x^{n-1}\cdot(x-2a)(x-2b)(x-2c)\dots\left\{x+\frac{a^2}{x-2a}+\frac{b^2}{x-2b}+\frac{c^2}{x-2c}+\dots\right\}.$$

The result of bordering the given determinants, by prefixing horizontally and vertically

$$0,\ 1,\ 1,\ 1,\ \ldots$$

to the first, and

$$0,\ a,\ b,\ c,\ \ldots$$

to the second, is a pair of determinants with the values

$$x^{n-1}\cdot(x-2a)(x-2b)(x-2c)\ \ldots\left\{\frac{1}{x-2a}+\frac{1}{x-2b}+\frac{1}{x-2c}+\ \ldots\right\},$$

$$-\,x^{n-1}\cdot(x-2a)(x-2b)(x-2c)\ \ldots\left\{\frac{a^2}{x-2a}+\frac{b^2}{x-2b}+\frac{c^2}{x-2c}+\ \ldots\right\}.$$

The case of the former value where

$$x = a+b+c+\ \ldots$$

is fully worked out for $n = 3$ and for $n = 4$.

In his text-book, which appeared the following year, he goes a little further (pp. 216–218), bordering axisymmetrically the first of the initial determinants above with the two lines

$$1,\ \ 1,\ \ \ldots,\ 1,\ \ 0,\ 0,$$
$$\beta_1,\ \beta_2,\ \ldots,\ \beta_n,\ 0,\ 0,$$

the evaluation being

$$x^{n-1}\cdot\{(x-2a)(x-2b)\ \ldots\}\begin{vmatrix} \frac{1}{x-2a}+\frac{1}{x-2b}+\ldots & \frac{\beta_1}{x-2a}+\frac{\beta_2}{x-2b}+\ldots \\ \frac{\beta_1}{x-2a}+\frac{\beta_2}{x-2b}+\ldots & \frac{\beta_1^{\,2}}{x-2a}+\frac{\beta_2^{\,2}}{x-2b}+\ldots \end{vmatrix}$$

and generalizing the second initial determinant so as to have

$$\begin{vmatrix} X_1 & a_2b_1 & a_3b_1 & \ldots \\ a_1b_2 & X_2 & a_3b_2 & \ldots \\ a_1b_3 & a_2b_3 & X_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n = \{(X_1-a_1b_1)(X_2-a_2b_2)\ \ldots\}\cdot\left\{1+\frac{a_1b_1}{X_1-a_1b_1}+\frac{a_2b_2}{X_2-a_2b_2}+\ \ldots\right\}.$$

He also notes (p. 214) the peculiar $2n$-line determinant whose odd rows have an a where they intersect the diagonal and a b elsewhere, and whose even rows have the reverse construction, the evaluation being

$$(-1)^{n-1}\cdot(n-1)\cdot(a-b)^n.$$

CHAPTER V.

ALTERNANTS, FROM 1860 TO 1879.

ALTHOUGH the attention given to alternants in the preceding twenty-year period was very considerable, greater, indeed, than in the case of any other special class except axisymmetric determinants, the present period shows a marked increase, the number of writings being almost doubled, and one or two of them being of unusual extent.

CLEBSCH, A. (1860): 'UN PROFESSEUR' (1860).

[Ueber symbolische Darstellung algebraischer Formen. *Crelle's Journ.*, lix. pp. 1–62.]

[Solution de la question 515. *Nouv. Annales de Math.*, (1) xix. pp. 181–182.]

In the first of these two papers (p. 41) there incidentally occurs the simple equality

$$\begin{vmatrix} a_1^2 & a_1b_1 & b_1^2 \\ a_2^2 & a_2b_2 & b_2^2 \\ a_3^2 & a_3b_3 & b_3^2 \end{vmatrix} = |a_1b_2| \cdot |a_1b_3| \cdot |a_2b_3|,$$

which is perhaps best viewed as the result of substituting b_1/a_1, b_2/a_2, b_3/a_3 for x, y, z respectively in the two forms of the difference-product of x, y, z.

The second paper merely reproduces Cauchy's mode of factorizing $|a_1^0 b_2^1 c_3^2 \ldots|$.

BORCHARDT, C. W. (1860).

[Ueber eine Interpolationsformel für eine Art symmetrischer Functionen und über deren Anwendungen. *Abhandl. . . . Akad. d. Wiss.* (Berlin), pp. 1–20; or *Gesammelte Werke*, pp. 153–172.]

From a different standpoint the interpolation-formula here dealt with may be viewed as giving an expression for the quotient of an alternating function $|\,F_1(x_1)\cdot F_2(x_2)\,.\,.\,.\,F_m(x_m)\,|$ by the difference-product of the variables. The theorem is that if none of the F's be of a higher degree than the $(n-1)^{\text{th}}$, and n be greater than m, then

$$\frac{|\,F_1(x_1)\cdot F_2(x_2)\,.\,.\,.\,F_m(x_m)\,|}{|\,x_1^0 x_2^1\,.\,.\,.\,x_m^{m-1}\,|} = \sum\left\{\frac{|\,F_1(a_1)\cdot F_2(a_2)\,.\,.\,.\,F_m(a_m)\,|}{|\,a_1^0 a^1\,.\,.\,.\,a_m^{m-1}\,|}\cdot\frac{R(x_1, x_2, \ldots, x_m;\ a_{m+1}, a_{m+2}, \ldots, a_n)}{R(a_1, a_2, \ldots, a_m;\ a_{m+1}, a_{m+2}, \ldots, a_n)}\right\},$$

where $R(x_1, x_2, \ldots, x_m;\ a_1, a_2, \ldots, a_n)$ stands for the product of the mn differences got by subtracting each quantity on the left of the semicolon from each quantity on the right, and where the Σ betokens a sum of $C_{n,m}$ terms, of which the first is given and the others differ from the first merely in having another set of m of the a's instead of the first set.

In regard to this it is appropriate first to note that there are $C_{n,m}$ different cases in which the equality is easily seen to hold, namely, the cases where the values given to the variables are any m of the quantities $a_1, a_2, \ldots, a_n$. Nothing more than substitution is required, all the $C_{n,m}$ terms on the right except one vanishing in each case, and that one being identical with the left-hand member. It should be noted too that the specified limitation respecting the degree of the F's is not here required.

In order to establish the *absolute* identity of the two members Borchardt uses n^2 times Lagrange's interpolation-theorem

$$\frac{F_r(z)}{\phi(z)} = \frac{F_r(a_1)}{\phi'(a_1)\cdot(z-a_1)} + \frac{F_r(a_2)}{\phi'(a_2)\cdot(z-a_2)} + \ldots + \frac{F_r(a_n)}{\phi'(a_n)\cdot(z-a_n)},$$

in which $\phi(z) = (z-a_1)(z-a_2)\,.\,.\,.\,(z-a_n)$, and which only holds

when $m < n$, and thus obtains from the multiplication-theorem of determinants the result

$$\frac{|\,F_1(x_1)\cdot F_2(x_2)\ldots F_m(x_m)\,|}{\phi(x_1)\cdot\phi(x_2)\ldots\phi(x_m)}$$

$$=\sum\left\{\frac{|\,F_1(a_1)\cdot F_2(a_2)\ldots F_m(a_m)\,|}{\phi'(a_1)\cdot\phi'(a_2)\ldots\phi'(a_m)}\cdot|\,(x_1-a_1)^{-1}(x_2-a_2)^{-1}\ldots(x_m-a_m)^{-1}\,|\right\}.$$

It only remains then to employ on the right the already known equalities

$$\phi'(a_1)\cdot\phi'(a_2)\ldots\phi'(a_m)=(-1)^{\frac{1}{2}m(m-1)}\cdot|\,a_1^0a_2^1\ldots a_m^{m-1}\,|^2\cdot R(a_1,\ldots,a_m;\,a_{m+1}\ldots a_n$$

and

$$|\,(x_1-a_1)^{-1}(x_2-a_2)^{-1}\ldots(x_m-a_m)^{-1}\,|=(-1)^{\frac{1}{2}m(m-1)}\cdot\frac{|\,x_1^0x_2^1\ldots x_m^{m-1}\,|\cdot|\,a_1^0a_2^1\ldots a_m^{m-1}|}{R(x_1,\ldots,x_m;\,a_1,\ldots,a_m)}$$

and on the left the self-evident identity

$$\phi(x_1)\cdot\phi(x_2)\ldots\phi(x_m)=R(x_1,\ldots,x_m;\,a_1,\,a_2,\ldots,a_n)$$
$$=R(x_1,\ldots,x_m;\,a_1,\ldots,a_m)\cdot R(x_1,\ldots,x_m;\,a_{m+1}\ldots a_n)$$

A remark added by Borchardt, and specially interesting from his standpoint, is to the effect that when m is taken equal to 1 the theorem degenerates into

$$F_1(x_1)=\sum F_1(a_1)\cdot\frac{R(x_1;\,a_2,\,a_3,\ldots,a_n)}{R(a_1;\,a_2,\,a_3,\ldots,a_n)},$$

that is, into

$$\frac{F_1(x_1)}{R(x_1;\,a_1,\,a_2,\ldots,a_n)}=\sum\frac{F_1(a_1)}{R(a_1;\,a_2,,\ldots,a_n)\cdot(x_1-a_1)},$$

which is exactly the simple interpolation-theorem used in the course of the demonstration.

On our side, however, it must be noted that Borchardt's theorem is of no practical use if we are seeking for the result of the performance of the actual division indicated in the left-hand member, as the equivalent offered by the other member involves not only the legitimate x's, but also the n extraneous quantities $a_1, a_2, \ldots, a_n$. For example, when $m = 2$, $n = 3$, $F_1(z) = bz$, $F_2(z) = cz^2$, we obtain for

$$\begin{vmatrix} bx_1 & cx_1^2 \\ bx_2 & cx_2^2 \end{vmatrix} \div (x_2-x_1),$$

which is manifestly equal to bcx_1x_2, the expression

$$bca_1a_2\frac{(x_1-a_3)(x_2-a_3)}{(a_1-a_3)(a_2-a_3)}+bca_1a_3\frac{(x_1-a_2)(x_2-a_2)}{(a_1-a_2)(a_3-a_2)}+bca_2a_3\frac{(x_1-a_1)(x_2-a_1)}{(a_2-a_1)(a_3-a_1)};$$

and, if n were greater, we should be offered a greater number of more complicated terms.

TRUDI, N. (1864).

[Intorno ad un determinante piu generale di quello che suol dirsi determinante delle radici di una equazione, ed alle funzioni simmetriche complete di questi radici. *Rendic. dell' Accad.* . . . (Napoli), pp. 121–134; or, with slightly different title, in *Giornale di Mat.*, ii. pp. 152–158, 180–186.]

The main theorem reached by Trudi is essentially the same as one of Jacobi's of 1841 (*Hist.*, i. p. 341), namely, where the quotient of an alternant by the corresponding difference-product is expressed as a determinant whose elements are complete homogeneous functions of the variables. These latter functions, which Jacobi denoted temporarily by C_1, C_2, . . . , are recognized by Trudi as the 'aleph' functions of Wronski (*Hist.*, ii. pp. 216–218), and are suitably represented by enclosing the set of variables within brackets and making the degree-number an exponent: for example,

$$(a, b, c)^1 = a+b+c,$$
$$(a, b, c)^2 = a^2+b^2+c^2+ab+ac+bc.$$

Starting with the self-evident development-formula,

$$(b, c, \ldots, l)^r = a^r + a^{r-1}(b, c, \ldots, l)^1 + a^{r-2}(b, c, \ldots, l)^2 + \ldots + (b, c, \ldots, l)^r,$$

and having thence shown that

$$(a, b, c, \ldots, l)^r = a(a, b, c, \ldots, l)^{r-1} + (b, c, \ldots, l)^r$$

and

$$(a, b, \ldots, i, k)^r - (a, b, \ldots, i, l)^r = (k-l)(a, b, \ldots, i, k, l)^{r-1},$$

he is prepared to deal with the determinant in question, namely,

$$|\, a^\alpha b^\beta c^\gamma d^\delta \ldots |.$$

Subtracting the first column from each of the others, he finds

$$\begin{vmatrix} a^\alpha & b^\alpha & c^\alpha & d^\alpha \\ a^\beta & b^\beta & c^\beta & d^\beta \\ a^\gamma & b^\gamma & c^\gamma & d^\gamma \\ a^\delta & b^\delta & c^\delta & d^\delta \end{vmatrix} = \begin{vmatrix} a^\alpha & (a,b)^{\alpha-1} & (a,c)^{\alpha-1} & (a,d)^{\alpha-1} \\ a^\beta & (a,b)^{\beta-1} & (a,c)^{\beta-1} & (a,d)^{\beta-1} \\ a^\gamma & (a,b)^{\gamma-1} & (a,c)^{\gamma-1} & (a,d)^{\gamma-1} \\ a^\delta & (a,b)^{\delta-1} & (a,c)^{\delta-1} & (a,d)^{\delta-1} \end{vmatrix} (b-a)(c-a)(d-a).$$

In the second place, by treating the new determinant in the same way, the second column being now the subtrahend, he removes the factors $c-b$, $d-b$; and finally there is obtained

$$\frac{|\,a^\beta b^\alpha c^\gamma d^\delta\,|}{|\,a^0 b^1 c^2 d^3\,|} = \begin{vmatrix} a^\alpha & (a,b)^{\alpha-1} & (a,b,c)^{\alpha-2} & (a,b,c,d)^{\alpha-3} \\ a^\beta & (a,b)^{\beta-1} & (a,b,c)^{\beta-2} & (a,b,c,d)^{\beta-3} \\ a^\gamma & (a,b)^{\gamma-1} & (a,b,c)^{\gamma-2} & (a,b,c,d)^{\gamma-3} \\ a^\delta & (a,b)^{\delta-1} & (a,b,c)^{\delta-2} & (a,b,c,d)^{\delta-3} \end{vmatrix}.$$

A variant form of this is got by showing that all the variables can without any substantial change be introduced into each column, namely,

$$\begin{vmatrix} (a,b,c,d)^\alpha & (a,b,c,d)^{\alpha-1} & (a,b,c,d)^{\alpha-2} & (a,b,c,d)^{\alpha-3} \\ (a,b,c,d)^\beta & (a,b,c,d)^{\beta-1} & (a,b,c,d)^{\beta-2} & (a,b,c,d)^{\beta-3} \\ (a,b,c,d)^\gamma & (a,b,c,d)^{\gamma-1} & (a,b,c,d)^{\gamma-2} & (a,b,c,d)^{\gamma-3} \\ (a,b,c,d)^\delta & (a,b,c,d)^{\delta-1} & (a,b,c,d)^{\delta-2} & (a,b,c,d)^{\delta-3} \end{vmatrix}.$$

Both forms agree with Jacobi's results, when in the latter m is put equal to n. Jacobi's procedure, however, is not nearly so attractive and convincing as Trudi's.

BALTZER, R. (1864).

[THEORIE UND ANWENDUNGEN DER DETERMINANTEN. 2te vermehrte Auflage. vi+224 pp. Leipzig.]

The section dealing with the difference-product (§ 10. pp. 70–92) is much fuller than in the previous edition. Most of the additions, however, have been already noticed under their original authors.

Attention is properly drawn to a theorem suggested by Borchardt in 1859 when dealing with Bezout's condensed eliminant (*Hist.*, ii. p. 148), namely, that if

$$\mathrm{F}(xy) = \sum (a_{ik} x^i y^k)_{i=0,1,\ldots,n-1}^{k=0,1,\ldots,n-1},$$

then

$$|\,\mathrm{F}(h_1k_1)\mathrm{F}(h_2k_2)\ldots\mathrm{F}(h_nk_n)\,|$$

$$= |\,a_{00}a_{11}\ldots a_{n-1,n-1}\,| \cdot \xi^{\frac{1}{2}}(h_1, h_2, \ldots, h_n) \cdot \xi^{\frac{1}{2}}(k_1, k_2, \ldots, k_n):$$

this is merely a double application of the multiplication-theorem. Leaving out one of the difference-products we should have another so-called theorem.

Cauchy's identity (*Hist.*, i. p. 345)*

$$f'(a_1)\cdot f'(a_2)\ldots f'(a_n) = (-1)^{\frac{1}{2}n(n-1)}\left|a_1^0 a_2^1 \ldots a_n^{n-1}\right|^2,$$

where $$f(x) = (x-a_1)(x-a_2)\ldots(x-a_n),$$

is exemplified by the case where the a's are the n^{th} roots of 1, the square on the right being then equal to

$$(-1)^{\frac{1}{2}(n-1)(n-2)}n^n.$$

As a contribution of his own, Baltzer brings forward

$$\left| a_1^0 a_2^1 \ldots a_{n-1}^{n-2} u_n \right| = \left(\frac{u_1}{f'(a_1)} + \frac{u_2}{f'(a_2)} + \ldots + \frac{u_n}{f'(a_n)}\right)\cdot \left| a_1^0 a_2^1 \ldots a_n^{n-1} \right|,$$

in order to introduce another result of Cauchy's, namely, that when

$$u_1, u_2, \ldots, u_n = a_1^r, a_2^r, \ldots, a_r^n$$

the bracketed expression † on the right is equal to 0 when r has any one of the values 0, 1, . . . , $n-2$: secondly, is equal to 1 when $r = n-1$: and, thirdly, is equal to the complete homogeneous function of $a_1, a_2, \ldots$ of the $(r-n+1)^{\text{th}}$ degree when r has a greater integral value.

It is also worth noting that he gives (§ 10. 10 and 17) expressions for the $(i, k)^{\text{th}}$ principal minor of

$$\left| a_1^0 a_2^1 \ldots a_n^{n-1} \right| \quad \text{and of} \quad \left| (\beta_1 - a_1)^{-1}(\beta_2 - a_2)^{-1} \ldots (\beta_n - a_n)^{-1} \right|,$$

namely

$$\frac{c_{i,n-k}}{f'(a_i)} \left| a_1^0 a_2^1 \ldots a_n^{n-1} \right|,$$

and

$$-\frac{f(\beta_i)}{g'(\beta_i)}\cdot\frac{g(a_k)}{f'(a_k)}\cdot\frac{1}{\beta_i - a_k}\left| (\beta_1 - a_1)^{-1}(\beta_2 - a_2)^{-1} \ldots (\beta_n - a_n)^{-1} \right|,$$

where $c_{i,n-k}$ is the coefficient of a_i^{n-k-1} in $f'(a_i)$, and

$$g(x) = (x-\beta_1)(x-\beta_2)\ldots(x-\beta_n).$$

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* Originally in *Journ. de l'éc. polyt.*, x. (1813), p. 485, pp. 497–502.

† It is also clear that in this particular case the bracketed expression is the coefficient of x^{-r-1} in

$$\frac{1}{(x-a_1)f'(a_1)} + \frac{1}{(x-a_2)f'(a_2)} + \ldots + \frac{1}{(x-a_n)f'(a_n)},$$

and therefore in $1/f(x)$, a result given by Jacobi and used above by Brioschi (*Hist.*, ii. p. 216).

To these may be added from another section (§ 3. 15) the result

$$\frac{\partial^{n-1}}{\partial a_1 \cdot \partial a_2 \cdots \partial a_{n-1}} \left| a_1^0 a_2^1 \cdots a_n^{n-1} \right| = (-1)^{n-1} \cdot (n-1)! \left| a_1^0 a_2^1 \cdots a_{n-1}^{n-2} \right|.$$

STERN, M. A. (1865).

[Ueber einen Satz aus der Determinantentheorie. *Crelle's Journ.*, lxvi. pp. 285–288.]

The theorem is that if ϕ_1, ϕ_2, . . . be functions of the first, second, . . . degree in the variable, and have 1 for the coefficient of the highest power in each, say

$$\begin{aligned}
\phi_1(x) &= x + a_{11}, \\
\phi_2(x) &= x^2 + a_{12}x + a_{22}, \\
&\cdots\cdots\cdots \\
\phi_{n-1}(x) &= x^{n-1} + a_{1,n-1}x^{n-2} + \ldots + a_{n-1,n-1},
\end{aligned}$$

then the determinant

$$\begin{vmatrix}
1 & \phi_1(x_1) & \phi_2(x_1) & \ldots & \phi_{n-1}(x_1) \\
1 & \phi_1(x_2) & \phi_2(x_2) & \ldots & \phi_{n-1}(x_2) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
1 & \phi_1(x_n) & \phi_2(x_n) & \ldots & \phi_{n-1}(x_n)
\end{vmatrix}$$

is equal to the difference product of $x_1, x_2, \ldots, x_n$, being in fact, although Stern does not say so, the difference-product multiplied row-wise by 1 in the form

$$\begin{vmatrix}
1 & . & . & . & . & . \\
a_{11} & 1 & . & . & . & . \\
a_{22} & a_{12} & 1 & . & . & . \\
a_{33} & a_{23} & a_{13} & 1 & . & . \\
. & . & . & . & . & .
\end{vmatrix}.$$

The theorem is clearly a case of that which we have spoken of under Baltzer (1864) as a "so-called theorem." By means of it the determinant

$$\left| C_{x_1,0}\ C_{x_2,1}\ C_{x_3,2} \ldots C_{x_n,n-1} \right|,$$

or

$$\begin{vmatrix} 1 & x_1 & \frac{1}{2}x_1(x_1-1) & \frac{1}{6}x_1(x_1-1)(x_1-2) & \dots \\ 1 & x_2 & \frac{1}{2}x_2(x_2-1) & \frac{1}{6}x_2(x_2-1)(x_2-2) & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & \frac{1}{2}x_n(x_n-1) & \frac{1}{6}x_n(x_n-1)(x_n-2) & \dots \end{vmatrix}$$

is at once seen to be equal to the difference-product divided by $1^{n-1}2^{n-2}3^{n-3}4^{n-4} \dots (n-1)^1$,—a result obtained the same year by Zeipel (see chap. xx.) when studying determinants with elements of the form $C_{x,r}$.

DIETRICH, M. (1865).

[Zur Theorie der Determinanten. *Archiv d. Math. u. Phys.*, xliv. pp. 344–355.]

Dietrich's subject is really the difference-product and related determinants, and the interest of the paper at the outset (pp. 344–348) lies in the order in which a number of previously known theorems are evolved. As an instance we may take the axisymmetric determinant

$$\begin{vmatrix} \frac{2a_1}{a_1+a_1} & \frac{2a_1}{a_1+a_2} & \dots & \frac{2a_1}{a_1+a_n} \\ \frac{2a_2}{a_2+a_1} & \frac{2a_2}{a_2+a_2} & \dots & \frac{2a_2}{a_2+a_n} \\ \dots & \dots & \dots & \dots \\ \frac{2a_n}{a_n+a_1} & \frac{2a_n}{a_n+a_2} & \dots & \frac{2a_n}{a_n+a_n} \end{vmatrix}, \quad \text{or} \quad \begin{vmatrix} r = 1, 2, \dots, n \\ \frac{2a_r}{a_r+a_s} \\ s = 1, 2, \dots, n \end{vmatrix}$$

as Dietrich writes it. This he reaches while on the way to Cauchy's double alternant of 1841 (*Hist.*, i. p. 345), whereas if in Cauchy's result he had put each negative β equal to the corresponding α he would have obtained

$$\begin{vmatrix} r = 1, 2, \dots, n \\ \frac{1}{a_r+a_s} \\ s = 1, 2, \dots, n \end{vmatrix} = \frac{1}{2^n \cdot a_1 a_2 \dots a_n} \left(\frac{\mathrm{D}}{\mathrm{S}}\right)^2,$$

where D is the product of the differences of the a's and S the product of the corresponding sums. There would then only have remained to multiply both sides by $2a_1 \cdot 2a_2 \dots 2a_n$.

The fresh matter consists of deductions from Cauchy's result just mentioned; for example,

$$\left| \begin{matrix} r = 1, 2, \ldots, n \\ \dfrac{1}{(a_s-\beta_1)(a_s-\beta_2)\ldots(a_s-\beta_r)} \\ s = 1, 2, \ldots, n \end{matrix} \right| = (-1)^{\frac{1}{2}n\,n-1)} \frac{|\, a_1^0 a_2^1 \ldots a_n^{n-1} \,|}{\mathrm{R}(a_1, a_2, \ldots, a_n;\ \beta_1, \beta_2, \ldots, \beta_n}$$

where in the denominator on the right we use Borchardt's notation of 1860. The sets of simultaneous linear equations corresponding to these new determinants are also dealt with.

SALMON, G. (1866).

[. . . . Modern Higher Algebra. 2nd edition. xv+296 pp. Dublin.]

In view of subsequent extensions three of Salmon's exercises are worth noting, namely,

$$\begin{vmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \sin\beta & \cos\beta \\ 1 & \sin\gamma & \cos\gamma \end{vmatrix} = 4\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2})\alpha-\gamma),$$

$$\begin{vmatrix} \cos\frac{1}{2}(\alpha-\beta) & \cos\frac{1}{2}(\alpha+\beta) & \sin\frac{1}{2}(\alpha+\beta) \\ \cos\frac{1}{2}(\beta-\gamma) & \cos\frac{1}{2}(\beta+\gamma) & \sin\frac{1}{2}(\beta+\gamma) \\ \cos\frac{1}{2}(\gamma-\alpha) & \cos\frac{1}{2}(\gamma+\alpha) & \sin\frac{1}{2}(\gamma+\alpha) \end{vmatrix} = \tfrac{1}{2} \begin{vmatrix} 1 & \sin\alpha & \cos\alpha \\ 1 & \sin\beta & \cos\beta \\ 1 & \sin\gamma & \cos\gamma \end{vmatrix},$$

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\cos\alpha \\ \sin\beta & \cos\beta & \sin\beta\cos\beta \\ \sin\gamma & \cos\gamma & \sin\gamma\cos\gamma \end{vmatrix} = 2\sin\tfrac{1}{2}(\alpha-\beta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\alpha-\gamma) \cdot\{\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)\}.$$

Attention might have been drawn by Salmon to the common factor

$$\sin\tfrac{1}{2}(\gamma-\beta)\cdot\sin\tfrac{1}{2}(\gamma-\alpha)\cdot\sin\tfrac{1}{2}(\beta-\alpha),$$

and to its analogue in algebraic alternants, namely, the difference-product. Were it not important to bring out this fact, the evaluations

$$\sum^{0}\sin(\alpha-\beta), \quad \tfrac{1}{2}\sum^{0}\sin(\alpha-\beta), \quad \tfrac{1}{2}\sum^{0}\sin 2\alpha\sin(\beta-\gamma),$$

might be considered more easily obtainable and in other ways preferable.

RUBINI, R. (1866).

[Su talune formule relative a determinanti. *Rendic. dell' Accad. delle Sci.* . . . . (Napoli), pp. 109–115; or *Giornale di Mat.*, iv. pp. 187–192.]

Rubini's fundamental result originates in the equatement of the two forms of solution of the set of linear equations (*Hist.*, ii. p. 155)

$$a_1^r x_1 + a_2^r x_2 + \ldots + a_n^r x_n = u^r \Big\}_{r=0}^{r=n-1},$$

namely, the solution

$$\left.\begin{aligned} x_1 &= \frac{f(u)}{(u-a_1)f'(a_1)} \\ x_2 &= \frac{f(u)}{(u-a_2)f'(a_2)} \\ &\ldots\ldots \end{aligned}\right\}, \quad \text{where } f(u) = (u-a_1)(u-a_2)\ldots(u-a_n),$$

and the solution

$$\left.\begin{aligned} x_1 &= \frac{\Delta_{a_1=u}}{\Delta} \\ x_2 &= \frac{\Delta_{a_2=u}}{\Delta} \\ &\ldots\ldots \end{aligned}\right\}, \quad \text{where } \Delta = |\, a_1^0 a_2^1 \ldots a_n^{n-1} \,|.$$

The result which evolves itself quite simply is

$$\Delta^{n-2}\{f(u)\}^{n-1} = (-1)^{\frac{1}{2}n(n-1)} \Delta_{a_1=u}\Delta_{a_2=u}\ldots\Delta_{a_n=u}.$$

The particular case where $u = 0$ is then deduced and the outcome formulated; and lastly, with some trouble (pp. 113–115) both members are expanded in ascending powers of u and the coefficients of like powers equated.

TARLETON, F. A. (1867).

[Question 2367. *Educ. Times*, xix. p. 280; *Math. from Educ. Times*, ix. pp. 69–70.]

Tarleton's theorem is that *the Jacobian of* Σa, $\Sigma a\beta$, $\Sigma a\beta\gamma$, . . . *with respect to* a, β, γ, . . . *is equal to* $\pm\zeta^{\frac{1}{2}}(a, \beta, \gamma, \ldots)$; and the proof is made to depend on changing the determinant into the form

$$\begin{vmatrix} 1 & \Sigma a - a & \Sigma a\beta - a\Sigma a + a^2 & \ldots \\ 1 & \Sigma a - \beta & \Sigma a\beta - \beta\Sigma a + \beta^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

We may note for ourselves (1) that if n be the number of variables, the connecting sign-factor is $(-1)^{\frac{1}{2}n(n-1)}$, the said determinant being equal to

$$\begin{vmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \dots \\ 1 & \beta & \beta^2 & \beta^3 & \dots \\ 1 & \gamma & \gamma^2 & \gamma^3 & \dots \\ 1 & \delta & \delta^2 & \delta^3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} \cdot \begin{vmatrix} 1 & . & . & . & \dots \\ \Sigma\alpha & -1 & . & . & \dots \\ \Sigma\alpha\beta & -\Sigma\alpha & 1 & . & \dots \\ \Sigma\alpha\beta\gamma & -\Sigma\alpha\beta & \Sigma\alpha & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

and (2) that, save for this question of sign, Cauchy's reasoning about $|\alpha^0\beta^1\gamma^2 \dots|$ might with equal effectiveness be applied here (*Hist.*, i. pp. 309–310).

COTTERILL, T. (1867).

[Question 2409 or 4830. *Educ. Times*, xx. p. 42; xxviii. p. 194; solution by J. Hammond in *Math. from Educ. Times*, xxviii. pp. 26–27.]

The theorem set for proof is that *if* (pxy) *be an alternating function of* p, x, y, *and the function be of such a nature that*

$$(\text{pxy})(\text{pzt}) + (\text{pyz})(\text{pxt}) + (\text{pzx})(\text{pyt}) = 0,$$

$$\textit{or, say,} \quad \phi(\text{p, xyzt}) = 0,$$

whatever p, x, y, z, t *may be, then*

$$(\text{acd})(\text{aef})(\text{bde})(\text{bcf}) - (\text{bcd})(\text{bef})(\text{ade})(\text{acf})$$

is an alternating function of a, b, c, d, e, f.

Neglecting Hammond's procedure and taking one of the fifteen pairs of letters that may be interchanged, say b and e, we see that the new function will be alternating with respect to the pair if

$$(acd)(aef)(bde)(bcf) - (bcd)(bef)(ade)(acf)$$
$$= -(acd)(abf)(edb)(ecf) + (ecd)(ebf)(adb)(acf),$$

that is, if

$$(acd)(bde)\{(aef)(bcf) + (abf)(cef)\} = (bef)(acf)\{(bcd)(ade) + (cde)(adb)\},$$

and therefore if

$$(aef)(bcf) + (abf)(cef) = (bef)(acf)$$
and
$$(bcd)(ade) + (cde)(adb) = (acd)(bde)$$

that is, if

$$\phi(f, abce) = 0 = \phi(d, abce):$$

and this by hypothesis is the case. In the same way it is readily found that the fourteen other interchanges necessitate only two other pairs of conditions, namely,

$$\phi(a, cdef) = 0 = \phi(b, cdef),$$
$$\phi(c, abdf) = 0 = \phi(e, adbf).$$

There is thus a considerable surplusage of hypotheses in the proposition as enunciated.

CLEBSCH, A., AND GORDAN, P. (1868).

[Ueber biternäre Formen mit contragredienten Variabeln. *Math. Annalen*, i. pp. 359–400.]

The procedure followed (pp. 390–391) in order to obtain the quotient of $|a^0b^hc^k|$ by $|a^0b^1c^2|$ is exactly that laid down by Jacobi in 1841, the result being of course

$$(a, b, c)^{h-1}(a, b, c)^{k-2} - (a, b, c)^{k-1}(a, b, c)^{h-2},$$

as we also know from Trudi (1864).

MERRIFIELD, C. W. (1870).

[A verification of the solution of a particular system of simultaneous equations. *O. C. and D. Messenger of Math.*, v. pp. 206–207.]

The system referred to is Lagrange's of 1775 (*Hist.*, ii. p. 155), and the procedure is fully indicated by the word 'verification.'

EUGENIO, V. (1871): FIORE, V. (1871).

[Quistioni. *Giornale di Mat.*, ix. p. 67.]

[Dimostrazioni d'una trasformazione di determinanti. *Giornale di Mat.*, x. p. 170.]

The questions concern the identities dealt with by Prouhet (1856), and well known prior to that date.

NAEGELSBACH, H. (1871).

[Ueber eine Classe symmetrischen Functionen. Sch.-Prog. 42 pp. Zweibrücken.]

This is an elaborate investigation of the quotient of $|\, a_1^{\alpha} a_2^{\beta} \ldots a_n^{\omega} \,|$ by the difference-product of the a's, the contents of its forty-two well-filled pages being divided into twenty-two sections with seventy-one numbered results. These last, however, are of very unequal importance, most of them in the early sections being already known although their authors are not named, a considerable sprinkling being trivial, and many, especially in the later sections, belonging strictly to other branches of algebra than determinants.*

The most important theorem is dealt with in §§ 6, 7 (pp. 8–11). Beginning with Bellavitis' result of 1857, namely,

$$\begin{vmatrix} a_1^0 & a_1^1 & \ldots & a_1^{h-1} & a_1^{h+1} & \ldots & a_1^n \\ a_2^0 & a_2^1 & \ldots & a_2^{h-1} & a_2^{h+1} & \ldots & a_2^n \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n^0 & a_n^1 & \ldots & a_n^{h-1} & a_n^{h+1} & \ldots & a_n^n \end{vmatrix} = \mathrm{C}_{n-h} \cdot \xi^{\frac{1}{2}},$$

where C_{n-h} is a function of the a's defined by the identical equation

$$(x-a_1)(x-a_2) \ldots (x-a_n) \equiv x^n - \mathrm{C}_1 x^{n-1} + \mathrm{C}_2 x^{n-2} - \ldots$$

and $\xi^{\frac{1}{2}}$ denotes the difference-product of the a's, he continues the use of Bellavitis' procedure. That is to say, by annexing an additional row and column he forms the determinant

$$\begin{vmatrix} a_1^0 & a_1^1 & \ldots & a_1^{h-1} & a_1^{h+1} & \ldots & a_1^n & a_1^{n+1} \\ a_2^0 & a_2^1 & \ldots & a_2^{h-1} & a_2^{h+1} & \ldots & a_2^n & a_2^{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n^0 & a_n^1 & \ldots & a_n^{h-1} & a_n^{h+1} & \ldots & a_n^n & a_n^{n+1} \\ x^0 & x^1 & \ldots & x^{h-1} & x^{h+1} & \ldots & x^n & x^{n+1} \end{vmatrix},$$

which from what precedes is known to be

$$= \mathrm{C}_{n-h+1}(a_1, a_2, \ldots, a_n, x) \cdot \xi^{\frac{1}{2}}(a_1, a_2, \ldots, a_n, x),$$

* The main results are clearly stated on pp. 149–152 of his paper "Studien zu Fürstenau's neuer Methode . . . " in the *Archiv d. Math. u. Phys.*, lix. pp. 147–192.

and therefore

$$= \{C_{n-h+1}(a_1, \ldots, a_n) + xC_{n-h}(a_1, \ldots, a_n)\}$$
$$\cdot \{(x-a_1)(x-a_2) \ldots (x-a_n) \cdot \xi^{\frac{1}{2}}(a_1, \ldots, a_n)\}$$
$$= (C_{n-h+1} + xC_{n-h}) \cdot (x^n - C_1 x^{n-1} + C_2 x^{n-2} - \ldots) \cdot \xi^{\frac{1}{2}}.$$

He then equates the coefficients of x^j in the two members of this, obtaining

$$(-1)^{n-j+1} \begin{vmatrix} a_1^0 & a_1^1 & \ldots & a_1^{h-1} & a_1^{h+1} & \ldots & a_1^{j-1} & a_1^{j+1} & \ldots & a_1^{n+1} \\ a_2^0 & a_2^1 & \ldots & a_2^{h-1} & a_2^{h+1} & \ldots & a_2^{j-1} & a_2^{j+1} & \ldots & a_2^{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n^0 & a_n^1 & \ldots & a_n^{h-1} & a_n^{h+1} & \ldots & a_n^{j-1} & a_n^{j+1} & \ldots & a_n^{n+1} \end{vmatrix}$$
$$= (-1)^{n-j} \{C_{n-h+1} C_{n-j} - C_{n-h} C_{n-j+1}\} \xi^{\frac{1}{2}},$$

and thence

$$\begin{vmatrix} a_1^0 & a_2^1 & \ldots & a_h^{h-1} & a_{h+1}^{h+1} & \ldots & a_{j-1}^{j-1} & a_j^{j+1} & \ldots & a_n^{n+1} \end{vmatrix} = \begin{vmatrix} C_{n+1-j} & C_{n-j} \\ C_{n+1-h} & C_{n-h} \end{vmatrix} \xi^{\frac{1}{2}}.$$

Following the same procedure with this second result as a basis, he next obtains

$$\begin{vmatrix} a_1^0 & a_2^1 & \ldots & a_h^{h-1} & a_{h+1}^{h+1} & \ldots & a_{j-1}^{j-1} & a_j^{j+1} & \ldots & a_{k-2}^{k-1} & a_{k-1}^{k+1} & \ldots & a_n^{n+2} \end{vmatrix}$$
$$= \begin{vmatrix} C_{n+2-k} & C_{n+1-k} & C_{n-k} \\ C_{n+2-j} & C_{n+1-j} & C_{n-j} \\ C_{n+2-h} & C_{n+1-h} & C_{n-h} \end{vmatrix} \xi^{\frac{1}{2}},$$

and thereafter in similar fashion

$$\begin{vmatrix} a_1^0 & a_2^1 & \ldots & a_h^{h-1} & a_{h+1}^{h+1} & \ldots & a_{j-1}^{j-1} & a_j^{j+1} & \ldots & a_{k-2}^{k-1} & a_{k-1}^{k+1} & \ldots & a_{l-3}^{l-1} & a_{l-2}^{l+1} & \ldots & a_n^{n+3} \end{vmatrix}$$
$$= \begin{vmatrix} C_{n+3-l} & C_{n+2-l} & C_{n+1-l} & C_{n-l} \\ C_{n+3-k} & C_{n+2-k} & C_{n+1-k} & C_{n-k} \\ C_{n+3-j} & C_{n+2-j} & C_{n+1-j} & C_{n-j} \\ C_{n+3-h} & C_{n+2-h} & C_{n+1-h} & C_{n-h} \end{vmatrix}.$$

The general theorem which Naegelsbach thus arrives at * is perhaps best enunciated in the form of a 'rule,' namely, *The given*

* Naegelsbach burdens himself with a troublesome sign-factor which, by two simple changes, we have rendered unnecessary.

alternant being $|\,a_0^1\,a_2^\beta\,a_3^\gamma\,\ldots\,a_n^\omega\,|$, *where* $\beta, \gamma, \ldots, \omega$ *are integers in ascending order of magnitude, write down in descending order the integral indices which do not appear in the series* $0, \beta, \gamma, \ldots, \omega$: *subtract each of the said indices from* ω, *taking the remainders for the suffixes of* C*'s for the first column of a determinant, and making the* C *suffixes of every other element* 1 *less than that of the corresponding* C *in the immediately preceding column. This determinant of* C*'s is equal to the quotient of* $|\,a_1^0\,a_2^\beta\,a_3^\gamma\,\ldots\,a_n^\omega\,|$ *by the difference-product of the* a*'s.* For example, the missing indices in $|\,a^0b^2c^5d^6\,|$ being 4, 3, 1, and the defects of these from 6 being 2, 3, 5, we have

$$\frac{|\,a^0b^2c^5d^6\,|}{|\,a^0b^1c^2d^3\,|} = \begin{vmatrix} C_2 & C_1 & C_0 \\ C_3 & C_2 & C_1 \\ C_5 & C_4 & C_3 \end{vmatrix} = \begin{vmatrix} \Sigma ab & \Sigma a & 1 \\ \Sigma abc & \Sigma ab & \Sigma a \\ . & abcd & \Sigma abc \end{vmatrix}.$$

A new departure of Naegelsbach's is to use Trudi's notation $(a, b, c, \ldots)^r$ in cases where the integer r is negative, the meaning being determined by assuming Cauchy's identity

$$(a_1, a_2, \ldots, a_n)^r = \frac{|\,a_1^0\ a_2^1\ \ldots\ a_{n-1}^{n-2}\ a_n^{n-1+r}\,|}{|\,a_1^0\ a_2^1\ \ldots\ a_{n-1}^{n-2}\ a_n^{n-1}\,|}$$

to hold for negative integral values of r. This manifestly entails that

$$(a_1, a_2, \ldots, a_n)^r = 0 \quad \text{for} \quad r = -1, -2, \ldots, -n+1,$$

and that when $r = -n, -n-1, \ldots$ non-zero values again appear. In the case of none of these negative values of r has the aleph-function definition

$$(a_1, a_2, \ldots, a_n)^r = \sum_{h_1+\ldots+h_n=r} a_1^{h_1} a_2^{h_2} \ldots a_n^{h_n}$$

any meaning. When, however, $r = -n, -n-1, \ldots$ Naegelsbach shows that the similar definition

$$(a_1, a_2, \ldots, a_n)^{-r} = \frac{(-1)^{n-1}}{a_1\ a_2\ \ldots\ a_n} \cdot \sum_{h_1+\ldots+h_n=r-n} a_1^{-h_1} a_2^{-h_2} \ldots a_n^{-h_n}$$

is in keeping with the above assumption. For example, when $n = 3$ and $r = 4$, the said assumption gives

$$(a,\ b,\ c)^{-4} = \begin{vmatrix} 1 & a & a^{-2} \\ 1 & b & b^{-2} \\ 1 & c & c^{-2} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \frac{1}{a^2b^2c^2} \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \frac{1}{a^2b^2c^2}(bc+ca+ab)$$

$$= \frac{1}{abc}(a^{-1}+b^{-1}+c^{-1}),$$

which clearly is the same as

$$\frac{(-1)^{3-1}}{abc}\sum_{h+k+l=1} a^{-h}b^{-k}c^{-l}.$$

His next most important theorem is reached in § 13 (p. 17), and concerns a determinant whose elements are aleph functions, namely,

$$\begin{vmatrix} [\kappa_1] & [\kappa_2] & \cdots & [\kappa_n] \\ [\kappa_1-\rho_2] & [\kappa_2-\rho_2] & \cdots & [\kappa_n-\rho_2] \\ [\kappa_1-\rho_3] & [\kappa_2-\rho_3] & \cdots & [\kappa_n-\rho_3] \\ \cdots & \cdots & \cdots & \cdots \\ [\kappa_1-\rho_n] & [\kappa_2-\rho_n] & \cdots & [\kappa_n-\rho_n] \end{vmatrix},$$

in which $[\kappa]$ stands for $(\alpha_1, \alpha_2, \ldots, \alpha_n)^\kappa$. By means of a previously established result each element is expressed as a sum of n products. The determinant is thus resolved into two determinants, one of which is complicated but is changeable into the same simple form as the other, the final result being

$$\begin{vmatrix} [\kappa_1] & [\kappa_2] & \cdots & [\kappa_n] \\ [\kappa_1-1] & [\kappa_2-1] & \cdots & [\kappa_n-1] \\ \cdots & \cdots & \cdots & \cdots \\ [\kappa_1-n+1] & [\kappa_2-n+1] & \cdots & [\kappa_n-n+1] \end{vmatrix}$$

$$\cdot \begin{vmatrix} [-\rho_n+n-1] & \cdots & [-\rho_2+n-1] & [n-1] \\ [-\rho_n+n-2] & \cdots & [-\rho_2+n-2] & [n-2] \\ \cdots & \cdots & \cdots & \cdots \\ [-\rho_n] & \cdots & [-\rho_2] & [0] \end{vmatrix}.$$

The common form is that which we have seen obtained above by Trudi as an equivalent for the quotient of an alternant by the difference-product of the variables, and we thus conclude that the given determinant must be equal to

$$\frac{|\ a_1^{\kappa_1}\ a_2^{\kappa_2}\ \ldots\ a_n^{\kappa_n}\ |}{|\ a_1^0\ a_2^1\ \ldots\ a_n^{n-1}\ |} \cdot \frac{|\ a_1^{-\rho_n+n-1}\ a_2^{-\rho_{n-1}+n-1}\ \ldots\ a_n^{n-1}\ |}{|\ a_1^0\ a_2^1\ \ldots\ a_n^{n-1}\ |}.$$

MERTENS, F. (1872).

[Auszug aus einem Schreiben .... *Crelle's Journ.*, lxxv. p. 264.]

The matter here dealt with is Borchardt's generating function of 1855 (*Hist.*, ii. pp. 173–175). In its place Mertens suggests

$$\frac{t_1^{n-1}t_2^{n-2}\ \ldots\ t_{n-1}^1 \cdot |\ t_1^0 t_2^1\ \ldots\ t_n^{n-1}\ | \cdot \mathrm{V}(t_1,\ t_2,\ \ldots,\ t_n)}{f(t_1) \cdot f(t_2)\ \ldots\ f(t_n)},$$

where $f(t) = (t-a_1)(t-a_2)\ \ldots\ (t-a_n)$, asserting that the coefficient of $(t_1t_2\ \ldots\ t_n)^{-1}$ in this is the symmetric function $\mathrm{V}(a_1,\ a_2,\ \ldots,\ a_n)$. As Borchardt's function had undeservedly withdrawn attention from Jacobi's result of 1841 (*Hist.*, i. pp. 336–339), namely, that the symmetric function

$$\sum \frac{\phi\,(a_1,\ a_2,\ \ldots,\ a_n)}{|\ a_1^0 a_2^1\ \ldots\ a_n^{n-1}\ |}$$

is the coefficient of $(t_1t_2\ \ldots\ t_n)^{-1}$ in

$$(-1)^{\frac{1}{2}n(n-1)} \cdot \frac{|\ t_1^0 t_2^1\ \ldots\ t_n^{n-1}\ | \cdot \phi(t_1,\ t_2,\ \ldots,\ t_n)}{f(t_1) \cdot f(t_2)\ \ldots\ f(t_n)}.$$

Mertens' closely analogous result has a double importance. In a sense Mertens' theorem is more general than Jacobi's: for the integral symmetric function, V, in the former is *any* such function, whereas in the latter it is one which arises from the division of an alternating function by the corresponding difference-product. Taking advantage of this and of the fact that

$$\sum \frac{\phi(a_1,\ a_2,\ \ldots,\ a_n)}{|\ a_1^0 a_2^1 \ldots a_n^{n-1}\ |} = \frac{\Sigma \pm \phi(a_1,\ a_2,\ \ldots,\ a_n)}{|\ a_1^0 a_2^1 \ldots a_n^{n-1}\ |},$$

we obtain by substituting the latter expression for $V(a_1, a_2, \ldots, a_n)$ an alternative form of Jacobi's function, namely,

$$\frac{t_1^{n-1}t_2^{n-2}\ldots t_{n-1}^{1}\cdot\sum\pm\phi(a_1, a_2, \ldots, a_n)}{f(t_1)\cdot f(t_2)\ldots f(t_n)}.$$

To understand the exact bearing of these forms on one another, it is important to consider a special case, say the case where

$$n = 2, \quad V = a_1^2+a_2^2.$$

Mertens' generating function then is

$$\frac{t_1\cdot(t_2-t_1)\cdot(t_1^2+t_2^2)}{(t_1-a_1)(t_1-a_2)\cdot(t_2-a_1)(t_2-a_2)},$$

which is equal to

$$(-t_1^4+t_1^3t_2-t_1^2t_2^2+t_1t_2^3)\cdot\left\{\frac{1}{t_1^2t_2^2}+\frac{\aleph_1}{t_1^3t_2^2}+\frac{\aleph_1}{t_1^2t_2^3}+\frac{\aleph_1^2}{t_1^3t_2^3}+\frac{\aleph_2}{t_1^4t_2^2}+\frac{\aleph_2}{t_1^2t_2^4}+\ldots\right\},$$

where the numerators are the aleph functions of a_1, a_2. In this the coefficient of $(t_1t_2)^{-1}$ is

$$\aleph_2+\aleph_2-\aleph_1^2,$$

$$i.e. \quad 2(a_1^2+a_1a_2+a_2^2)-(a_1+a_2)^2,$$

as it should be. Jacobi's theorem, on the other hand, we can only utilize when we so choose his asymmetric ϕ that

$$\sum\frac{\phi(a_1, a_2)}{a_2-a_1}\equiv a_1^2+a_2^2.$$

Doing this by putting $\phi(a_1, a_2)\equiv a_2^3-a_2^2a_1$ we obtain, in place of Mertens' numerator,

$$(-1)\cdot(t_2-t_1)\cdot(t_2^3-t_2^2t_1),$$

$$i.e. \quad -t_2^4+2t_2^3t_1-t_2^2t_1^2,$$

and therefore, for the coefficient of $(t_1t_2)^{-1}$,

$$2\aleph_2-\aleph_1^2,$$

as before. In the third place, Borchardt's generating function in its original form is

$$\frac{1}{t_1-a_1}\cdot\frac{1}{t_2-a_2}+\frac{1}{t_1-a_2}\cdot\frac{1}{t_2-a_1},$$

which is found to be

$$\frac{2}{t_1t_2}+(a_1+a_2)\left(\frac{1}{t_1^2t_2}+\frac{1}{t_1t_2^2}\right)+(a_1^2+a_1^2)\left(\frac{1}{t_1^3t_2}+\frac{1}{t_1t_2^3}\right)+\frac{2a_1a_2}{t_1^2t_2^2}+\ldots,$$

where $a_1^2+a_2^2$ is the coefficient of $(t_1t_2)^{-3}(t_1^2+t_2^2)$. Lastly, Borchardt's derived form * is

$$\frac{\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}Z}{Z},$$

where $Z=(t_2-t_1)/(t_1^2-c_1t_1+c_2)(t_2^2-c_1t_2+c_2)$. This, when the tiresome operations indicated are performed, becomes

$$\frac{2t_1t_2-c_1(t_1+t_2)+2c_2}{(t_1^2-c_1t_1+c_2)(t_2^2-c_1t_2+c_2)},$$

which is exactly what is obtained by performing the simple addition specified in the original form. Also, since its denominator is identical with the denominator in Mertens' form, we readily see that the cofactor of either term of $(t_1t_2)^{-3}(t_1^2+t_2^2)$ is

$$2\aleph_2-c_1\aleph_1,$$

as it ought to be.

GORDAN, P. (1873).

[Ueber den grössten gemeinsamen Factor. *Math. Annalen*, vii. pp. 433–448.]

Gordan here makes an effective application of Brill's theorem of 1871 regarding two arrays so related to one another that the principal minors of the one are proportional to those of the other. A simple example or two will make the general procedure clear.

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* The following writings concern this form—

1856. Cayley, A. A memoir on the symmetric functions . . . . *Philos. Transac. R. Soc.* (London), cxlvii. pp. 489–499; or *Collected Math. Papers*, ii. pp. 417–439 (see pp. 421–423).

1864. Baltzer, R. Theorie und Anwendung . . . 2te Aufl. pp. 24–25.

1875. Bruno, F. Faà di. Sur la fonction génératrice de Borchardt. *Crelle's Journ.*, lxxxi. pp. 217–219 (cf. Baltzer, 1864).

1876. Kostka, C. Ueber Borchardt's Function. *Crelle's Journ.*, lxxxii. pp. 212–229.

If α, β, γ be common roots of the two quartic equations

$$\left.\begin{aligned} a_1x^4+a_2x^3+a_3x^2+a_4x+a_5 &= 0 \\ b_1x^4+b_2x^3+b_3x^2+b_4x+b_5 &= 0 \end{aligned}\right\}$$

it at once follows that we have two arrays

$$\begin{array}{ccccc|ccccc} \alpha^4 & \alpha^3 & \alpha^2 & \alpha & 1 & a_1 & a_2 & a_3 & a_4 & a_5 \\ \beta^4 & \beta^3 & \beta^2 & \beta & 1 & b_1 & b_2 & b_3 & b_4 & b_5 \\ \gamma^4 & \gamma^3 & \gamma^2 & \gamma & 1, & & & & & \end{array}$$

satisfying the conditions of the said theorem, and that therefore the ten alternants formable from the array on the left are proportional to

$$|a_4b_5|, \quad -|a_3b_5|, \quad \ldots., \quad |a_1b_2|.$$

Similarly taking four roots all belonging to one quartic equation we should obtain the old result that the five alternants of the array

$$\begin{array}{ccccc} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \beta & \beta^2 & \beta^3 & \beta^4 \\ 1 & \gamma & \gamma^2 & \gamma^3 & \gamma^4 \\ 1 & \delta & \delta^2 & \delta^3 & \delta^4 \end{array}$$

are proportional to

$$a_1, \quad -a_2, \quad a_3, \quad -a_4, \quad a_5,$$

and therefore to

$$1, \quad \Sigma\alpha, \quad \Sigma\alpha\beta, \quad \Sigma\alpha\beta\gamma, \quad \alpha\beta\gamma\delta.$$

In the next place, by taking the same quartic and using it thrice, the two arrays would be

$$\begin{array}{ccccccc|ccccccc} \alpha^6 & \alpha^5 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha & 1 & . & . & a_1 & a_2 & a_3 & a_4 & a_5 \\ \beta^6 & \beta^5 & \beta^4 & \beta^3 & \beta^2 & \beta & 1 & . & a_1 & a_2 & a_3 & a_4 & a_5 & . \\ \gamma^6 & \gamma^5 & \gamma^4 & \gamma^3 & \gamma^2 & \gamma & 1 & a_1 & a_2 & a_3 & a_4 & a_5 & . & ., \\ \delta^6 & \delta^5 & \delta^4 & \delta^3 & \delta^2 & \delta & 1, & & & & & & & \end{array}$$

and we should have, for example,

$$\frac{|\alpha^6\beta^5\gamma^2 1|}{|\alpha^3\beta^2\gamma 1|} = -\begin{vmatrix} a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \\ a_3 & a_4 & . \end{vmatrix} \div \begin{vmatrix} . & . & a_1 \\ . & a_1 & a_2 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{a_2}{a_1} & \frac{a_4}{a_1} \\ \frac{a_2}{a_1} & a_3 & a_5 \\ \frac{a_3}{a_1} & \frac{a_4}{a_1} & . \end{vmatrix} = \begin{vmatrix} 1 & -\Sigma\alpha & -\Sigma\alpha\beta\gamma \\ -\Sigma\alpha & \Sigma\alpha\beta & \alpha\beta\gamma\delta \\ \Sigma\alpha\beta & -\Sigma\alpha\beta\gamma & . \end{vmatrix}$$

in accordance with Naegelsbach's theorem. Of much less note is the last of the series of deductions, exemplified by

$$\begin{Vmatrix} 1 & -x & . & . & . \\ . & 1 & -x & . & . \\ . & . & 1 & -x & . \\ . & . & . & 1 & -x \end{Vmatrix} = 1,\ -x,\ x^2,\ -x^3,\ x^4.$$

Thenceforward (pp. 436–448) Gordan occupies himself with the main subject of his paper, showing first how Sylvester's bigradient may be transformed into Euler's product of differences (*Hist.*, ii. pp. 369–370, 374–375) and passing on to the analogous trigradient arising from three given equations in x.

WEIHRAUCH, K. (1874).

[Zur Determinantenlehre. *Zeitschrift f. Math. u. Phys.*, xix. pp. 354–360.]

The second section (pp. 359–360) establishes the determinant form for the difference-product by a gradational proof, $(n-1)^{\text{th}}$ order to n^{th}

MALET, J. C. (1874).

[On certain symmetric functions of the roots of an algebraic equation. *Transac. R. Irish Acad.* (Dublin), xxv. pp. 337–342.]

Malet's result is essentially the same as Naegelsbach's of 1871, and is obtained in the same way. He evades the difficulty of the sign-factor.

GARBIERI, G. (1874).

[I Determinanti, con . . . . xiii+267 pp. Bologna.]

Garbieri establishes the already known (1864) identity

$$\begin{vmatrix} w_1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ w_2 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ w_n & a_n & a_n^2 & \dots & a_n^{n-1} \end{vmatrix}$$

$$= (-1)^{n-1} \left| a_1^1 a_2^2 \dots a_n^n \right| \cdot \left\{ \frac{w_1}{a_1 f'(a_1)} + \frac{w_2}{a_2 f'(a_2)} + \dots + \frac{w_n}{a_n f'(a_n)} \right\},$$

where $$f(x) = (x-a_1)(x-a_2) \dots (x-a_n).$$

A simple mode of verification consists in comparing the cofactor of any one of the w's on the one side with its cofactor on the other.

A similar expansion is obtained for the determinant got by substituting the w's in any other column of $|\, a_1^0 a_2^1 \ldots a_n^{n-1} \,|$ than the first.

BELTRAMI, E. (1874).

[I Determinanti, con . . . . , per G. Garbieri. pp. 169–171.]

Beltrami carries us a step further than the so-called theorem noted by Baltzer (1864). Instead of multiplying $|\, a_{00} a_{11} \ldots a_{n-1,\,n-1} \,|$ by $|\, a_1^0 a_2^1 \ldots a_n^{n-1} \,|$ he (or Garbieri) multiplies

$$\begin{vmatrix} a_{00} & a_{01} & \ldots & a_{0n} \\ a_{10} & a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n-1,0} & a_{n-1,1} & \ldots & a_{n-1,n} \\ c_n & c_{n-1} & \ldots & c_0 \end{vmatrix} \text{ by } \begin{vmatrix} 1 & a_1 & a_1^2 & \ldots & a_1^n \\ 1 & a_2 & a_2^2 & \ldots & a_2^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a_n & a_n^2 & \ldots & a_n^n \\ 1 & x & x^2 & \ldots & x^n \end{vmatrix},$$

where the c's are defined by the identical equation

$$(x-a_1)(x-a_2) \ldots (x-a_n) \equiv c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + \ldots;$$

and denoting

$$\begin{aligned} &a_{00} + a_{01}\xi + a_{02}\xi^2 + \ldots + a_{0n}\xi^n \quad \text{by } f_1(\xi), \\ &a_{10} + a_{11}\xi + a_{12}\xi^2 + \ldots + a_{1n}\xi^n \quad \text{by } f_2(\xi), \\ &\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\ &a_{n-1,0} + a_{n-1,1}\xi + a_{n-1,2}\xi^2 + \ldots + a_{n-1,n}\xi^n \quad \text{by } f_n(\xi), \end{aligned}$$

he obtains for his product

$$\begin{vmatrix} f_1(a_1) & f_1(a_2) & \ldots & f_1(a_n) & f_1(x) \\ f_2(a_1) & f_2(a_2) & \ldots & f_2(a_n) & f_2(x) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f_n(a_1) & f_n(a_2) & \ldots & f_n(a_n) & f_n(x) \\ 0 & 0 & \ldots & 0 & c_0 x^n + c_1 x^{n-1} + \ldots \end{vmatrix}.$$

As, however, the multiplier is $\xi^{\frac{1}{2}}(a_1, a_2, \ldots a_n, x)$, and therefore

$$\begin{aligned} &= (x-a_n)(x-a_{n-1}) \ldots (x-a_2)(x-a_1) \cdot \xi^{\frac{1}{2}}(a_1, a_2, \ldots, a_n) \\ &= (x^n + c_1 x^{n-1} + \ldots + c_n) \cdot \xi^{\frac{1}{2}}(a_1, a_2, \ldots, a_n), \end{aligned}$$

division at once gives us

$$\frac{|f_1(a_1)\cdot f_2(a_2)\ldots f_n(a_n)|}{\zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_n)} = \begin{vmatrix} a_{00} & a_{01} & \ldots & a_{0n} \\ a_{10} & a_{11} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n-1,0} & a_{n-1,1} & \ldots & a_{n-1,n} \\ c_n & c_{n-1} & \ldots & c_0 \end{vmatrix},$$

a result which we may advantageously formulate for ourselves as follows: *If the* n*-line alternant constructed from* n *variables,* a_1, $a_2, \ldots, a_n$ *and* n *univarial functions of the* n^{th} *degree,* $f_1, f_2, \ldots, f_n$, *be divided by the difference-product of the said variables, the quotient is expressible as a determinant of the* $(n+1)^{th}$ *order, whose first* n *rows consist of the coefficients of the functions and whose last row consists of the coefficients of the powers of* x *in the expansion of*

$$(x-a_1)(x-a_2)\ldots(x-a_n),$$

the order of writing the coefficients in every case being that corresponding to the ascending order of the powers.

For example, the given functions being

$$a_0+a_1x+a_2x^2+a_3x^3, \quad b_0+b_1x+b_2x^2+b_3x^3, \quad c_0+c_1x+c_2x^2,$$

and the given variables λ, μ, ν, we have

$$\frac{\begin{vmatrix} a_0+a_1\lambda+a_2\lambda^2+a_3\lambda^3 & a_0+a_1\mu+a_2\mu^2+a_3\mu^3 & a_0+a_1\nu+a_2\nu^2+a_3\nu^3 \\ b_0+b_1\lambda+b_2\lambda^2+b_3\lambda^3 & b_0+b_1\mu+b_2\mu^2+b_3\mu^3 & b_0+b_1\nu+b_2\nu^2+b_3\nu^3 \\ c_0+c_1\lambda+c_2\lambda^2 & c_0+c_1\mu+c_2\mu^2 & c_0+c_1\nu+c_2\nu^2 \end{vmatrix}}{(\nu-\mu)(\nu-\lambda)(\mu-\lambda)}$$

$$= \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ -\lambda\mu\nu & \Sigma\lambda\mu & -\Sigma\lambda & 1 \end{vmatrix}.$$

KOSTKA, C. (1875).

[Ueber die Bestimmung von symmetrischen Functionen der Wurzeln einer algebraischen Gleichung durch deren Coefficienten. *Crelle's Journ.*, lxxxi. pp. 281–289.]

Kostka, like Malet in the preceding year, discovers for himself Naegelsbach's theorem of 1871, establishing it in a very simple

and attractive way, which we may illustrate by the example already used, namely, $|\,a^0b^2c^5d^6\,|$. Writing this in the form

$$-\begin{vmatrix} a^0 & a^1 & a^2 & a^3 & a^4 & a^5 & a^6 \\ b^0 & b^1 & b^2 & b^3 & b^4 & b^5 & b^6 \\ c^0 & c^1 & c^2 & c^3 & c^4 & c^5 & c^6 \\ d^0 & d^1 & d^2 & d^3 & d^4 & d^5 & d^6 \\ . & 1 & . & . & . & . & . \\ . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . \end{vmatrix},$$

he performs the operations

$$\begin{aligned} &\mathrm{col}_7 - C_1\,\mathrm{col}_6 + C_2\,\mathrm{col}_5 - C_3\,\mathrm{col}_4, \\ &\mathrm{col}_6 - C_1\,\mathrm{col}_5 + C_2\,\mathrm{col}_4 - C_3\,\mathrm{col}_3, \\ &\mathrm{col}_5 - C_1\,\mathrm{col}_4 + C_2\,\mathrm{col}_3 - C_3\,\mathrm{col}_2, \end{aligned}$$

which give the result

$$-\begin{vmatrix} a^0 & a^1 & a^2 & a^3 & . & . & . \\ b^0 & b^1 & b^2 & b^3 & . & . & . \\ c^0 & c^1 & c^2 & c^3 & . & . & . \\ d^0 & d^1 & d^2 & d^3 & . & . & . \\ . & 1 & . & . & -C_3 & C_4 & . \\ . & . & . & 1 & -C_1 & C_2 & -C_3 \\ . & . & . & . & 1 & -C_1 & C_2 \end{vmatrix},$$

that is,

$$|\,a^0b^1c^2d^3\,| \cdot \begin{vmatrix} C_3 & C_4 & . \\ C_1 & C_2 & C_3 \\ 1 & C_1 & C_2 \end{vmatrix},$$

and thence, as before,

$$\frac{|\,a^0b^2c^5d^6\,|}{|\,a^0b^1c^2d^3\,|} = \begin{vmatrix} C_2 & C_1 & 1 \\ C_3 & C_2 & C_1 \\ . & C_4 & C_3 \end{vmatrix}.$$

He also makes the first rediscovery of Schweins' multiplication-theorem of 1825, and then takes the fresh step of using it to express any symmetric function as a sum of determinants. Thus, since Schweins' theorem gives

$$\begin{aligned} &|\,a^0b^1c^2d^3\,| \cdot (a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2) \\ &= |\,a^1b^2c^3d^4\,| - |\,a^0b^2c^3d^5\,| + |\,a^0b^1c^4d^5\,|, \end{aligned}$$

we have

$$\sum a^2b^2 = \frac{|\,a^1b^2c^3d^4\,|}{|\,a^0b^1c^2d^3\,|} - \frac{|\,a^0b^2c^3d^5\,|}{|\,a^0b^1c^2d^3\,|} + \frac{|\,a^0b^1c^4d^5\,|}{|\,a^0b^1c^2d^3\,|},$$

and therefore, by Naegelsbach's theorem,

$$\sum a^2b^2 = C_4 - \begin{vmatrix} C_1 & 1 \\ C_4 & C_3 \end{vmatrix} + \begin{vmatrix} C_2 & C_1 \\ C_3 & C_2 \end{vmatrix}.$$

The accuracy of this is readily verified, for the right-hand member

$$\begin{aligned} &= 2C_4 - 2C_1C_3 + C_2^2 \\ &= 2abcd - 2(\Sigma a^2bc + 4abcd) + (\Sigma a^2b^2 + 2\Sigma a^2bc + 6abcd) \\ &= \Sigma a^2b^2. \end{aligned}$$

The rest of the interesting paper deals with certain special cases and applications.

SCHRÖDER, E. (1875).

[Ueber v. Staudt's Rechnung mit Würfen und verwandte Processe. § 6. *Math. Annalen*, x. pp. 297–301.]

Schröder's result is implicitly included in Beltrami's, from which it can be deduced by putting one of the given variables equal to 0. Thus, putting $\nu = 0$ in the example given under Beltrami, we obtain

$$\begin{vmatrix} a_1+a_2\lambda+a_3\lambda^2 & a_1+a_2\mu+a_3\mu^2 & a^0 \\ b_1+b_2\lambda+b_3\lambda^2 & b_1+b_2\mu+b_3\mu^2 & b^0 \\ c_1+c_2\lambda & c_1+c_2\mu & c^0 \end{vmatrix} \div (\mu-\lambda)$$

$$= \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & . \\ . & \lambda\mu & -\lambda-\mu & 1 \end{vmatrix},$$

which is an example of Schröder's.

VALERIANO, V. (1876).

[Alcune notevoli applicazioni della induzione matematica. § 6. *Giornale di Mat.*, xv. pp. 41–45.]

The subject is the factorizing of

$$|\,a_1^0a_2^1 \ldots a_n^{n-1}\,| \quad \text{and} \quad |\,a_1^0a_2^1 \ldots a_{n-1}^{n-2}a_n^n\,|.$$

Brioschi (1854) might have been referred to.

BRUNO, F. FAÀ DI (1875, 1876).

[Sur la fonction génératrice de Borchardt. *Crelle's Journ.*, lxxxi. pp. 217–219.]

[Théorie des Formes Binaires. xvi+320 pp. Turin.]

For the function in question, dealt with above under Mertens, (1872), Bruno obtains a fresh expression,

$$\frac{1}{f(t_1)\ \ldots\ f(t_n)\cdot\xi^{\frac{1}{2}}(t_1,\ \ldots,\ t_n)} \cdot \begin{vmatrix} f'(t_1) & f(t_1)-t_1f'(t_1) & \ldots & (n-1)t_1^{n-2}f(t_1)-t_1^{n-1}f'(t_1) \\ f'(t_2) & f(t_2)-t_2f'(t_2) & \ldots & (n-1)t_2^{n-2}f(t_2)-t_2^{n-1}f'(t_2) \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n .$$

To illustrate its superiority over Borchardt's, he applies both to the case where the given equation is a quadratic, pointing out, as Cayley had done in 1856, that Borchardt's form was 'not a very convenient one for calculation.' He also applies his own to the case of the cubic, the result being an interesting exercise in the use of alternants.

In his book the section (pp. 38–50) devoted to the subject contains a verification by Cayley of Borchardt's equality (*Hist.*, ii. p. 174)

$$D = T \cdot \Delta.$$

FERRERS, N. M. (1876): MALET, J. C. (1876): HARRIS, H. W. (1877).

[An Elementary Treatise on Trilinear Coordinates. 3rd edition (pp. 59–73, 172, 179). xiv+184 pp. London.]

[Question 5144. *Educ. Times*, xxiv. p. 212; *Math. from Educ. Times*, xxviii. pp. 51–53.]

[Question 5473. *Educ. Times*, xxx. p. 197; lv. p. 437; *Math. from Educ. Times*, (2) iii. p. 82.]

Both of the questions noted concern $|\ a^0b^mc^nd^p\ |$, the one dealing with the multiplication of it by $a+b+c+d$, and therefore answered by Schweins in 1825, and the other giving the three-line com-

pound determinant equal to $|\ a^0b^mc^nd^p\ | \cdot |\ a^0b^1c^2d^3\ |$, and therefore answered by Bazin in 1854 (*Hist.*, ii. p. 207).

All that Ferrers gives is the easily reached result *

$$4\ |\ a^0b^1c^2d^3e^6\ | - 6\ |\ a^0b^1c^2d^4e^5\ | = |\ a^0b^1c^2d^3e^4\ | \cdot \Sigma(a-b)^2.$$

SALMON, G. (1876).

[. . . . Modern Higher Algebra. 3rd edition. xx+318 pp. Dublin.]

A note by Cayley (pp. 290–292) deals with determinants of the form

$$\begin{vmatrix} \phi(x) & \phi(y) & \dots \\ \psi(x) & \psi(y) & \dots \\ \dots & \dots & \dots \end{vmatrix},$$

to which, oddly enough, the name 'rational functional determinants' is given, because $\phi, \psi, \dots$ are taken to be rational integral functions. A method of Jacobi's (*Hist.*, i. p. 341) is illustrated by showing that

$$\begin{vmatrix} a^\alpha & a^\beta & a^\gamma \\ b^\alpha & b^\beta & b^\gamma \\ c^\alpha & c^\beta & c^\gamma \end{vmatrix} = \xi^{\frac{1}{2}}(a,b,c) \begin{vmatrix} (a,b,c)^\alpha & (a,b,c)^{\alpha-1} & (a,b,c)^{\alpha-2} \\ (a,b,c)^\beta & (a,b,c)^{\beta-1} & (a,b,c)^{\beta-2} \\ (a,b,c)^\gamma & (a,b,c)^{\gamma-1} & (a,b,c)^{\gamma-2} \end{vmatrix}.$$

KOSTKA, C. (1876).

[Ueber Borchardt's Function. *Crelle's Journ.*, lxxxii. pp. 212–229.]

The main interest here is connected with the generating function of 1855†: but as the investigation of this involves the frequent use of determinants of the form

$$\begin{vmatrix} c_h & c_k & c_l & \dots \\ c_{h-\delta} & c_{k-\delta_1} & c_{l-\delta_1} & \dots \\ c_{h-\delta_2} & c_{k-\delta_2} & c_{l-\delta_2} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

where c_r is defined by the identity

$$(x-t_1)(x-t_2) \dots (x-t_n) \equiv x^n - c_1x^{n-1} + c_2x^{n-2} - \dots,$$

* *Cambridge Univ. Exam. Papers*, iv. (1875), p. 220.

† In the list above (p. 150), bearing on this subject, should be inserted :—

1860. Meier, F. Sur une nouvelle fonction génératrice des fonctions symétriques. *Bull de l'Acad. . . . de Belgique* (2) x. pp. 608–631.

the development of such determinants in terms of the t's comes in for considerable attention in the second section (pp. 216–224).

FRANKE, E. (1876).

[Ueber den Ausdruck, welcher im Fall gleicher Wurzeln an die Stelle der *Vandermonde*'schen alternirenden Function tritt. *Crelle's Journ.*, lxxxiii. pp. 65–71.]

If the roots of such an equation as

$$x^5 - p_1x^4 + p_2x^3 - p_3x^2 + p_4x - p_5 = 0$$

be all unequal, we have already seen how a set of five equations is obtained for the determination of the p's in terms of the roots (*Hist.*, ii. pp. 168–169, 175–176, 181–182). When the roots are not all unequal, differentiation is necessary in order to obtain the required set: for example, if the roots be

$$a,\ a,\ b,\ b,\ b,$$

the set is

$$\left.\begin{array}{rrrrrl} a^5 - p_1a^4 & + p_2a^3 & - p_3a^2 & + p_4a - p_5 & = & 0 \\ 5a^4 - p_1 \cdot 4a^3 & + p_2 \cdot 3a^2 & - p_3 \cdot 2a & + p_4 & = & 0 \\ b^5 - p_1b^4 & + p_2b^3 & - p_3b^2 & + p_4b - p_5 & = & 0 \\ 5b^4 - p_1 \cdot 4b^3 & + p_2 \cdot 3b^2 & - p_3 \cdot 2b & + p_4 & = & 0 \\ 20b^3 - p_1 \cdot 12b^2 & + p_2 \cdot 6b & - p_3 \cdot 2 & & = & 0 \end{array}\right\}.$$

Now Franke's problem is in effect the solution of such a set when the given equation has

$$\begin{array}{l} \alpha \text{ roots each equal to } a, \\ \beta \text{ roots each equal to } b, \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \end{array}$$

and $\alpha + \beta + \ldots$ equal to n. He successfully plods his way through the maze, and formulates at some length the result. The case where *all* the roots are alike is used as a test, the common denominator of the p's then degenerating into

$$(n-1)!\,(n-2)!\ldots 2!\,1! \quad \text{or} \quad (n-1)^1(n-2)^2\ldots 2^{n-2}1^{n-1},$$

as we know it ought to do.

DOSTOR, G. (1877).

[Éléments de la Theorie des Déterminants, avec. . . . xxxi+352 pp. Paris.]

The following result is established by Dostor (pp. 87–88) at considerable length,

$$\begin{vmatrix} a^3 & (a+x)^3 & (2a+x)^3 \\ b^3 & (b+x)^3 & (2b+x)^3 \\ c^3 & (c+x)^3 & (2c+x)^3 \end{vmatrix} = 3x^3 \cdot \xi^{\frac{1}{2}}(a,\ b,\ c) \cdot \{x^2\Sigma a + 3x\Sigma ab + 6abc\}.$$

We may add that if, in the last column, the coefficient 2 were attached to the x's and not to the a's, the right-hand member of the identity would be

$$-6x^3 \cdot \xi^{\frac{1}{2}}(a,\ b,\ c) \cdot \{2x^2\Sigma a + 3x\Sigma ab + 3abc\}.$$

CAYLEY, A. (1877).

[On the general differential equation $\partial x/\sqrt{\mathrm{X}} - \partial y/\sqrt{\mathrm{Y}} = 0$, where X, Y are the same quartic functions of x, y respectively. *Proceed. London Math. Soc.*, viii. pp. 184–199; or *Collected Math. Papers*, ix. pp. 592–608.]

Here it is the observation on which the investigation rests that is of interest to us, namely, that

$$\begin{vmatrix} x^2 & x & 1 & \sqrt{\mathrm{X}} \\ y^2 & y & 1 & \sqrt{\mathrm{Y}} \\ z^2 & z^2 & 1 & \sqrt{\mathrm{Z}} \\ w^2 & w & 1 & \sqrt{\mathrm{W}} \end{vmatrix} = 0$$

is a particular integral of the equation

$$\frac{\partial x}{\sqrt{\mathrm{X}}} + \frac{\partial y}{\sqrt{\mathrm{Y}}} + \frac{\partial z}{\sqrt{\mathrm{Z}}} + \frac{\partial w}{\sqrt{\mathrm{W}}} = 0,$$

and consequently is the general integral of

$$\frac{\partial x}{\sqrt{\mathrm{X}}} + \frac{\partial y}{\sqrt{\mathrm{Y}}} = 0$$

when z and w are constants and enter only as one constant.

FROBENIUS, G., AND STICKELBERGER, L. (1877).

[Zur Theorie der elliptischen Functionen. *Crelle's Journ.*, lxxxiii. pp. 175–179.]

So far as the pure theory of alternants is concerned, the only fresh point in this paper is the consideration of a determinant of the form

$$\begin{vmatrix} . & 1 & \dots & 1 \\ 1 & \psi(u_0+v_0) & \dots & \psi(u_0+v_n) \\ . & . & . & . \\ 1 & \psi(u_n+v_0) & \dots & \psi(u_n+v_n) \end{vmatrix}.$$

The investigation is conducted on the understanding that

$$\psi(u) \equiv \frac{\partial \log \sigma(u)}{\partial u},$$

where $\sigma(\ )$ is the function-symbol introduced by Weierstrass, and where therefore $-\psi'(u)$ defines Weierstrass' other function $\wp(u)$. The result obtained is shown to include Hermite's equivalent of 1876 for the alternant

$$\begin{vmatrix} 1 & \wp(u_0) & \wp'(u_0) & \dots & \wp^{(n-1)}(u_0) \\ 1 & \wp(u_1) & \wp'(u_1) & \dots & \wp^{(n-1)}(u_1) \\ . & . & . & . & . \\ 1 & \wp(u_n) & \wp'(u_n) & \dots & \wp^{(n-1)}(u_n) \end{vmatrix}.$$

(*Crelle's Journ.*, lxxxii. p. 143) and Kiepert's of 1872 for the persymmetric determinant $\mathrm{P}\big(\wp'(u)\cdot\wp''(u), \dots, \wp^{(2n-1)}(u)\big)$.

BORCHARDT, C. W. (1878).

[Zur Theorie der Elimination und Kettenbruch-Entwickelung. *Abhandl. . . . Akad. d. Wiss.* (Berlin), 1878, Nr. 1, pp. 1–17; or *Gesammelte Werke.*]

This is based on a theorem of Rosenham's of 1845, which, as not involving determinants, was not given by us when dealing with his paper (*Hist.*, ii. pp. 160–161). It is to the effect that the resultant of the equations

$$f(x) = 0, \qquad g(x) = 0$$

of the m^{th} and n^{th} degrees respectively, is

$$\sum \frac{f(a_1) \dots f(a_n)\cdot g(a_{n+1}) \dots g(a_{n+m})}{\mathrm{R}(a_1, \dots, a_n;\ a_{n+1}, \dots, a_{n+m})},$$

where $R(\ldots;\ldots)$ indicates the product of the differences got by subtracting each quantity on the left of the semicolon from each quantity on the right, and where the Σ betokens a sum of $C_{n+m,n}$ terms of which the first is given and the others differ from the first merely in having another set of n of the α's instead of the first set.

For example, the resultant of

$$ax+b = 0 \quad \text{and} \quad cx^2+dx+e = 0$$

is

$$\frac{(a\alpha+b)(a\beta+b)\cdot(c\gamma^2+d\gamma+e)}{(\gamma-\alpha)(\gamma-\beta)} + \frac{(a\alpha+b)(a\gamma+b)\cdot(c\beta^2+d\beta+e)}{(\beta-\alpha)(\beta-\gamma)} + \frac{(a\beta+b)(a\gamma+b)\cdot(c\alpha^2+d\alpha+e)}{(\alpha-\beta)(\alpha-\gamma)}.$$

As a mere help towards acceptance it may be pointed out that the expression is manifestly of the 0^{th} degree in each of the α's, of the n^{th} degree in the coefficients of $f(x)$, and of the m^{th} degree in the coefficients of $g(x)$.

There is herefrom suggested to Borchardt the finding of expressions of like kind for the other 'simplified remainders' whose vanishing is the necessary and sufficient condition for the existence of a common factor of $f(x)$, $g(x)$; and, this being accomplished, he uses his paper of 1860 to show how older expressions for these 'remainders,' such as Cayley's and Sylvester's, may be reproduced. He then passes on to the case (§ 2) where $f(x)$, $g(x)$ are of the same degree. Here he not only obtains his result in the same manner as before, but, denoting

$$\frac{f(x)\,g(y)-f(y)\,g(x)}{y-x} \quad \text{by} \quad F(x, y),$$

and recalling Bezout's 'abridged method,' he arrives at a second and quite unlike expression. Equatement being thus possible, there is evolved the very noteworthy identity in alternants

$$\frac{|\,F(a_1, a_{\nu+1}) \ldots F(a_\nu, a_{2\nu})\,|}{\Delta^{\frac{1}{2}}(a_1, \ldots, a_\nu)\cdot\Delta^{\frac{1}{2}}(a_{\nu+1}, \ldots, a_{2\nu})} = (-1)^{\frac{1}{2}\nu(\nu-1)}\sum\frac{f(a_1)\ldots f(a_\nu)\cdot g(a_{\nu+1})\ldots g(a_{2\nu})}{R(a_1, \ldots, a_\nu;\ a_{\nu+1}, \ldots, a_{2\nu})}.$$

The remaining half of the paper is concerned with the related question in continued fractions.

GARBIERI, G. (1878).

[Nuovo teorema algebrico e sua speciale applicazione. . . . *Giornale di Mat.*, xvi. pp. 1–17.]

Garbieri's first theorem is an extension of Beltrami's of 1874, the degree of the f's being no longer kept from exceeding that of the alternant, namely, the n^{th}. If the excess of the one over the other be h, so that

$f_{r+1}(\xi)$ now stands for $a_{r0}+a_{r1}\xi+\ldots+a_{r,n+h}\xi^{n+h}$,

then

$$\frac{|f_1(a_1)\cdot f_2(a_2)\ \ldots\ f_n(a_n)|}{\xi^{\frac{1}{2}}(a_1,\ a_2,\ \ldots,\ a_n)}$$

$$= \begin{vmatrix} a_{00} & a_{01} & a_{02} & \ldots & a_{0n} & a_{0,n+1} & \ldots & a_{0,n+h} \\ a_{10} & a_{11} & a_{12} & \ldots & a_{1n} & a_{1,n+1} & \ldots & a_{1,n+h} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \ldots & a_{n-1,n} & a_{n-1,n+1} & \ldots & a_{n-1,n+h} \\ c_n & c_{n-1} & c_{n-2} & \ldots & c_0 & . & \ldots & . \\ . & c_n & c_{n-1} & \ldots & c_1 & c_0 & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ . & . & . & . & c_h & c_{h-1} & \ldots & c_0 \end{vmatrix},$$

The mode of proof is quite similar to that previously employed; that is to say, the right-hand member is multiplied rowwise by $\xi^{\frac{1}{2}}(a_1,\ a_2,\ \ldots,\ a_n,\ y_0,\ y_1,\ \ldots,\ y_h)$ in the form $|\ a_1^0 a_2^1\ \ldots\ y_h^{n+h}\ |$, and the result being found to be a determinant which is manifestly resolvable into

$$|f_1(a_1)\cdot f_2(a_2)\ \ldots\ f_n(a_n)|\cdot\phi(y_0)\,\phi(y_1)\ \ldots\ \phi(y_h)\cdot\xi^{\frac{1}{2}}(y_0,\ y_1,\ \ldots,\ y_h),$$

it is only necessary to make use of the further fact that the multiplier itself is resolvable into

$$\xi^{\frac{1}{2}}(a_1,\ a_2,\ \ldots,\ a_n)\cdot\phi(y_0)\,\phi(y_1)\ \ldots\ \phi(y_h)\cdot\xi^{\frac{1}{2}}(y_0,\ y_1,\ \ldots,\ y_h).$$

A suggested alternative mode of proof is not essentially different from Mansion's of the following year.

Of the previously obtained results which Garbieri notes as being included in his own, the most important is Naegelsbach's of 1871. In illustration of the relationship between the two, let us put in the above

$$f_1(x)=x^0,\quad f_2(x)=x^2,\quad f_3(x)=x^5,\quad f_4(x)=x^6.$$

We then have from Garbieri

$$\frac{|\,a^0\beta^2\gamma^5\delta^6\,|}{\xi^{\frac{1}{2}}(a,\ \beta,\ \gamma,\ \delta)} = \begin{vmatrix} 1 & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & 1 \\ a\beta\gamma\delta & -\Sigma a\beta\gamma & \Sigma a\beta & -\Sigma a & 1 & . & . \\ . & a\beta\gamma\delta & -\Sigma a\beta\gamma & \Sigma a\beta & -\Sigma a & 1 & . \\ . & . & a\beta\gamma\delta & -\Sigma a\beta\gamma & \Sigma a\beta & -\Sigma a & 1 \end{vmatrix},$$

from which, manifestly, there comes on the right

$$-\begin{vmatrix} -\Sigma a\beta\gamma & -\Sigma a & 1 \\ a\beta\gamma\delta & \Sigma a\beta & -\Sigma a \\ . & -\Sigma a\beta\gamma & \Sigma a\beta \end{vmatrix};$$

and this is Naegelsbach's result.

Garbieri's next theorem concerns the double alternant formable from the bivarial function

$$\sum (a_{ik}x^i y^k)_{i=0,\,1,\,\ldots,\,n+h}^{k=0,\,1,\,\ldots,\,n+h},$$

and two sets of variables

$$a_1,\ a_2,\ \ldots,\ a_n \quad \text{and} \quad \beta_1,\ \beta_2,\ \ldots,\ \beta_n.$$

Calling the function $F(x, y)$, Garbieri affirms that

$$\frac{|\,F(a_1\beta_1)\cdot F(a_2\beta_2)\ \ldots\ F(a_n\beta_n)\,|}{\xi^{\frac{1}{2}}(a_1,\ a_2,\ \ldots,\ a_n)\cdot \xi^{\frac{1}{2}}(\beta_1,\ \beta_2,\ \ldots,\ \beta_n)}$$

$$= (-1)^{h+1}\begin{vmatrix} a_{00} & a_{01} & \ldots & a_{0n} & a_{0,n+1} & \ldots & a_{0,n+h} & c'_n & . & . & \ldots & . \\ a_{10} & a_{11} & \ldots & a_{1n} & a_{1,n+1} & \ldots & a_{1,n+h} & c'_{n-1} & c'_n & . & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n0} & a_{n1} & \ldots & a_{nn} & a_{n,n+1} & \ldots & a_{n,n+h} & c'_0 & c'_1 & c'_2 & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n+h,0} & a_{n+h,1} & \ldots & a_{n+h,n} & a_{n+h,n+1} & \ldots & a_{n+h,n+h} & . & . & . & \ldots & c'_0 \\ c_n & c_{n-1} & \ldots & c_0 & . & \ldots & . & . & . & . & \ldots & . \\ . & c_n & \ldots & c_1 & c_0 & \ldots & . & . & . & . & \ldots & . \\ . & . & \ldots & c_2 & c_1 & \ldots & . & . & . & . & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ . & . & \ldots & c_h & c_{h-1} & \ldots & c_0 & . & . & . & \ldots & . \end{vmatrix}$$

where the c's and c''s are defined by the identical equations

$$(x-a_1)(x-a_2)\ \ldots\ (x-a_n) \equiv c_0x^n+c_1x^{n-1}+\ \ldots\ +c_n,$$
$$(x-\beta_1)(x-\beta_2)\ \ldots\ (x-\beta_n) \equiv c'_0x^n+c'_1x^{n-1}+\ \ldots\ +c'_n.$$

The mode of proof preferred is similar to that chosen in the case of the first theorem, save that the right-hand member is now multiplied by the *two* difference-products

$$\xi^{\frac{1}{2}}(a_1, a_2, \ldots, a_n, y_0, y_1, \ldots, y_h), \quad \xi^{\frac{1}{2}}(\beta_1, \beta_2, \ldots, \beta_n, z_0, z_1, \ldots, z_h)$$

in succession, and that prior to multiplication the determinant form of each of these is raised in the usual way from the $(n+h+1)^{\text{th}}$ order to the $(n+2h+2)^{\text{th}}$. Another proof, however, is indicated, dependent on the use of the first theorem; and it is also pointed out that the second theorem includes the first.

Finally, note is taken of Borchardt's special case of 1859, where $h=-1$; and the fact that Cauchy's result of 1841, though not included, may be deduced with ease by putting

$$\mathrm{F}(x, y) = \frac{x\psi(x)-y\psi(y)}{x-y},$$

and

$$\psi(x) = (x-\beta_1)(x-\beta_2)\ \ldots\ (x-\beta_n).$$

CROCCHI, L. (1878).

[Sopra le funzioni Aleph ed il determinante di Cauchy. *Giornale di Mat.*, xvii. pp. 218–231, 380.]

After a study of Trudi's paper of 1864 regarding the resolution of $|\ x_1^{\alpha}\ x_2^{\beta}\ x_3^{\gamma}\ x_4^{\delta}\ |$, Crocchi is led to hazard the assertion that for a similar reason any Jacobian ought to be divisible by the difference-product of the variables, and that the search for the quotient may be furthered by means of the relation

$$\left|\frac{\partial f_1}{\partial x_1}\cdot\frac{\partial f_2}{\partial x_2}\ \ldots\ \frac{\partial f_n}{\partial x_n}\right| = \left|\frac{\partial f_1}{\partial y_1}\cdot\frac{\partial f_2}{y_2}\ \ldots\ \frac{\partial f_n}{\partial y_n}\right|\cdot\left|\frac{\partial y_1}{\partial x_1}\cdot\frac{\partial y_2}{\partial x_2}\ \ldots\ \frac{\partial y_n}{\partial x_n}\right|.$$

Nothing is said regarding the f's being symmetrical with respect to the x's. As an illustration of his suggested process for finding the quotient he takes Trudi's case, choosing for this end

$$f_1 = s_{\alpha+1}\div(\alpha+1), \qquad y_1 = s_1\div 1,$$
$$f_2 = s_{\beta+1}\div(\beta+1), \qquad y_2 = s_2\div 2,$$
$$f_3 = s_{\gamma+1}\div(\gamma+1), \qquad y_3 = s_3\div 3,$$
$$f_4 = s_{\delta+1}\div(\delta+1), \qquad y_4 = s_4\div 4,$$

where $s_\mu = x_1^\mu + x_2^\mu + x_3^\mu + x_4^\mu$. This, it will be observed, ensures that

$$\left| \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \frac{\partial f_3}{\partial x_3} \frac{\partial f_4}{\partial x_4} \right| = \left| x_1^\alpha x_2^\beta x_3^\gamma x_4^\delta \right|,$$

and

$$\left| \frac{\partial y_1}{\partial x_1} \frac{\partial y_2}{\partial x_2} \frac{\partial y_3}{\partial x_3} \frac{\partial y_4}{\partial x_4} \right| = \left| x_1^0 x_2^1 x_3^2 x_4^3 \right|;$$

and there remains the trouble—which is still ample—of solving four sets of four simultaneous equations of the type

$$\frac{\partial f_h}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_k} + \frac{\partial f_h}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_k} + \frac{\partial f_h}{\partial y_3} \cdot \frac{\partial y_3}{\partial x_k} + \frac{\partial f_h}{\partial y_4} \cdot \frac{\partial y_4}{\partial x_k} = \frac{\partial f_h}{\partial x_k}$$

in order to obtain the corresponding values of the elements of

$$\left| \frac{\partial f_1}{\partial y_1} \frac{\partial f_2}{\partial y_2} \frac{\partial f_3}{\partial y_3} \frac{\partial f_4}{\partial y_4} \right|.$$

At the end of the paper there is given another illustrative result worth noting, namely, *The Jacobian of the first* n *Aleph functions of a set of* n *variables is equal to the difference-product of the variables.* This may be appropriately grouped with Tarleton's of 1867.

WOLSTENHOLME, J. (1878).

[MATHEMATICAL PROBLEMS. . . . 2nd edition. x+480 pp. London.]

Here (p. 276) we have two additional examples of alternants whose elements are goniometric functions, namely,

$$\left| \begin{array}{lll} 1 & \cos(\beta+\gamma) & \sin^2\tfrac{1}{2}(\beta-\theta) \sin^2\tfrac{1}{2}(\gamma-\theta) \\ 1 & \cos(\gamma+\alpha) & \sin^2\tfrac{1}{2}(\gamma-\theta) \sin^2\tfrac{1}{2}(\alpha-\theta) \\ 1 & \cos(\alpha+\beta) & \sin^2\tfrac{1}{2}(\alpha-\theta) \sin^2\tfrac{1}{2}(\beta-\theta) \end{array} \right|$$

$$= 2\sin\tfrac{1}{2}(\gamma-\beta) \sin\tfrac{1}{2}(\gamma-\alpha) \sin\tfrac{1}{2}(\beta-\alpha) \cdot \{2\sin 2\theta + 2\sin(\alpha+\beta+\gamma-\theta) - \Sigma\sin(\beta+\gamma)\},$$

and

$$\left| \begin{array}{lll} 1 & \cos(\alpha+\theta) & \sin^2\tfrac{1}{2}(\beta-\theta) \sin^2\tfrac{1}{2}(\gamma-\theta) \\ 1 & \cos(\beta+\theta) & \sin^2\tfrac{1}{2}(\gamma-\theta) \sin^2\tfrac{1}{2}(\alpha-\theta) \\ 1 & \cos(\gamma+\theta) & \sin^2\tfrac{1}{2}(\alpha-\theta) \sin^2\tfrac{1}{2}(\beta-\theta) \end{array} \right|$$

$$= 2\sin\tfrac{1}{2}(\gamma-\beta) \sin\tfrac{1}{2}(\gamma-\alpha) \sin\tfrac{1}{2}(\beta-\alpha) \cdot \{2\sin 2\theta + \sin(\alpha+\beta+\gamma-\theta) - \Sigma\sin(\alpha+\theta)\}.$$

CAYLEY, A. (1876): SCOTT, R. F. (1879).

[Theorem in Trigonometry. *Messenger of Math.*, v. p. 164.]

[Note on a theorem of Professor Cayley's. *Messenger of Math.*, viii. pp. 155–157.]

By substituting the proper exponential expressions for sines and cosines, Scott proves that

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\lambda+\alpha)\sin(\mu+\alpha)\sin(\nu+\alpha) \\ \sin\beta & \cos\beta & \sin(\lambda+\beta)\sin(\mu+\beta)\sin(\nu+\beta) \\ \sin\gamma & \cos\gamma & \sin(\lambda+\gamma)\sin(\mu+\gamma)\sin(\nu+\gamma) \end{vmatrix}$$

$$= \sin(\gamma-\beta)\sin(\gamma-\alpha)\sin(\beta-\alpha)\cdot\sin(\alpha+\beta+\gamma+\lambda+\mu+\nu),$$

which becomes Cayley's result when $\alpha+\beta+\gamma+\lambda+\mu+\nu=0$. In his text-book of the following year he gives the similar example

$$\begin{vmatrix} 1 & \tan\alpha & \sin 2\alpha \\ 1 & \tan\beta & \sin 2\beta \\ 1 & \tan\gamma & \sin 2\gamma \end{vmatrix}$$

$$= -2\cdot\sin(\gamma-\beta)\sin(\gamma-\alpha)\sin(\beta-\alpha)\cdot\frac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}.$$

We may note in passing that these may with advantage be assimilated to Wolstenholme's example by changing the alternating factor on the right into

$$2^3\sin\tfrac{1}{2}(\gamma-\beta)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\beta-\alpha)$$
$$\cdot\cos\tfrac{1}{2}(\gamma-\beta)\cos\tfrac{1}{2}(\gamma-\alpha)\cos\tfrac{1}{2}(\beta-\alpha).$$

Scott adds four additional results, namely,

$$\begin{vmatrix} 1 & \cos\alpha & \sin\alpha & \sin 3\alpha \\ 1 & \cos\beta & \sin\beta & \sin 3\beta \\ 1 & \cos\gamma & \sin\gamma & \sin 3\gamma \\ 1 & \cos\delta & \sin\delta & \sin 3\delta \end{vmatrix}$$

$$= 2^5\,\Pi\sin\tfrac{1}{2}(\alpha-\beta)$$
$$\cdot\left\{\cos\tfrac{1}{2}(3\alpha+\beta+\gamma+\delta)+\cos\tfrac{1}{2}(\alpha+3\beta+\gamma+\delta)+\ldots\right\};$$

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin 2\alpha & \cos 2\alpha \\ \sin\beta & \cos\beta & \sin 2\beta & \cos 2\beta \\ \sin\gamma & \cos\gamma & \sin 2\gamma & \cos 2\gamma \\ \sin\delta & \cos\delta & \sin 2\delta & \cos 2\delta \end{vmatrix}$$

$$= 2^5 \Pi \sin \tfrac{1}{2}(\alpha-\beta) \cdot \{\cos \tfrac{1}{2}(\alpha+\beta-\gamma-\delta) + \cos \tfrac{1}{2}(\alpha+\gamma-\beta-\delta) + \cos \tfrac{1}{2}(\alpha+\delta-\beta-\gamma)\};$$

$$\begin{vmatrix} 1 & \sin a_1 & \cos a_1 & \sin 2a_1 & \cos 2a_1 & \dots & \sin na_1 & \cos na_1 \\ 1 & \sin a_2 & \cos a_2 & \sin 2a_2 & \cos 2a_2 & \dots & \sin na_2 & \cos na_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \sin a_{2n+1} & \dots & \dots & \dots & \dots & \dots & \cos na_{2n+1} \end{vmatrix}$$

$$= (-1)^n 4^{nn} \Pi \sin \tfrac{1}{2}(a_j - a_i), \quad \text{where}^* \ j > i;$$

and

$$\begin{vmatrix} \cosh \alpha & \sinh \alpha & \cosh 3\alpha \\ \cosh \beta & \sinh \beta & \cosh 3\beta \\ \cosh \gamma & \sinh \gamma & \cosh 3\gamma \end{vmatrix}$$

$$= 4 \sinh(\gamma-\beta) \sinh(\gamma-\alpha) \sinh(\beta-\alpha) \cdot \cosh(\alpha+\beta+\gamma).$$

The second of these four he generalized later in the same year (*Messenger of Math.*, viii. p. 184), the angles α, β, γ, δ then giving place to a_1, a_2, . . . , a_{2n}, the multiples 1, 2 to 1, 2, . . . , n and the evaluation to

$$2^{2n^2-2n+1} \cdot \Pi \sin \tfrac{1}{2}(a_k - a_h)_{k>h} \cdot \sum \cos \tfrac{1}{2}(a_1 + a_2 + \dots + a_n - a_{n+1} - \dots - a_{2n}).$$

SCOTT, R. F. (1879).

[Notes on determinants. *Messenger of Math.*, viii. pp. 182–187.]

The results given by Scott at the outset (§ 3) are those assigned above to Bellavitis (1857) and Naegelsbach (1871). He then passes on to *double* alternants, opening with Zehfuss' theorem of 1859 and a companion-theorem of his own, namely,

$$\begin{vmatrix} (a_1-\beta_1)^n & \dots & (a_1-\beta_n)^n \\ (a_2-\beta_1)^n & \dots & (a_2-\beta_n)^n \\ \dots & \dots & \dots \\ (a_n-\beta_1)^n & \dots & (a_n-\beta_n)^n \end{vmatrix} = \frac{(n)_1 (n)_2 \dots (n)_n}{1 \cdot 2 \dots n} \zeta_1^{\frac{1}{2}} \zeta_2^{\frac{1}{2}} \begin{vmatrix} a_1-\beta_1 & \dots & a_1-\beta_n \\ a_2-\beta_1 & \dots & a_2-\beta_n \\ \dots & \dots & \dots \\ a_n-\beta_1 & \dots & a_n-\beta_n \end{vmatrix}^+$$

where $(n)_r$ stands for $n(n-1)\dots(n-r+1)/1 \cdot 2 \dots r$, and the suffixes attached to $\zeta^{\frac{1}{2}}$ imply that the a's are the variables in the

* We have annexed the sign-factor.

one case and the β's in the other. Unfortunately, no proof is given of this latter important result. The next two theorems are really cases of the two preceding, namely, the cases where the variables are

$$\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n,\ x$$
$$\beta_1,\ \beta_2,\ \ldots,\ \beta_n,\ x;$$

and the final pair are

$$\begin{vmatrix} (\alpha_1-\beta_1)^n & \ldots & (\alpha_1-\beta_n)^n & 1 \\ \cdot & \cdot & \cdot & \cdot \\ (\alpha_n-\beta_1)^n & \ldots & (\alpha_n-\beta_n)^n & 1 \\ 1 & \ldots & 1 & . \end{vmatrix} = (n)_1(n_2)\ \ldots\ (n)_{n-1}\cdot\xi_1^{\frac{1}{2}}\xi_2^{\frac{1}{2}},$$

$$\begin{vmatrix} (\alpha_1-\beta_1)^{n+1} & \ldots & (\alpha_1-\beta_n)^{n+1} & 1 \\ \cdot & \cdot & \cdot & \cdot \\ (\alpha_n-\beta_1)^{n+1} & \ldots & (\alpha_n-\beta_n)^{n+1} & 1 \\ 1 & \ldots & 1 & . \end{vmatrix} = \frac{(n+1)_1(n+2)_2\ \ldots\ (n+1)_n}{1\cdot 2\ \ldots\ n}\xi_1^{\frac{1}{2}}\xi_2^{\frac{1}{2}}\cdot\mathbf{P},$$

where $$\mathbf{P} = \begin{vmatrix}^{+} & & & ^{+} \\ \alpha_1-\beta_1 & \ldots & \alpha_1-\beta_n & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_n-\beta_1 & \ldots & \alpha_n-\beta_n & 1 \\ 1 & \ldots & 1 & . \end{vmatrix}.$$

MUIR, T. (1879).

[General theorems on determinants. *Transac. R. Soc. Edinburgh*, xxix. pp. 47–54.]

On p. 54 there is given a result more general than Malet's of 1876 but less general than Schweins' of 1825, the real point of interest being that it is made dependent on a theorem for the multiplication of *any* n-line determinant by *any* n-termed expression.

MUIR, T. (1879).

[Preliminary note on alternants. *Proceed. R. Soc. Edinburgh*, x. pp. 102–103.]

The note describes itself as an abstract of a paper on the general subject of the quotient of an alternant by the difference-product

of the variables. As, however, previous to printing, the main part of the paper was found to be covered by Garbieri's of the preceding year the manuscript was withheld (*Proceed. R. Soc. Edinburgh*, xiv. p. 433).

[The word 'alternant' appears here for the first time as the designation of a special form of determinant. Having been told that I had been forestalled in the usage by Sylvester, I called it from the outset "Sylvester's word." My informant, however, was probably in error, as some years afterwards I found Sylvester employing it to denote a peculiar form of operator in the theory of reciprocants. (*American Journ. of Math.*, ix. pp. 130–132,* published in January 1887.)]

HAHN, J. (1879).

[Untersuchung der Kegelschnittnetze deren *Jacobi*'sche oder *Hermite*'sche Form identisch verschwindet. *Math. Annalen*, xv. pp. 111–121.]

One of the identities here used (p. 117), which may be written

$$2\,|a_1b_2|\,|b_1c_2|\,|c_1a_2|\cdot|x_1y_2|\,|y_1z_2|\,|z_1x_2| = \begin{vmatrix} |a_1x_2|^2 & |a_1y_2|^2 & |a_1z_2|^2 \\ |b_1x_2|^2 & |b_1y_2|^2 & |b_1z_2|^2 \\ |c_1x_2|^2 & |c_1y_2|^2 & |c_1z_2|^2 \end{vmatrix},$$

it is best to view as a simple case of Zehfuss' theorem of 1859 regarding a double alternant (*Hist.*, ii. p. 190), the product

$$|a_1b_2|\cdot|b_1c_2|\cdot|c_1a_2|$$

being equal to

$$\begin{vmatrix} a_2^2 & a_1a_2 & a_1^2 \\ b_2^2 & b_1b_2 & b_1^2 \\ c_2^2 & c_1c_2 & c_1^2 \end{vmatrix},$$

i.e. $$a_2^2b_2^2c_2^2 \begin{vmatrix} \left(\frac{a_1}{a_2}\right)^0 \left(\frac{b_1}{b_2}\right)^1 \left(\frac{c_1}{c_2}\right)^2 \end{vmatrix},$$

as we have seen above (p. 132).

* *Collected Math. Papers*, iv. pp. 415–417.

MANSION, P. (1879).

[On rational functional determinants. *Messenger of Math.*, ix. pp. 30–32.]

Mansion re-investigates in an interesting way the theorems of Naegelsbach and Garbieri, using alternants of only three lines. He first considers the evanescent determinant

$$\begin{vmatrix} \alpha^m & \alpha^n & \alpha^p \mathrm{F}(\alpha) \\ \beta^m & \beta^n & \beta^p \mathrm{F}(\beta) \\ \gamma^m & \gamma^n & \gamma^p \mathrm{F}(\gamma) \end{vmatrix},$$

where

$$\mathrm{F}(x) \equiv x^3 - c_1 x^2 + c_2 x - c_3 \equiv (x-\alpha)(x-\beta)(x-\gamma),$$

and therefrom virtually obtains the identity

$$|\alpha^m\beta^n\gamma^{p+3}| - c_1|\alpha^m\beta^n\gamma^{p+2}| + c_2|\alpha^m\beta^n\gamma^{p+1}| - c_3|\alpha^m\beta^n\gamma^p| = 0.$$

By putting in this $m = 0$, $n = 1$, and in succession $p = 0, 1, 2, 3$, four equations arise, from which

$$\frac{|\alpha^0\beta^1\gamma^3|}{|\alpha^0\beta^1\gamma^2|}, \quad \frac{|\alpha^0\beta^1\gamma^4|}{|\alpha^0\beta^1\gamma^2|}, \quad \frac{|\alpha^0\beta^1\gamma^5|}{|\alpha^0\beta^1\gamma^2|}, \quad \frac{|\alpha^0\beta^1\gamma^6|}{|\alpha^0\beta^1\gamma^2|}$$

are obtained in terms of the c's. Similarly, by putting $m = 0$, $n = 2$, and in succession $p = 0, 1, 2, 3$, other four equations arise which give similar expressions for the quotients of

$$|\alpha^0\beta^2\gamma^3|, \quad |\alpha^0\beta^2\gamma^4|, \quad |\alpha^0\beta^2\gamma^5|, \quad |\alpha^0\beta^2\gamma^6|$$

by $|\alpha^0\beta^1\gamma^2|$. On this investigation Naegelsbach's general theorem is made to rest.

Next, taking the determinant

$$\begin{vmatrix} \phi(\alpha) & \psi(\alpha) & \chi(\alpha) \\ \phi(\beta) & \psi(\beta) & \chi(\beta) \\ \phi(\gamma) & \psi(\gamma) & \chi(\gamma) \end{vmatrix},$$

where

$$\begin{aligned} \phi(x) &= \mathrm{A}_0 + \mathrm{A}_1 x + \mathrm{A}_2 x^2 + \ldots + \mathrm{A}_6 x^6, \\ \psi(x) &= \mathrm{B}_0 + \mathrm{B}_1 x + \mathrm{B}_2 x^2 + \ldots + \mathrm{B}_6 x^6, \\ \chi(x) &= \mathrm{C}_0 + \mathrm{C}_1 x + \mathrm{C}_2 x^2 + \ldots + \mathrm{C}_6 x^6, \end{aligned}$$

he compares the cofactor of $|A_m B_n C_p|$ in it with the cofactor of the same in

$$\begin{vmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ B_0 & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\ C_0 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\ -c_3 & c_2 & -c_1 & 1 & . & . & . \\ . & -c_3 & c_2 & -c_1 & 1 & . & . \\ . & . & -c_3 & c_2 & -c_1 & 1 & . \\ . & . & . & -c_3 & c_2 & -c_1 & 1 \end{vmatrix} \quad |\alpha^0\beta^1\gamma^2|,$$

and, ascertaining by Naegelsbach's theorem that they are identical, he views Garbieri's theorem as established.

It is important to note in passing that the latent identity to which we have drawn attention and which may be readily formulated for any number of variables can also be proved by means of the extension of Schweins' multiplication-theorem.

BERG, F. J. VAN DEN (1879).

[Ontwikkeling van eenige algebraïsche en van daarmede gelijkvormige goniometrische identiteiten. *Verslagen . . . . Akad. van Wetensch.* (Amsterdam), (2) xiv. pp. 340–359.]

Van den Berg starts with an identity which in effect is an assertion that *the difference-product* alternant

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \dots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \dots & a_2^{n-1} \\ . & . & . & . & . \end{vmatrix}$$

is not altered in substance by changing the last column into

$$\begin{aligned} &(a_1-b_1)(a_1-b_2)\dots(a_1-b_{n-1}) \\ &(a_2-b_1)(a_2-b_2)\dots(a_2-b_{n-1}) \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \end{aligned}$$

where, be it noted, the number of binomial factors in each element of the substituted column is 1 less than the number of columns in the determinant. His next step is to consider the result of a substitution which differs from this in having n binomial factors in each

element of the substituted column, the outcome being that *the new determinant divided by the old is equal to*

$$\Sigma a - \Sigma b.$$

For example,

$$\begin{vmatrix} 1 & a_1 & (a_1-b_1)(a_1-b_2)(a_1-b_3) \\ 1 & a_2 & (a_2-b_1)(a_2-b_2)(a_2-b_3) \\ 1 & a_3 & (a_3-b_1)(a_3-b_2)(a_3-b_3) \end{vmatrix} = \begin{vmatrix} 1 & a_1 & a_1^3-a_1^2\Sigma b+a_1\Sigma b_1b_2-b_1b_2b_3 \\ 1 & a_2 & a_2^3-a_2^2\Sigma b+a_2\Sigma b_1b_2-b_1b_2b_3 \\ 1 & a_3 & a_3^3-a_3^2\Sigma b+a_3\Sigma b_1b_2-b_1b_2b_3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a_1 & a_1^3 \\ 1 & a_2 & a_2^3 \\ 1 & a_3 & a_3^3 \end{vmatrix} - \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} \Sigma b$$

$$= \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} (\Sigma a - \Sigma b).$$

By similarly substituting in the last column of

$$|a_1^0a_2^1a_3^3a_4^4|, \quad |a_1^0a_2^2a_3^3a_4^4|, \quad |a_1^1a_2^2a_3^3a_4^4|,$$

we obtain in the same way determinants which when divided by $|a^0a_2^1a_3^2a_4^3|$ give respectively the quotients

$$\sum a_1a_2 - \sum b_1b_2,$$

$$\sum a_1a_2a_3 - \sum b_1b_2b_3,$$

$$a_1a_2a_3a_4 - b_1b_2b_3b_4,$$

the series of resulting identities degenerating into a well-known series of early date on making the b's vanish. The already partially known non-determinant forms of these identities are next given, being derived by expanding the determinants in terms of the elements of what we have called the substituted column and their complementary minors. For example,

$$\frac{(a_1-b_1)(a_1-b_2)}{(a_1-a_2)(a_1-a_3)} + \frac{(a_2-b_1)(a_2-b_2)}{(a_2-a_3)(a_2-a_1)} + \frac{(a_3-b_1)(a_3-b_2)}{(a_3-a_1)(a_3-a_2)} = 1,$$

$$\frac{(a_1-b_1)(a_1-b_2)(a_1-b_3)}{(a_1-a_2)(a_1-a_3)} + \frac{(a_2-b_1)(a_2-b_2)(a_2-b_3)}{(a_2-a_3)(a_2-a_1)}$$

$$+ \frac{(a_3-b_1)(a_3-b_2)(a_3-b_3)}{(a_3-a_1)(a_3-a_2)} = \Sigma a - \Sigma b.$$

Van den Berg's second section (pp. 345–356) concerns identities similar in form to the foregoing but having angle-functions in place of the a's and b's. He first, however, proves the general proposition that *if* $f(z)$ *stand for* $(z-a_1)(z-a_2)\ .\ .\ .\ (z-a_n)$, *then*

$$\frac{\left|a_1^0 a_2^1 a_3^2 \ .\ .\ .\ a_{n-1}^{n-2}\phi(a_n)\right|}{\left|a_1^0 a_2^1 a_3^2 \ .\ .\ .\ a_{n-1}^{n-2} a_n^{n-1}\right|} = \textit{coefficient of } \frac{1}{z} \textit{ in } \frac{\phi(z)}{f(z)},$$

illustrating it by the case, already dealt with, in which $n = 3$ and $\phi(z) = (z-b_1)(z-b_2)(z-b_3)$, and where therefore

$$\frac{\phi(z)}{f(z)} = \frac{z^3 - z^2\Sigma b + z\Sigma b_1 b_2 - b_1 b_2 b_3}{z^3 - z^2\Sigma a + z\Sigma a_1 a_2 - a_1 a_2 a_3},$$

and the coefficient of $\frac{1}{z}$ is equal to

$$\Sigma a - \Sigma b.$$

Then, in the expression

$$\sum \frac{(a_1-b_1)(a_1-b_2)(a_1-b_3)}{(a_1-a_2)(a_1-a_3)},$$

given above as an equivalent for $\Sigma a - \Sigma b$, $\tan a_r$, $\tan \beta_r$ are substituted for a_r, b_r respectively, and the said general proposition with other assistance is used to show that

$$\sum \frac{\sin(a_1-\beta_1)\sin(a_1-\beta_2)\sin(a_1-\beta_3)}{\sin(a_1-a_2)\sin(a_1-a_3)} = \sin(\Sigma a - \Sigma\beta).$$

The three determinants dealt with that have angle-functions for elements are (1)

$$\begin{vmatrix} \sin^{n-1}a_1 & \sin^{n-2}a_1\cos a_1 & \ .\ .\ .\ . & \sin a_1\cos^{n-2}a_1 & \cos^{n-1}a_1 \\ \sin^{n-1}a_2 & \sin^{n-2}a_2\cos a_2 & \ .\ .\ .\ . & \sin a_2\cos^{n-2}a_2 & \cos^{n-1}a_2 \\ \ .\ .\ . & \ .\ .\ . & \ .\ .\ .\ . & \ .\ .\ . & \ .\ .\ . \end{vmatrix};$$

(2) the determinant formed from this by substituting

$$\sin(a_1-\beta_1)\sin(a_1-\beta_2)\ .\ .\ .\ .\sin(a_1-\beta_n),$$
$$\sin(a_2-\beta_1)\sin(a_2-\beta_2)\ .\ .\ .\ .\sin(a_2-\beta_n),$$
$$.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .$$

for the first column and lowering the powers of the cosines in the other columns by 1; and (3) the determinant formed in the same way save that the powers of the cosines are increased by 1. The

first determinant is stated to be equal to the product of the sines of the differences of the angles, that is, to

$$\sin(\alpha_1-\alpha_2)\sin(\alpha_1-\alpha_3)\ .\ .\ .\ .\ \sin(\alpha_{n-1}-\alpha_n);$$

the second and third to have the first for a factor with

$$\sin(\Sigma\alpha-\Sigma\beta)$$

for the cofactor in the one case, and

$$\cos\alpha_1\cos\alpha_2\ .\ .\ .\ \cos\alpha_n\cdot\cos\beta_1\cos\beta_2\ .\ .\ .\ \cos\beta_n\cdot(\Sigma\tan\alpha-\Sigma\tan\beta)$$

in the other.

The last section (pp. 356–359) is of less importance than the others, the only statement worth noting being that

$$\begin{vmatrix} \sin^{2r+1}\alpha_1 & \sin\alpha_1 & \cos\alpha_1 \\ \sin^{2r+1}\alpha_2 & \sin\alpha_2 & \cos\alpha_2 \\ \sin^{2r+1}\alpha_3 & \sin\alpha_3 & \cos\alpha_3 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \sin^{2r}\alpha_1 & \sin^2\alpha_1 & \cos^2\alpha_1 \\ \sin^{2r}\alpha_2 & \sin^2\alpha_2 & \cos^2\alpha_2 \\ \sin^{2r}\alpha_3 & \sin^2\alpha_3 & \cos^2\alpha_3 \end{vmatrix}$$

are divisible by

$$\sin(\alpha_1-\alpha_2)\sin(\alpha_1-\alpha_2)\sin(\alpha_2-\alpha_3),$$

the latter, indeed, divisible by

$$\begin{vmatrix} \sin^4\alpha_1 & \sin^2\alpha_1 & \cos^2\alpha_1 \\ \sin^4\alpha_2 & \sin^2\alpha_2 & \cos^2\alpha_2 \\ \sin^4\alpha_3 & \sin^2\alpha_3 & \cos^2\alpha_3 \end{vmatrix},$$

and therefore by

$$(\sin^2\alpha_1-\sin^2\alpha_2)(\sin^2\alpha_1-\sin^2\alpha_3)(\sin^2\alpha_2-\sin^2\alpha_3),$$

which equals

$$\sin(\alpha_1-\alpha_2)\sin(\alpha_1-\alpha_3)\sin(\alpha_2-\alpha_3)\cdot\sin(\alpha_1+\alpha_2)\sin(\alpha_1+\alpha_3)\sin(\alpha_2+\alpha_3).$$

CHAPTER VI.

COMPOUND DETERMINANTS, FROM 1862 TO 1880.

STILL greater progress than in the case of alternants has here to be chronicled. Not only is the number of writings more than treble the number for the preceding twenty-year period, but there is also a large proportion of them of considerable importance.

Compound determinants of the special type associated with Wronskians (*Hist.*, ii. pp. 227–228) will be found referred to in chap. viii.

ZEHFUSS, G. (1862).

[Zwei Sätze über Determinanten. *Zeitschrift f. Math. u. Phys.*, vii. pp. 436–439.]

Zehfuss' first theorem is one of the extreme cases of Bazin's of the year 1851, namely,

$$\begin{vmatrix} |\alpha_1\beta_2\gamma_3\ldots\lambda_n| & |\alpha_1\alpha_2\gamma_3\ldots\lambda_n| & \ldots. & |\alpha_1\beta_2\ldots\alpha_n| \\ |b_1\beta_2\gamma_3\ldots\lambda_n| & |\alpha_1 b_2\gamma_3\ldots\lambda_n| & \ldots. & |\alpha_1\beta_2\ldots b_n| \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots \\ |l_1\beta_2\gamma_3\ldots\lambda_n| & |\alpha_1 l_2\gamma_3\ldots\lambda_n| & \ldots. & |\alpha_1\beta_2\ldots l_n| \end{vmatrix}$$
$$= |\alpha_1 b_2\ldots l_n|\cdot|\alpha_1\beta_2\ldots\lambda_n|^{n-1}.$$

His proof is a lengthened way of saying that if we multiply the compound determinant by $|\alpha_1\beta_2\gamma_3\ldots\lambda_n|$ in row-by-column fashion we obtain a determinant having for its elements the elements of $|\alpha_1 b_2\ldots l_n|$, each multiplied by $|\alpha_1\beta_2\gamma_3\ldots\lambda_n|$, and being therefore equal to

$$|\alpha_1 b_2\ldots l_n|\cdot|\alpha_1\beta_2\ldots\lambda_n|^n.$$

The theorem is used to prove Jacobi's theorem regarding a minor of the adjugate.

FRANKE, E. (1862): BORCHARDT, C. W. (1863).

[Ueber Determinanten aus Determinanten. *Crelle's Journal*, lxi. pp. 350–355.]

The subject is the m^{th} compound and any minor of it, and there is obtained not only Sylvester's theorem regarding the former,

$$||\,a_{1n}\,||_m = |\,a_{1n}\,|^{(n-1)_{m-1}},$$

but a theorem including Spottiswoode's results in reference to the latter and corresponding to Jacobi's theorem regarding any minor of the adjugate. Franke's mode of proof is the gradational,—making any case dependent on the preceding case—and, though greatly superior to Spottiswoode's, is not very attractive. Besides it is overshadowed by two footnotes of the editor's.

In the one footnote Borchardt deals with a k-line minor of the m^{th} compound in the manner initiated by Cayley in 1843 in the case of the $(n-1)^{\text{th}}$ compound. Retaining the first k rows of the compound and substituting zeros for all the non-diagonal elements of the remaining rows and units for the diagonal elements, he then multiplies this new determinant by the $(n-m)^{\text{th}}$ compound, and from the result, with the help of the above-mentioned theorem of Sylvester's, obtains the theorem which we may formulate for ourselves thus: *The first* k*-line coaxial minor of the* m^{th} *compound of* $|\,\text{a}_{1\text{n}}\,|$ *is the product of the complementary of the corresponding minor of the* $(\text{n}-\text{m})^{th}$ *compound by* $|\,\text{a}_{1\text{n}}\,|^{\text{k}-\text{C}_{\text{n}}\ _{1,\text{m}}}$.

In the other footnote Borchardt goes back to Cauchy's result of 1812 (*Hist.*, i. p. 118) regarding the product of two complementary compounds,

$$||\,a_{1n}\,||_k \cdot ||\,a_{1n}\,||_{n-k} = |\,a_{1n}\,|^{(n)_k},$$

and by a mere consideration of dimensions assigns to each its proper power of the original determinant.

JANNI, G. (1863).

[Nota sopra i determinanti minori di un dato determinante. *Giornale di Mat.*, i. pp. 270–275.]

The subject is the same as Franke's of the preceding year. First there is given a lengthy but fresh proof of Sylvester's theorem that

the m^{th} compound of $|\,a_{1n}\,|$ is equal to $|\,a_{1n}\,|^{C_{n-1,m-1}}$. The proof, though not worth reproducing, is interesting as suggesting consideration of the relation between the m^{th} compound of $|\,a_{1n}\,|$ and the m^{th} compound of the equivalent determinant

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} & . & . & \dots & . \\ a_{21} & a_{22} & \dots & a_{2n} & . & . & \dots & . \\ . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{nn} & . & . & \dots & . \\ . & . & \dots & . & 1 & . & \dots & . \\ . & . & \dots & . & . & 1 & \dots & . \\ . & . & . & . & . & . & . & . \\ . & . & \dots & . & . & . & \dots & 1 \end{vmatrix}.$$

For example, the second compound of

$$\begin{vmatrix} a_1 & a_2 & a_3 & . \\ b_1 & b_2 & b_3 & . \\ c_1 & c_2 & c_3 & . \\ . & . & . & 1 \end{vmatrix}$$

is the product of the second compound of $|\,a_1b_2c_3\,|$ by $|\,a_1b_2c_3\,|$.

Janni's mode of dealing with a k-line minor of the m^{th} compound is practically the same as Borchardt's.

SYLVESTER, J. J. (1863).

[On a question of compound arrangement. *Proceed. R. Soc. London*, xii. pp. 561–563 ; or *Collected Math. Papers*, ii. pp. 325–326.]

[On a theorem relating to polar umbræ. *Proceed. R. Soc. London*, xii. pp. 563–565 ; or *Collected Math. Papers*, ii. pp. 327–328.]

[Sequel to the theorems relating to 'canonic roots,' . . . . *Philos. Magazine*, (4) xxv. pp. 453-460 ; or *Collected Math. Papers*, ii. pp. 331–337.]

All these papers refer hurriedly and somewhat obscurely to '*double* determinants,' the last of the three indicating the line of investigation in which the idea had originated, and the two others being bye-products of the result.

An 'observation' appended to the first says, "A double determinant means the resultant of a system of $m+n-1$ homogeneous

equations each containing mn terms and linear in respect to each of two systems of m and n variables taken separately, but of the second order in respect to the variables of these two systems taken collectively." It is also stated to be representable by an ordinary determinant of the $(m+n-1)^{\text{th}}$ order* whose elements are themselves "sums of simple determinants of the $(m+n-1)^{\text{th}}$ order." Cayley's resultant of the year 1854 is referred to as the particular case where $m, n = 2, 2$.

The theorem of the next paper is spoken of as the auxiliary by means of which he obtains "the resultant of a lineo-linear system of equations in its most perfect form. It is easy," he adds, "to obtain two different solutions, each of them unsymmetrical in respect of the data of the question," but this theorem effects "a conversion and fusion of each of these into one and the same determinant, symmetrical in all its relations to the data."

An important part of the third paper concerns the elimination of u, v, w and x, y from the equations

$$\left.\begin{aligned}(bu+b'v+b''w)y-(au+a'v+a''w)x&=0\\(cu+c'v+c''w)y-(bu+b'v+b''w)x&=0\\(du+d'v+d''w)y-(cu+c'v+c''w)x&=0\\(eu+e'v+e''w)y-(du+d'v+d''w)x&=0\end{aligned}\right\},$$

the eliminant found being a compound determinant of the third order, which we may write in the form

$$\begin{vmatrix}|1456| & |1246|-|1345| & |1234|\\ |2456| & |1256|-|2345| & |1235|\\ |3456| & |1356|-|2346| & |1236|\end{vmatrix},$$

if we use $|rstu|$ to stand for the determinant whose columns are the r^{th}, s^{th}, t^{th}, u^{th} columns of the 4-by-6 array of given coefficients.

We may add for ourselves in passing that this would still be the eliminant if the 24 coefficients were all different. Further, it may be noted that though Sylvester in this last paper viewed the discovery of double determinants as "the dawn of a new epoch in the history of modern algebra," he never afterwards returned to the subject.

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* This cannot be correct, because the "resultant is of the degree

$$(m+n-1)!/(m-1)!(n-1)!$$

in respect of the given coefficients."

CAYLEY, A. (1865).

[Solution of a problem of elimination. *Quart. Journ. of Math.*, viii. pp. 183–185; or *Collected Math. Papers*, vi. pp. 40–42.]

The given set of equations

$$\begin{Vmatrix} x^4 & x^3y & x^2y^2 & xy^3 & y^4 \\ a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \end{Vmatrix} = 0$$

is replaced by

$$\left.\begin{aligned} x^4 &= \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3, \\ x^3y &= \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3, \\ x^2y^2 &= \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3, \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\},$$

and this, after division and substitution of k for x/y, by

$$\left.\begin{aligned} \lambda_1(a_1 - kb_1) + \lambda_2(a_2 - kb_2) + \lambda_3(a_3 - kb_3) &= 0 \\ \lambda_1(b_1 - kc_1) + \lambda_2(b_2 - kc_2) + \lambda_3(b_3 - kc_3) &= 0 \\ \lambda_1(c_1 - kd_1) + \lambda_2(c_2 - kd_2) + \lambda_3(c_3 - kd_3) &= 0 \\ \lambda_1(d_1 - ke_1) + \lambda_2(d_2 - ke_2) + \lambda_3(d_3 - ke_3) &= 0 \end{aligned}\right\},$$

in which the eliminands are k, λ_1, λ_2, λ_3. Using on the equations of the last set the multipliers

$$|\, b_1c_2d_3 \,|, \quad -|\, a_1c_2d_3 \,|, \quad |\, a_1b_2d_3 \,|, \quad -|\, a_1b_2c_3 \,|,$$

obtainable from columns 1, 2, 3 of the array

$$\begin{matrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 & c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ d_1 & d_2 & d_3 & e_1 & e_2 & e_3 \end{matrix}$$

appearing in the set, we obtain, after addition and division by k,

$$4123 \cdot \lambda_1 + 5123 \cdot \lambda_2 + 6123 \cdot \lambda_3 = 0,$$

where 4123 stands for the determinant whose columns are columns 4, 1, 2, 3 of the said array. Similarly, by taking our multipliers from columns 4, 5, 6, there results

$$1456 \cdot \lambda_1 + 2456 \cdot \lambda_2 + 3456 \cdot \lambda_3 = 0.$$

Again, by going through three similar operations, the sets of columns used being 156, 426, 453, we obtain on addition

$$\begin{aligned}&(-k\cdot 4156\cdot\lambda_1+\ 2156\cdot\lambda_2+\ 3156\cdot\lambda_3)\\+&(\ \ 1426\cdot\lambda_1-k\cdot 5246\cdot\lambda_2+\ 3426\cdot\lambda_3)\\+&(\ \ 1453\cdot\lambda_1+\ 2453\cdot\lambda_2-k\cdot 6453\cdot\lambda_3)\ =\ 0,\end{aligned}$$

which, with the help of our second result, can be cleared of k and becomes

$$(1426+1453)\lambda_1+(2156+2453)\lambda_2+(3156+3426)\lambda_3=0.$$

The desired resultant equation thus is, as in Sylvester's paper of 1863,

$$\begin{vmatrix} 4123 & 1426+1453 & 1456 \\ 5123 & 2453+2156 & 2456 \\ 6123 & 3156+3426 & 3456 \end{vmatrix} = 0.$$

For the next higher case of the problem the corresponding equation is also given.

REISS, M. (1867).

[Beiträge zur Theorie der Determinanten. viii+113 pp. Leipzig.]

The second section (pp. 25–54) of this important memoir is devoted to compound determinants whose elements are equigrade minors having for each row one invariable set of row-numbers, and for each column one invariable set of column-numbers. To this special class belong almost all the compound determinants hitherto met with. From the definition it is evident that every member of the class can without risk of mistake be specified by giving its diagonal term : for example, we can safely put

$$\left|\begin{vmatrix}13\\24\end{vmatrix}\cdot\begin{vmatrix}35\\46\end{vmatrix}\cdot\begin{vmatrix}51\\62\end{vmatrix}\right| \text{ for } \left|\begin{matrix}\begin{vmatrix}13\\24\end{vmatrix} & \begin{vmatrix}13\\46\end{vmatrix} & \begin{vmatrix}13\\62\end{vmatrix}\\ \begin{vmatrix}35\\24\end{vmatrix} & \begin{vmatrix}35\\46\end{vmatrix} & \begin{vmatrix}35\\62\end{vmatrix}\\ \begin{vmatrix}51\\24\end{vmatrix} & \begin{vmatrix}51\\46\end{vmatrix} & \begin{vmatrix}51\\62\end{vmatrix}\end{matrix}\right|$$

and

$$\left|\begin{vmatrix}234\\234\end{vmatrix}\cdot\begin{vmatrix}134\\134\end{vmatrix}\cdot\begin{vmatrix}124\\124\end{vmatrix}\cdot\begin{vmatrix}123\\123\end{vmatrix}\right| \text{ for the adjugate of } \begin{vmatrix}1234\\1234\end{vmatrix}.$$

The first eight pages constitute a general introduction to compound determinants, including much on the subject of notation and nomenclature, the notation being as a rule imperfectly suggestive and somewhat ungainly. The next four pages deal with three already known theorems, namely, first, the theorems regarding the k^{th} compound,—a theorem primarily formulated by Spottiswoode * in 1853, but obtainable also from Sylvester's theorem of March 1851 by putting $n = 0$ (*Hist.*, ii. pp. 204, 196); second, the 'extensional' of this, that is to say, the said theorem of March 1851; and, third, Franke's theorem of 1862 regarding a minor of the k^{th} compound. The last appears in the unattractive form

$$\overset{k}{c}\begin{pmatrix} l_1 & \dots & l_m \\ \lambda_1 & \dots & \lambda_m \end{pmatrix} = \mathrm{EC}^{m-\frac{n-1}{k}} \cdot \overset{k}{\gamma}\begin{pmatrix} l_{m+1} & \dots & l_{\mathrm{N}^0} \\ \lambda_{m+1} & \dots & \lambda_{\mathrm{N}^0} \end{pmatrix},$$

which may be partially explained by saying that C is the generating or basic determinant

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix},$$

that the left-hand member is an m-line minor of the k^{th} compound of C, that this minor's cofactor in the compound is

$$\mathrm{E} \cdot \overset{k}{c}\begin{pmatrix} l_{m+1} & \dots & l_{\mathrm{N}^0} \\ \lambda_{m+1} & \dots & \lambda_{\mathrm{N}^0} \end{pmatrix},$$

that the compound itself is

$$\overset{k}{c}\begin{pmatrix} 1, & 2, & \dots, & \mathrm{N}^0 \\ 1, & 2, & \dots, & \mathrm{N}^0 \end{pmatrix},$$

that E is an undetermined sign-factor, that N^0 stands for

$$n(n-1) \dots (n-k+1)/k!,$$

that $\binom{n-1}{k}$ stands for $(n-1)(n-2) \dots (n-k)/k!$, and that $\overset{k}{\gamma}(\quad)$ stands for the cofactor of $\overset{k}{c}(\quad)$ in C. By way of illustration there is taken the case where $n = 5$, $k = 2$, $m = 7$, and the particular 7-line minor is

$$\begin{vmatrix} \begin{vmatrix} 14 \\ 12 \end{vmatrix} & \begin{vmatrix} 15 \\ 13 \end{vmatrix} & \begin{vmatrix} 24 \\ 14 \end{vmatrix} & \begin{vmatrix} 25 \\ 15 \end{vmatrix} & \begin{vmatrix} 34 \\ 23 \end{vmatrix} & \begin{vmatrix} 35 \\ 24 \end{vmatrix} & \begin{vmatrix} 45 \\ 25 \end{vmatrix} \end{vmatrix}.$$

* Incorrectly, ν' standing for $(n-1)_i$, not for $(n-1)_{i-1}$.

The complementary minor of this in its own compound being

$$-\left|\begin{vmatrix}12\\34\end{vmatrix}\quad\begin{vmatrix}13\\35\end{vmatrix}\quad\begin{vmatrix}23\\45\end{vmatrix}\right|,$$

and in the complementary compound being

$$-\left|\begin{vmatrix}345\\125\end{vmatrix}\quad\begin{vmatrix}245\\124\end{vmatrix}\quad\begin{vmatrix}145\\123\end{vmatrix}\right|,$$

the result obtained from the theorem is

$$-\begin{vmatrix}12345\\12345\end{vmatrix}\cdot\left|\begin{vmatrix}345\\125\end{vmatrix}\quad\begin{vmatrix}245\\124\end{vmatrix}\quad\begin{vmatrix}145\\123\end{vmatrix}\right|.$$

The second factor here, however, being equal to

$$\left|\begin{vmatrix}345\\512\end{vmatrix}\quad\begin{vmatrix}245\\412\end{vmatrix}\quad\begin{vmatrix}145\\312\end{vmatrix}\right|,$$

is observed to be an ‘ extensional ’ of

$$\begin{vmatrix}321\\543\end{vmatrix},$$

and therefore to be equal to

$$\begin{vmatrix}32145\\54312\end{vmatrix}\cdot\begin{vmatrix}45\\12\end{vmatrix}^2.$$

Reiss thereupon goes a little beyond Franke and gives

$$\left|\begin{vmatrix}14\\12\end{vmatrix}\quad\begin{vmatrix}15\\13\end{vmatrix}\quad\begin{vmatrix}24\\14\end{vmatrix}\quad\begin{vmatrix}25\\15\end{vmatrix}\quad\begin{vmatrix}34\\23\end{vmatrix}\quad\begin{vmatrix}35\\24\end{vmatrix}\quad\begin{vmatrix}45\\25\end{vmatrix}\right| = -\begin{vmatrix}12345\\12345\end{vmatrix}^2\cdot\begin{vmatrix}45\\12\end{vmatrix}^2.$$

An investigation then follows (pp. 37–50) for the purpose of enabling us in every case where it is possible to take this additional step and others like it; in other words, for the purpose of resolving any minor of any compound of C into factors which are C and minors of C. It is duly recognised that on the application of Franke's theorem we may not always be led to a minor like

$$\left|\begin{vmatrix}345\\125\end{vmatrix}\quad\begin{vmatrix}245\\124\end{vmatrix}\quad\begin{vmatrix}145\\123\end{vmatrix}\right|,$$

which is fully resolvable,—that, for example, the said application might bring us down to such a compound determinant as

$$\left|\begin{vmatrix}12\\12\end{vmatrix}\quad\begin{vmatrix}34\\34\end{vmatrix}\right|,$$

which is irresolvable, or to such a compound determinant as

$$\left|\begin{vmatrix}12\\12\end{vmatrix}\ \begin{vmatrix}13\\13\end{vmatrix}\ \begin{vmatrix}14\\23\end{vmatrix}\right|,$$

which is equal to 0. Thus, the 9-line coaxial minor

$$\left|\begin{vmatrix}1234\\1234\end{vmatrix}\ \begin{vmatrix}1235\\1235\end{vmatrix}\ \begin{vmatrix}1245\\1245\end{vmatrix}\ \begin{vmatrix}1345\\1345\end{vmatrix}\ \begin{vmatrix}1356\\1356\end{vmatrix}\ \begin{vmatrix}1456\\1456\end{vmatrix}\ \begin{vmatrix}2345\\2345\end{vmatrix}\ \begin{vmatrix}2356\\2356\end{vmatrix}\ \begin{vmatrix}2456\\2456\end{vmatrix}\right|$$

of the 4th compound of $\begin{vmatrix}123456\\123456\end{vmatrix}$ is by Franke's theorem equal to

$$\begin{vmatrix}123456\\123456\end{vmatrix}^{9-(5)_4}\cdot\left|\begin{vmatrix}45\\45\end{vmatrix}\ \begin{vmatrix}35\\35\end{vmatrix}\ \begin{vmatrix}34\\34\end{vmatrix}\ \begin{vmatrix}25\\25\end{vmatrix}\ \begin{vmatrix}15\\15\end{vmatrix}\ \begin{vmatrix}12\\12\end{vmatrix}\right|,$$

and by the same theorem

$$\left|\begin{vmatrix}12\\12\end{vmatrix}\ \begin{vmatrix}15\\15\end{vmatrix}\ \begin{vmatrix}25\\25\end{vmatrix}\ \begin{vmatrix}34\\34\end{vmatrix}\ \begin{vmatrix}35\\35\end{vmatrix}\ \begin{vmatrix}45\\45\end{vmatrix}\right| = \begin{vmatrix}12345\\12345\end{vmatrix}^{6-(4)_2}\cdot\left|\begin{vmatrix}245\\245\end{vmatrix}\ \begin{vmatrix}235\\235\end{vmatrix}\ \begin{vmatrix}145\\145\end{vmatrix}\ \begin{vmatrix}135\\135\end{vmatrix}\right|$$

so that the given minor is equal to

$$\begin{vmatrix}123456\\123456\end{vmatrix}^{4}\cdot\left|\begin{vmatrix}135\\135\end{vmatrix}\ \begin{vmatrix}145\\145\end{vmatrix}\ \begin{vmatrix}235\\235\end{vmatrix}\ \begin{vmatrix}245\\245\end{vmatrix}\right|.$$

When, however, on perceiving that the compound determinant here is an extensional, we try to proceed further by using Franke's theorem on

$$\left|\begin{vmatrix}13\\13\end{vmatrix}\ \begin{vmatrix}14\\14\end{vmatrix}\ \begin{vmatrix}23\\23\end{vmatrix}\ \begin{vmatrix}24\\24\end{vmatrix}\right|,$$

we are brought up against the irresolvable factor

$$\left|\begin{vmatrix}12\\12\end{vmatrix}\ \begin{vmatrix}34\\34\end{vmatrix}\right|,$$

and thus have to be content with the result

$$\begin{vmatrix}123456\\123456\end{vmatrix}^{4}\cdot\begin{vmatrix}12345\\12345\end{vmatrix}\cdot\left|\begin{vmatrix}125\\125\end{vmatrix}\ \begin{vmatrix}345\\345\end{vmatrix}\right|\cdot\begin{vmatrix}5\\5\end{vmatrix},$$

which still contains a compound determinant. Notwithstanding this narrowing of the field, however, pains are taken to classify cases for treatment, to formulate a general result when full resolvability is known to be possible, and to give a rule for anticipating the final factors by counting the number of occurrences of suffixes in the specification of the given determinant. Thus, the result obtained by repeated applications of Franke's theorem as follows :

$$\begin{vmatrix} 123 & 124 & 125 & 126 & 134 & 135 & 136 & 145 & 234 & 235 & 236 & 245 & 345 \\ 123 & 125 & 126 & 134 & 135 & 136 & 145 & 156 & 235 & 236 & 256 & 345 & 356 \end{vmatrix}$$

$$= \begin{vmatrix} 123456 \\ 123456 \end{vmatrix}^3 \cdot \begin{vmatrix} 235 & 234 & 135 & 134 & 125 & 124 & 123 \\ 356 & 235 & 156 & 136 & 135 & 125 & 123 \end{vmatrix}$$

$$= -\begin{vmatrix} 123456 \\ 123456 \end{vmatrix}^3 \cdot \begin{vmatrix} 12345 \\ 12356 \end{vmatrix}^3 \cdot \begin{vmatrix} 23 & 13 & 12 \\ 35 & 15 & 13 \end{vmatrix}$$

$$= -\begin{vmatrix} 123456 \\ 123456 \end{vmatrix}^3 \cdot \begin{vmatrix} 12345 \\ 12356 \end{vmatrix}^3 \cdot \begin{vmatrix} 123 \\ 135 \end{vmatrix}^2,$$

is held to be also involved in the simple fact that the row-suffixes of the generating determinant occur in the given compound determinant

8, 8, 8, 6, 6, 3

times respectively, and the column-suffixes

8, 6, 8, 3, 8, 6

times respectively.

The Third Section (pp. 55–99) is concerned with compound determinants of a different type and with the relations of these to those of the Second Section. An example of the new type is

$$\begin{vmatrix} \begin{vmatrix} 12345 \\ 12567 \end{vmatrix} & \begin{vmatrix} 12345 \\ 12568 \end{vmatrix} & \begin{vmatrix} 12345 \\ 12569 \end{vmatrix} & \begin{vmatrix} 12345 \\ 12578 \end{vmatrix} & \begin{vmatrix} 12345 \\ 12579 \end{vmatrix} & \begin{vmatrix} 12345 \\ 12589 \end{vmatrix} \\ \begin{vmatrix} 12345 \\ 13567 \end{vmatrix} & \begin{vmatrix} 12345 \\ 13568 \end{vmatrix} & \begin{vmatrix} 12345 \\ 13569 \end{vmatrix} & \begin{vmatrix} 12345 \\ 13578 \end{vmatrix} & \begin{vmatrix} 12345 \\ 13579 \end{vmatrix} & \begin{vmatrix} 12345 \\ 13589 \end{vmatrix} \\ \begin{vmatrix} 12345 \\ 14567 \end{vmatrix} & \begin{vmatrix} 12345 \\ 14568 \end{vmatrix} & \begin{vmatrix} 12345 \\ 14569 \end{vmatrix} & \begin{vmatrix} 12345 \\ 14578 \end{vmatrix} & \begin{vmatrix} 12345 \\ 14579 \end{vmatrix} & \begin{vmatrix} 12345 \\ 14589 \end{vmatrix} \\ \begin{vmatrix} 12345 \\ 23567 \end{vmatrix} & \begin{vmatrix} 12345 \\ 23568 \end{vmatrix} & \begin{vmatrix} 12345 \\ 23569 \end{vmatrix} & \begin{vmatrix} 12345 \\ 23578 \end{vmatrix} & \begin{vmatrix} 12345 \\ 23579 \end{vmatrix} & \begin{vmatrix} 12345 \\ 23589 \end{vmatrix} \\ \begin{vmatrix} 12345 \\ 24567 \end{vmatrix} & \begin{vmatrix} 12345 \\ 24568 \end{vmatrix} & \begin{vmatrix} 12345 \\ 24569 \end{vmatrix} & \begin{vmatrix} 12345 \\ 24578 \end{vmatrix} & \begin{vmatrix} 12345 \\ 24579 \end{vmatrix} & \begin{vmatrix} 12345 \\ 24589 \end{vmatrix} \\ \begin{vmatrix} 12345 \\ 34567 \end{vmatrix} & \begin{vmatrix} 12345 \\ 34568 \end{vmatrix} & \begin{vmatrix} 12345 \\ 34569 \end{vmatrix} & \begin{vmatrix} 12345 \\ 34578 \end{vmatrix} & \begin{vmatrix} 12345 \\ 34579 \end{vmatrix} & \begin{vmatrix} 12345 \\ 34589 \end{vmatrix} \end{vmatrix},$$

the essential characteristics being that the set of row-suffixes never alters throughout the whole of the elements, and the set of column-suffixes attached to any element contains a fixed number of suffixes which remain invariable throughout the row and a fixed number which remain invariable throughout the column to which the element belongs. Such a determinant cannot any longer be safely denoted by its diagonal term ; but if we insert some mark to separate

each set of column-suffixes into its two parts, we obtain a quite simple and satisfactory notation: for example, the above determinant may be appropriately denoted by

$$\left|\begin{vmatrix}12\,345\\12,567\end{vmatrix}\ \begin{vmatrix}12\,345\\13,568\end{vmatrix}\ \begin{vmatrix}12\,345\\14,569\end{vmatrix}\ \begin{vmatrix}12\,345\\23,578\end{vmatrix}\ \begin{vmatrix}12\,345\\24,579\end{vmatrix}\ \begin{vmatrix}12\,345\\34,589\end{vmatrix}\right|,$$

if the requisite convention be adopted with reference to the use of the comma. Both kinds of compound determinants appear in the simple identity

$$\begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}\cdot\left|\begin{vmatrix}b_4c_5|&|c_4a_5|&|a_4b_5|\\|b_6c_7|&|c_6a_7|&|a_6b_7|\\|b_8c_9|&|c_8a_9|&|a_8b_9|\end{vmatrix}\right|=\left|\begin{vmatrix}a_1b_4c_5|&|a_1b_6c_7|&|a_1b_8c_9|\\|a_2b_4c_5|&|a_2b_6c_7|&|a_2b_8c_9|\\|a_3b_4c_5|&|a_3b_6c_7|&|a_3b_8c_9|\end{vmatrix}\right|,$$

and this in the notations referred to is

$$|a_1b_2c_3|\cdot\left|\begin{matrix}|b_4c_5&|c_6a_7&|a_8b_9|\end{matrix}\right|=\left|\begin{matrix}|a_1,b_4c_5&|a_2,b_6c_7&|a_3,b_8c_9\end{matrix}\right|.$$

On occasion the invariable set of row-suffixes may be temporarily dispensed with: this is in effect what is done when the principal minors of an oblong array are denoted by the numbers of their columns only. It is the latter contracted notation which was used in 1851 by Bazin in his statement of the first theorem regarding such determinants (*Hist.*, ii. pp. 206–208).

Reiss' notation is of course very different from the foregoing, both in principle and in appearance, and considerable space is again devoted to it. Six pages have to be gone through before his first theorem is reached. This theorem, however, is fundamental, and deserves every attention. It may be viewed as giving the resolution of a special compound determinant of the second kind into two factors which are compound determinants of the first kind. We shall take it in a slightly different light, and for the sake of clearness give a formal enunciation of it in words, detailing fully the construction of the determinants involved:—*If the combinations of* n *letters* k *at a time be taken, and each combination with any* k *suffixes be used to denote a determinant, and the whole of such determinants be made the diagonal of a compound determinant and be understood to suggest as usual the other elements of the latter, then,* D *being this determinant and* D′ *a determinant similar in construction save that the combinations are taken* n—k *at a time and are furnished from a*

different collection of suffixes, the product DD′ *is expressible as a compound determinant in which the* (i, j)th *element is a determinant denoted by all the* n *letters accompanied by the* ith *of the first sets of suffixes and the* (n−j)th *of the second sets.* The proof is a mere application of the multiplication-theorem, with the assistance of Laplace's expansion-theorem. Thus, if $n=5$, $k=2$ and the letters be a, b, c, d, e, we first form the ten combinations ab, ac, . . . , de, then the determinants $|\,a_1b_2\,|$, $|\,a_3c_4\,|$, . . . by taking any suffixes whatever, and finally from these the compound determinant

$$\left|\,|\,a_1b_2\,|\quad|\,a_3c_4\,|\quad|\,a_5d_6\,|\,\ldots\,\right|,$$

so that we have

$$\mathrm{D} \equiv \left|\begin{array}{ccccc} |\,a_1b_2\,| & |\,a_1c_2\,| & |\,a_1d_2\,| & \ldots & |\,d_1e_2\,| \\ |\,a_3b_4\,| & |\,a_3c_4\,| & |\,a_3d_4\,| & \ldots & |\,d_3e_4\,| \\ |\,a_5b_6\,| & |\,a_5c_6\,| & |\,a_5d_6\,| & \ldots & |\,d_5e_6\,| \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right|_{10}.$$

Similarly, we form the combinations abc, abd, . . . , cde, writing them for present convenience in the reverse order, then the determinants $\begin{vmatrix} c\,d\,e \\ 1'\,2'\,3' \end{vmatrix}$, $\begin{vmatrix} b\,d\,e \\ 4'\,5'\,6' \end{vmatrix}$, . . . , and finally have

$$\mathrm{D}' \equiv \left|\begin{array}{ccccc} \begin{vmatrix} c\,d\,e \\ 1'\,2'\,3' \end{vmatrix} & \begin{vmatrix} b\,d\,e \\ 4'\,5'\,6' \end{vmatrix} & \begin{vmatrix} b\,c\,e \\ 7'\,8'\,9' \end{vmatrix} & \ldots \end{array}\right|$$

$$\equiv \left|\begin{array}{ccccc} \begin{vmatrix} c\,d\,e \\ 1'\,2'\,3' \end{vmatrix} & \begin{vmatrix} b\,d\,e \\ 1'\,2'\,3' \end{vmatrix} & \begin{vmatrix} b\,c\,e \\ 1'\,2'\,3' \end{vmatrix} & \ldots \\ \begin{vmatrix} c\,d\,e \\ 4'\,5'\,6' \end{vmatrix} & \begin{vmatrix} b\,d\,e \\ 4'\,5'\,6' \end{vmatrix} & \begin{vmatrix} b\,c\,e \\ 4'\,5'\,6' \end{vmatrix} & \ldots \\ \begin{vmatrix} c\,d\,e \\ 7'\,8'\,9' \end{vmatrix} & \begin{vmatrix} b\,d\,e \\ 7'\,8'\,9' \end{vmatrix} & \begin{vmatrix} b\,c\,e \\ 7'\,8'\,9' \end{vmatrix} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}\right|.$$

The result of the multiplication of these, after altering the signs of the 2nd, 4th, 6th, 9th columns in one of them, is

$$\mathrm{DD}' = \left|\begin{array}{ccc} \begin{vmatrix} a\,b\,c\,d\,e \\ 1\,2\,1'\,2'\,3' \end{vmatrix} & \begin{vmatrix} a\,b\,c\,d\,e \\ 1\,2\,4'\,5'\,6' \end{vmatrix} & \ldots \\ \begin{vmatrix} a\,b\,c\,d\,e \\ 3\,4\,1'\,2'\,3' \end{vmatrix} & \begin{vmatrix} a\,b\,c\,d\,e \\ 3\,4\,4'\,5'\,6' \end{vmatrix} & \ldots \\ \ldots & \ldots & \ldots \end{array}\right|_{10}.$$

To state the general theorem in symbols is not at all an easy matter: Reiss puts it (p. 61) in the form

$$c^{nk}_{\nu\kappa}\begin{pmatrix} A & a & l_1 & \dots & l_{K^0} \\ & a' & \lambda_1 & \dots & \lambda_{K^0} \end{pmatrix}$$

$$= si\begin{pmatrix} K^0 \dots 1 \\ 1 \quad \dots K^0 \end{pmatrix}(-1)^{\frac{1}{2}k\kappa K^0} \cdot c^{k}\begin{pmatrix} k+\kappa & A & 1 & \dots & K^0 \\ n & a' & l_1 & \dots & l_{K^0} \end{pmatrix} \cdot c^{\kappa}\begin{pmatrix} k+\kappa & A & 1 & \dots & K^0 \\ \nu & a' & \lambda_1 & \dots & \lambda_{K^0} \end{pmatrix},$$

where the left-hand member is the compound determinant

$$\begin{vmatrix} c^{nk}_{\nu\kappa}\begin{pmatrix} A & a & l_1 \\ & a' & \lambda_1 \end{pmatrix} & c^{nk}_{\nu\kappa}\begin{pmatrix} A & a & l_1 \\ & a' & \lambda_2 \end{pmatrix} & \dots \\ c^{nk}_{\nu\kappa}\begin{pmatrix} A & a & l_2 \\ & a' & \lambda_1 \end{pmatrix} & c^{nk}_{\nu\kappa}\begin{pmatrix} A & a & l_2 \\ & a' & \lambda_2 \end{pmatrix} & \dots \\ \dots & \dots & \dots \end{vmatrix},$$

whose typical element $c^{nk}_{\nu\kappa}\begin{pmatrix} A & a & r \\ & a' & \rho \end{pmatrix}$ stands for

$$\begin{pmatrix} A_1 & \dots & \dots & \dots & A_{k+\kappa} \\ a_{r_1} & \dots & a_{r_k} & a_{\rho_1} \dots & a_{\rho_\kappa} \end{pmatrix}.$$

If instead of the ten entirely different pairs of suffixes

$$12,\ 34,\ 56,\ \dots\ .$$

there were taken the ten pairs

$$12,\ 13,\ 14,\ \dots\ ,$$

which are combinations of 1, 2, 3, 4, 5, the compound determinant reached, D, would be the second compound of $|\, a_1b_2c_3d_4e_5\,|$, and could therefore be replaced by $|\, a_1b_2c_3d_4e_5\,|^4$. Similarly, if the sets of suffixes $1'2'3'$, $4'5'6'$, ... were the triads obtainable from $1'$, $2'$, $3'$, $4'$, $5'$, we should have

$$D' = \begin{vmatrix} a & b & c & d & e \\ 1'2'3' & 4'5' & & & \end{vmatrix}^6;$$

and if both these restrictions in the selection of suffixes were made, the resulting special case of the theorem would be, on leaving out the common set of row-letters,

$$\big|\,|\,12,1'2'3'\,|\;|\,13,1'2'4'\,|\,\dots|\,45,3'4'5'\,|\,\big| = |\,12345\,|^4 \cdot |\,1'2'3'4'5'6'\,|^6.$$

This exemplifies a very interesting theorem stated by Reiss (p. 62) in the form

$$c^{k+\kappa\ k}_{k+\kappa\ \kappa}\begin{pmatrix} A & a & 1 \dots K^0 \\ & a' & 1 \dots K^0 \end{pmatrix} = E \cdot \begin{pmatrix} A & 1 \dots k+\kappa \\ a' & 1 \dots k+\kappa \end{pmatrix}^{\frac{(k+\kappa-1)}{\kappa}} \begin{pmatrix} A & 1 \dots k+\kappa \\ a' & 1 \dots k+\kappa \end{pmatrix}^{\frac{(k+\kappa-1)}{k}},$$

where

$$E = si\begin{pmatrix} K^0 & \dots & 1 \\ 1 & \dots & K^0 \end{pmatrix}(-1)^{\frac{1}{2}k\kappa K^0},$$

but apparently altogether lost sight of until rediscovered and enunciated in a very different guise by Picquet in 1878. It should be noted, however, that as regards one case of it Reiss had been forestalled by Bazin in 1851 (*Hist.*, ii. pp. 206–208). The exact connection between the two theorems, Bazin's and the so-called Picquet's, is probably best expressed by saying that, if we specialize Bazin's by removing the 'extension' and specialize Picquet's by putting $k = 1$, we shall reach the same result.

The general theorem, Reiss notes, may be sometimes applied with success in treating the problem of resolvability put forward in the Second Section, the appropriate cases of course being determinants of the type to which the two factors in the theorem belong. Thus the determinant

$$\begin{vmatrix} \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \end{vmatrix} & \begin{vmatrix} a & b & d \\ 1 & 2 & 4 \end{vmatrix} & \begin{vmatrix} a & c & d \\ 1 & 5 & 6 \end{vmatrix} & \begin{vmatrix} b & c & d \\ 7 & 8 & 9 \end{vmatrix} \end{vmatrix}$$

being of this type, we take a determinant of the twin type

$$\begin{vmatrix} d & c & b & a \\ 4 & 3 & 2 & 1 \end{vmatrix},$$

and finding that their product is equal to

$$\begin{vmatrix} \begin{vmatrix} a & b & c & d \\ 1 & 2 & 3 & 4 \end{vmatrix} & . & . & . \\ & \begin{vmatrix} a & b & c & d \\ 1 & 2 & 4 & 3 \end{vmatrix} & . & . \\ \begin{vmatrix} a & b & c & d \\ 1 & 5 & 6 & 4 \end{vmatrix} & \begin{vmatrix} a & b & c & d \\ 1 & 5 & 6 & 3 \end{vmatrix} & \begin{vmatrix} a & b & c & d \\ 1 & 5 & 6 & 2 \end{vmatrix} & . \\ \begin{vmatrix} a & b & c & d \\ 7 & 8 & 9 & 4 \end{vmatrix} & \begin{vmatrix} a & b & c & d \\ 7 & 8 & 9 & 3 \end{vmatrix} & \begin{vmatrix} a & b & c & d \\ 7 & 8 & 9 & 2 \end{vmatrix} & \begin{vmatrix} a & b & c & d \\ 7 & 8 & 9 & 1 \end{vmatrix} \end{vmatrix},$$

we have

$$\begin{vmatrix} \begin{vmatrix} a & b & c \\ 1 & 2 & 3 \end{vmatrix} & \begin{vmatrix} a & b & d \\ 1 & 2 & 4 \end{vmatrix} & \begin{vmatrix} a & c & d \\ 1 & 5 & 6 \end{vmatrix} & \begin{vmatrix} b & c & d \\ 7 & 8 & 9 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} a & b & c & d \\ 1 & 2 & 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} a & b & c & d \\ 1 & 2 & 5 & 6 \end{vmatrix} \cdot \begin{vmatrix} a & b & c & d \\ 1 & 7 & 8 & 9 \end{vmatrix}.$$

Similarly it may be shown that

$$\begin{vmatrix} \begin{vmatrix}1234\\1234\end{vmatrix} & \begin{vmatrix}1235\\123a\end{vmatrix} & \begin{vmatrix}1245\\12\beta\gamma\end{vmatrix} & \begin{vmatrix}1345\\1\delta\epsilon\xi\end{vmatrix} & \begin{vmatrix}2345\\\eta\theta\iota\kappa\end{vmatrix} \end{vmatrix} = \begin{vmatrix}12345\\1234a\end{vmatrix} \cdot \begin{vmatrix}12345\\123\beta\gamma\end{vmatrix} \cdot \begin{vmatrix}12345\\12\delta\epsilon\xi\end{vmatrix} \cdot \begin{vmatrix}12345\\1\eta\theta\iota\kappa\end{vmatrix},$$

and so on in general, a first requirement being that the k of the theorem be equal to $n-1$. Again, the given determinant being

$$\begin{vmatrix} \begin{vmatrix}12\\12\end{vmatrix} & \begin{vmatrix}13\\13\end{vmatrix} & \begin{vmatrix}14\\14\end{vmatrix} & \begin{vmatrix}23\\56\end{vmatrix} & \begin{vmatrix}24\\57\end{vmatrix} & \begin{vmatrix}34\\58\end{vmatrix} \end{vmatrix},$$

Reiss selects the twin form

$$\begin{vmatrix} \begin{vmatrix}34\\56\end{vmatrix} & \begin{vmatrix}24\\26\end{vmatrix} & \begin{vmatrix}23\\25\end{vmatrix} & \begin{vmatrix}14\\16\end{vmatrix} & \begin{vmatrix}13\\15\end{vmatrix} & \begin{vmatrix}12\\12\end{vmatrix} \end{vmatrix},$$

which, being a second compound, is equal to

$$\begin{vmatrix}1234\\1256\end{vmatrix}^3,$$

and, finding that the product of the two has a vanishing column, concludes that

$$\begin{vmatrix} \begin{vmatrix}12\\12\end{vmatrix} & \begin{vmatrix}13\\13\end{vmatrix} & \begin{vmatrix}14\\14\end{vmatrix} & \begin{vmatrix}23\\56\end{vmatrix} & \begin{vmatrix}24\\57\end{vmatrix} & \begin{vmatrix}34\\58\end{vmatrix} \end{vmatrix} = 0.$$

Much of the space that follows (pp. 67–89) is occupied with the question of the vanishing of compound determinants of the two kinds, the requisite criteria being found for a number of special forms. The results are mainly deductions from the fundamental theorem or its corollaries.

The concluding portion (pp. 89–99) of the Section is devoted to the establishment of a theorem regarding any minor of the determinant

$$c\begin{matrix}k+\kappa & k\\ k+\kappa & \kappa\end{matrix}\left(\mathrm{A}\begin{matrix}a & 1 & \ldots & \mathrm{K}^0\\ a' & 1 & \ldots & \mathrm{K}^0\end{matrix}\right),$$

which we have seen to be the subject of the so-called Picquet's theorem. The result obtained is related to the latter theorem exactly as Franke's is related to Sylvester's, and, as Reiss points out, degenerates into Franke's theorem simultaneously with the degeneration of Picquet's into Sylvester's.

In an Appendix (pp. 100–113) Reiss makes a variety of applications to Geometry.

KRONECKER, L. (1869).

[Bemerkungen zur Determinanten-Theorie. *Crelle's Journ.*, lxxii. pp. 152–175; or *Werke*, i. pp. 237–269.]

The theorem which the first of the notes concerns is § 4. 7 of Baltzer's second edition (see above, p. 14), and is to the effect that *if the determinant* $|\,a_{11}a_{22}\ldots a_{mm}\,|$ *be bordered in* n^2 *different ways, the typical resulting determinant being*

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1m} & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2m} & a_{2k} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mm} & a_{mk} \\ a_{h1} & a_{h2} & \ldots & a_{hm} & \cdot \end{vmatrix} \quad \textit{or } B_{hk}, \textit{ say,}$$

where h *and* k *are any one of the first* n *integers and* $n > m$, *then all the* $(m+1)$*-line minors of* $|\,B_{1n}\,|$ *must vanish.* The proof consists in expressing each B in terms of the elements of its last row and their complementary minors, and then pointing out that any one of the $(m+1)$-line minors in question, as thus altered, can be seen to be expressible as the product of two vanishing determinants.

Hesse's compound determinant like $|\,B_{1n}\,|$ may be worth recalling (*Hist.*, ii. pp. 130–132).

HUNYADY, E. (1875, 1876).

[A kúpszeleten fekvö hat pont feltételi egyenletének különbözö alakjairól. *Értekezések a math. tudományok köréből* (Budapest), iv. 6 (23 pp.)]

[Ueber die verschiedenen Formen der Bedingungsgleichung, welche ausdrückt, dass sechs Punkte auf einem Kegelschnitte liegen. *Crelle's Journ.*, lxxxiii. pp. 76–85.]

What concerns us here is a three-line determinant which is *doubly* compound, that is to say, a determinant whose elements are themselves compound determinants. If we put

$$\left.\begin{array}{ll} X_{h,k} & \text{for } |\,y_h z_k\,| \\ Y_{h,k} & \text{for } |\,z_h x_k\,| \\ Z_{h,k} & \text{for } |\,x_h y_k\,| \end{array}\right\} \text{ and } |\,Y_{hk}Z_{mn}\,| \text{ for } Y_{hk}Z_{mn} - Y_{mn}Z_{hk},$$

it may be written

$$\begin{vmatrix} |Y_{12}Z_{45}| & |Z_{12}X_{45}| & |X_{12}Y_{45}| \\ |Y_{34}Z_{61}| & |Z_{34}X_{61}| & |X_{34}Y_{61}| \\ |Y_{56}Z_{23}| & |Z_{56}X_{23}| & |X_{56}Y_{23}| \end{vmatrix}$$ and be denoted by Δ.

Multiplying rowwise by

$$\begin{vmatrix} X_{12} & Y_{12} & Z_{12} \\ X_{34} & Y_{34} & Z_{34} \\ X_{45} & Y_{45} & Z_{45} \end{vmatrix},$$

we obtain

$$\Delta \cdot |X_{12}Y_{34}Z_{45}| = \begin{vmatrix} & |X_{12}Y_{45}Z_{34}| & \\ |X_{34}Y_{61}Z_{12}| & & |X_{34}Y_{61}Z_{45}| \\ |X_{56}Y_{23}Z_{12}| & |X_{56}Y_{23}Z_{34}| & |X_{56}Y_{23}Z_{45}| \end{vmatrix},$$

whence

$$\begin{aligned} \Delta &= \begin{vmatrix} |X_{34}Y_{61}Z_{12}| & |X_{34}Y_{61}Z_{45}| \\ |X_{56}Y_{23}Z_{12}| & |X_{56}Y_{23}Z_{45}| \end{vmatrix} \\ &= \begin{vmatrix} |x_1y_2z_6| \cdot |x_1y_3z_4| & -|x_3y_4z_5| \cdot |x_1y_4z_6| \\ -|x_1y_2z_3| \cdot |x_2y_5z_6| & |x_4y_5z_6| \cdot |x_2y_3z_5| \end{vmatrix} \\ &= |x_1y_2z_6| \cdot |x_1y_3z_4| \cdot |x_2y_3z_5| \cdot |x_4y_5z_6| \\ &\quad -|x_1y_2z_3| \cdot |x_1y_4z_6| \cdot |x_2y_5z_6| \cdot |x_3y_4z_5|. \end{aligned}$$

The vanishing of the left-hand member is in co-ordinate geometry the expression of Pascal's theorem, and the vanishing of the right-hand member is the analogous equivalent of a theorem of Pappus or Desargues. The proof of the identity of the two members may be interestingly compared with Cayley's procedure of 1843 (*Hist.*, ii. pp. 10–13).

Using other multipliers than $|X_{12}Y_{34}Z_{45}|$, for example

$$|X_{12}Y_{34}Z_{61}|, \quad |X_{12}Y_{23}Z_{56}|, \quad \ldots,$$

Hunyady obtains the requisite variety of similar expressions for Δ.

It has also to be noted that by taking a multiplier in which no suffix is repeated an equally interesting result is reached: for example

$$\begin{aligned} \Delta \cdot |X_{12}Y_{34}Z_{56}| &= \overline{125} \cdot \overline{126} \cdot \overline{134} \cdot 234 \cdot \overline{356} \cdot \overline{456} \\ &\quad - \overline{123} \cdot \overline{124} \cdot 156 \cdot \overline{256} \cdot \overline{345} \cdot \overline{346}, \end{aligned}$$

where 125, 126, ... stand for $|x_1y_2z_5|$, $|x_1y_2z_6|$, ...

SCHRÖDER, E. (1875).

[Ueber v. Staudt's Rechnung mit Würfen und verwandte Processe. *Math. Annalen*, x. pp. 289–317.]

Schröder's second subsidiary theorem connected with determinants is given (§ 7. pp. 301–305) as a result in elimination, and is to the effect that the eliminant of the set of equations

$$\left.\begin{vmatrix} . & x_1 & x_2 & \dots & x_n \\ b_{1s} & a_{11} & a_{12} & \dots & a_{1n} \\ b_{2s} & a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . & . \\ b_{ns} & a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = 0 \right\}_{s=1}^{s=n}$$

is $|\, b_{11}b_{22}\dots b_{nn}\,|$. If each equation of the set were written in the usual way with each unknown accompanied only by its own coefficient, the eliminant would of course appear as a compound determinant, and would on examination be found to be of the type dealt with by Bazin in 1851, and thus equal to

$$|\, b_{11}b_{22}\dots b_{nn}\,| \cdot |\, a_{11}a_{22}\dots a_{nn}\,|^{n-1}.$$

Schröder's procedure is of a different character.

D'OVIDIO, E. (1876, 1877).

[Nota sui determinanti di determinanti. *Atti ... Accad. delle Sci.* (Torino), xi. pp. 949–956.]

[Addizioni alla nota sui determinanti di determinanti. *Atti ... Accad. delle Sci.* (Torino), xii. pp. 331–333.]

The results in D'Ovidio's 'Note' are mainly rediscoveries, as he himself points out in the 'Additions.' One of them, however, is less manifestly so than the others, and deserves enunciation in its new form, namely, *If in every possible way* μ *of the last* λ *rows and* μ *of the last* λ *columns of* $|\, a_{1n}\,|$, *where of course* $n > \lambda > \mu$, *be deleted, the determinant whose elements are the minors thus resulting is equal to*

$$|\, a_{1,\,n-\lambda}\,|^{(\lambda-1)\mu-1} \cdot |\, a_{1n}\,|^{(\lambda-1)\mu}.$$

The elements in question being of the $(n-\mu)^{\text{th}}$ order, the said determinant of them must be a minor of the $(n-\mu)^{\text{th}}$ compound of $|\, a_{1n}\,|$, and as such is noteworthy in being expressible in terms of a power

of the 'generating' determinant $|\,a_{1n}\,|$ and a power of a minor of the latter. A little examination will show that the theorem is an 'extensional,' and further inquiry will also make clear that it is not essentially different from Sylvester's of March 1851 (*Hist.*, ii. pp. 194–196), the m, n, r in the latter corresponding to λ, $n-\lambda$, $\lambda-\mu$.

On the other hand, the companion theorem of § iii. (pp. 955–956) is new. We may formulate it for ourselves as follows: *If every possible μ-line minor of* $|\,a_{1n}\,|$ *be taken whose rows are not all included in the last* λ *rows and whose columns are not all included in the last* λ *columns, the determinant of these minors is equal to*

$$|\,a_{1,n-\lambda}\,|^{(\lambda-1)_\mu-1}\cdot|\,a_{1n}\,|^{(n-1)_\mu-1-(\lambda-1)_\mu-1}.$$

Here, as in the other, the determinant dealt with is a minor of a compound of $|\,a_{1n}\,|$, the compound being now the μ^{th} and the order of its minor $(n)_\mu-(\lambda)_\mu$; whereas, in the preceding theorem, the compound is the $(n-\mu)^{\text{th}}$ and the order of its minor $(\lambda)_\mu$.

In his 'additions' D'Ovidio passes on to a third determinant got from that of the immediately preceding theorem by substituting for each element its complementary; this, he says, is equal to

$$|\,a_{n-\lambda+1,n}\,|^{(\lambda-1)_\mu-1}\cdot|\,a_{1n}\,|^{(n-1)_\mu-(\lambda)_\mu},$$

where $|\,a_{n-\lambda+1,n}\,|$ is the complementary of $|\,a_{1,n-\lambda}\,|$. He also gives the ratios of two pairs of other determinants; namely, the determinant, F say, whose every element is obtained by deleting μ rows and μ columns belonging either all to the last λ rows and λ columns of $|\,a_{1n}\,|$ or all to the first $n-\lambda$ rows and $n-\lambda$ columns; next, the determinant, G say, with elements obtained in similar fashion, but the condition of choice now being that at least one row and one column shall belong to each of the two specified sets; and, finally, the two determinants, F′, G′ say, got from these by substituting for each element its complementary. The ratios given are

$$\mathrm{G}'/\mathrm{F} = |\,a_{1n}\,|^{(n-1)_\mu-1-(\lambda)_\mu-(n-\lambda)_\mu},$$
$$\mathrm{G}/\mathrm{F}' = |\,a_{1n}\,|^{(n-1)_\mu-(\lambda)_\mu-(n-\lambda)_\mu}.$$

The order of F is manifestly $(\lambda)_\mu+(n-\lambda)_\mu$, and that of G

$$(n)_\mu-(\lambda)_\mu-(n-\lambda)_\mu.$$

TANNER, H. W. LLOYD (1877).

[Question 5247. *Educ. Times*, xxx. p. 20; *Math. from Educ. Times*, xxviii. pp. 41–43.]

The theorem here proved is not essentially different from Bazin's of 1851, but the proofs given are both new, and the second, by G. S. Carr, is noteworthy. When applied to the first example given by us under Bazin (*Hist.*, ii. pp. 206–208) and somewhat improved, it is as follows, $|A_1B_2C_3D_4|$ being the adjugate of 1234:

$$\begin{vmatrix} 5234 & 6234 \\ 1534 & 1634 \end{vmatrix} = \begin{vmatrix} a_5A_1+b_5B_1+c_5C_1+d_5D_1 & a_6A_1+b_6B_1+c_6C_1+d_6D_1 \\ a_5A_2+b_5B_2+c_5C_2+d_5D_2 & a_6A_2+b_6B_2+c_6C_2+d_6D_2 \end{vmatrix}$$

$$= \left\| \begin{matrix} A_1B_1C_1D_1 \\ A_2B_2C_2D_2 \end{matrix} \right\| . \left\| \begin{matrix} a_5b_5c_5d_5 \\ a_6b_6c_6d_6 \end{matrix} \right\|$$

$$= |A_1B_2| \cdot |a_5b_6| + \ldots . + |C_1D_2| \cdot |c_5d_6|$$

$$= |a_1b_2c_3d_4| \{ |c_3d_4| \cdot |a_5b_6| + \ldots . + |a_3b_4| \cdot |c_5d_6| \}$$

$$= |a_1b_2c_3d_4| \cdot |a_5b_6c_3d_4|.$$

MERTENS, F. (1877).

[Sätze über Determinanten und Anwendung derselben zum Beweise der Sätze von Pascal und Brianchon. *Crelle's Journ.*, lxxxiv. pp. 355–359.]

Mertens might quite appropriately have adopted the title of Hunyady's paper (1875–6), their common subject being the eliminant-forms connected with six points on a conic. The one form dealt with by both writers is the doubly compound determinant

$$\left| |Y_{12}Z_{45}| \; |Z_{23}X_{56}| \; |X_{34}Y_{61}| \right| \text{ or } \Delta.$$

The other forms which Mertens deals with are

$$\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & y_1z_1 & z_1x_1 & x_1y_1 \\ x_2^2 & y_2^2 & z_2^2 & y_2z_2 & z_2x_2 & x_2y_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_6^2 & y_6^2 & z_6^2 & y_6z_6 & z_6x_6 & x_6y_6 \end{vmatrix} \text{ or, P say,}$$

and

$$\begin{vmatrix} x_1x_2 & y_1y_2 & z_1z_2 & y_1z_2+y_2z_1 & z_1x_2+z_2x_1 & x_1y_2+x_2y_1 \\ x_2x_3 & y_2y_3 & z_2z_3 & y_2z_3+y_3z_2 & z_2x_3+z_3x_2 & x_2y_3+x_3y_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_6x_1 & y_6y_1 & z_6z_1 & y_6z_1+y_1z_6 & z_6x_1+z_1x_6 & x_6y_1+x_1y_6 \end{vmatrix} \text{ or, Q say;}$$

and the relations established by him are

$$\Delta = -P = Q.$$

The basis of this relationship he finds to be the fact that all three expressions are homogeneous and of the second degree in each of the six sets of coordinates, and all vanish if one set of coordinates be made the same as another set. That Q vanishes on putting

$$x_5, \quad y_5, \quad z_5 = x_1, \quad y_1, \quad z_1,$$

he proves by performing the operation

$$\text{row}_1 \cdot \overline{346} - \text{row}_3 \cdot \overline{246} + \text{row}_4 \cdot \overline{236} - \text{row}_5 \cdot \overline{234},$$

the first row then becoming a row of zeros.

He also gives an interesting result regarding a quasi-adjugate of P, namely, that *if in* P *there be substituted for the elements of* $|x_1y_2z_3|$ *the elements of the adjugate* $|X_1Y_2Z_3|$ *and for the elements of* $|x_4y_5z_6|$ *the elements of the adjugate* $|X_4Y_5Z_6|$, *the resulting* P *is* $-|x_1y_2z_3|^2 \, |x_4y_5z_6|^2$ *times the original* P. This he establishes by making the substitution not in P, but in its equivalent $-\Delta$.

SCHOLTZ, A. (1877–8).

[Sechs Punkte eines Kegelschnittes. *Archiv d. Math. u. Phys.*, lxii. (1878), pp. 317–324; or, in Magyar, *Müegyetemi Lapok*, ii. (1877), pp. 65– .]

Scholtz's object is quite similar to that of Hunyady and Mertens, neither of whom he mentions. He brings forward no new eliminant-form, but his exposition is specially clear and full, and his mode of establishing the identity of the expressions which we have denoted by

$$-P, \quad \Delta, \quad D$$

is his own. As auxiliaries he uses, after proof, the case of Schläfli's theorem of 1851, in which $n, r = 3, 2$, and the theorem concerning a 3-by-6 array,

$$\begin{vmatrix} |y_\alpha z_\beta| & |z_\alpha x_\beta| & |x_\alpha y_\beta| \\ |y_\gamma z_\delta| & |z_\gamma x_\delta| & |x_\gamma y_\delta| \\ |y_\epsilon z_\zeta| & |z_\epsilon x_\zeta| & |x_\epsilon y_\zeta| \end{vmatrix} = |x_\alpha y_\beta z_\epsilon| \cdot |x_\gamma y_\delta z_\zeta| - |x_\alpha y_\beta z_\zeta| \cdot |x_\gamma y_\delta z_\epsilon|,$$

a special case of which had been employed by Mertens, and which in general form is probably due to Hunyady (see below).

FROBENIUS, G. (1878).

[Ueber die schiefe Invariante einer bilinearen oder quadratischen Form. *Crelle's Journ.*, lxxxvi. pp. 44–71.]

By 'Sylvester's Determinantensatz,' to which § 3 (pp. 53–54) is devoted, Frobenius means the first theorem of March 1851. The proof adduced is far from attractive: thus, in the case of the example before given by us (*Hist.*, ii. p. 61), he has to multiply

$$|\, a_1 b_2 c_3 d_4 e_5 f_6 g_7 \,|$$

laboriously by $|\, a_1 b_2 c_3 \,|^6$ in the form

$$\begin{array}{|cccccccc|}
|\, b_2c_3 \,| & -|\, b_1c_3 \,| & |\, b_1c_2 \,| & . & . & . & . \\
|\, a_2c_3 \,| & |\, a_1c_3 \,| & -|\, a_1c_2 \,| & . & . & . & . \\
a_2b_3 \,| & -|\, a_1b_3 \,| & |\, a_1b_2 \,| & . & . & . & . \\
{}_4b_2c_3 \,| & |\, a_1b_4c_3 \,| & |\, a_1b_2c_4 \,| & -|\, a_1b_2c_3 \,| & . & . & . \\
{}_5b_2c_3 \,| & |\, a_1b_5c_3 \,| & |\, a_1b_2c_5 \,| & . & -|\, a_1b_2c_3 \,| & . & . \\
{}_6b_2c_3 \,| & |\, a_1b_6c_3 \,| & |\, a_1b_2c_6 \,| & . & . & -|\, a_1b_2c_3 \,| & . \\
{}_7b_2c_3 \,| & |\, a_1b_7c_3 \,| & |\, a_1b_2c_7 \,| & . & . & . & -|\, a_1b_2c_3 \,|
\end{array}$$

and then divide the result by $|\, a_1 b_2 c_3 \,|^3$.

PICQUET, H. (1878).

[Sur le déterminant dont les éléments sont tous les mineurs possibles d'ordre donné d'un déterminant donné. *Comptes Rendus* . . . *Acad. des. Sci.* (Paris), lxxxvi. pp. 310–312.]

[Analyse combinatoire des déterminants. *Comptes Rendus* . . . *Acad. des Sci.* (Paris), lxxxvi. pp. 1118–1119.]

[Mémoire sur l'application du calcul des combinaisons à la théorie des déterminants. *Journ. de l'Éc. Polyt.*, Cah. xlv. pp. 201–243.]

Picquet's short communications to the Academy of Sciences contain the enunciations of nine theorems at first supposed by him to be all new: the memoir, on the other hand, is an exposition of what he believed to be the whole subject, with his own theorems set in their appropriate places. It (the memoir) is orderly arranged and clearly written, and, although among the more important of

the writer's predecessors Reiss and D'Ovidio are lost sight of, it remains a valuable monograph.

In the first twelve pages there is little that is fresh. We may note merely that formal proof is given (pp. 205–206) of the proposition that *the* m^{th} *compound of* $|\,a_{1n}\,|$ *is not altered in substance if to each of its elements be prefixed* $+$ *or* $-$ *as determined by the sign-rule associated with Laplace's expansion-theorem*; *that* (Cor. p. 208) *the* m^{th} *compound is equal to the* $(n-m+1)^{th}$ *compound*: and that (Cor. p. 211) *to every minor of the order* $(n-1)_m$ *in the* m^{th} *compound there is an equivalent minor in the complementary compound.* In connection with the second of these we may remark that the 1st and n^{th} compound are not only equal but are identical.

The first thing noteworthy is a preliminary theorem, which, although formally proved, rests manifestly on the simple fact that *the* m^{th} *compound of a minor of* $|\,a_{1n}\,|$ *is a minor of the* m^{th} *compound of* $|\,a_{1n}\,|$. It is to the effect that *any* m*-line minor of* $|\,a_{1n}\,|$ *is expressible as the* $(m-1)_t^{th}$ *root of a minor of the* $(m-t)^{th}$ *compound of* $|\,a_{1n}\,|$. This is followed by the related result (p. 213) that *if* M_h *be an* h*-line minor of* $|\,a_{1n}\,|$, M'_{n-h} *being its complementary, and there be formed all the minors of the* $(h+k)^{th}$ *order which contain neither all the rows nor all the columns of* M_h, *then the determinant whose elements are these minors is equal to*

$$(M'_{n-h})^{(n-h-1)_k} \cdot |\,a_{1n}\,|^{(n-1)_{h+k-1}-(n-h)_k}.$$

It ought to be noted, although Picquet views the matter from a different standpoint, that we have here again an instance of a minor of the m^{th} compound being expressible as a product of a power of the generating determinant by a power of one of its minors.

What is next reached is the interesting theorem (no. 8), which we have already spoken of under Reiss as the so-called 'Picquet's theorem.' As now stated, it is to the effect that *if every set of* q *columns of* $|\,a_{1n}\,|$ *be replaced by every set of* q *columns of* $|\,b_{1n}\,|$, *the determinant,* ∇ *say, of the square array of determinants thus obtained is equal to*

$$|\,a_{1n}\,|^{(n-1)_q} \cdot |\,b_{1n}\,|^{(n-1)_{q-1}}.$$

A mode of proof is at once suggested if it be observed that the first factor of the result is equal to the $(n-q)^{th}$ compound of $|\,a_{1n}\,|$ and the second to the q^{th} compound of $|\,b_{1n}\,|$. If in the formation

of the elements of this ' mixed n^{th} compound,' if we may so call it, the two basic determinants were to change places we should have another determinant ∇', which would consequently be equal to

$$|\, b_{1n} \,|^{(n-1)_q} \cdot |\, a_{1n} \,|^{(n-1)_{q-1}};$$

so that there would result

$$\nabla\nabla' = \left\{ |\, a_{1n} \,| \cdot |\, b_{1n} \,| \right\}^{(n)_q},$$

as an analogue to Cauchy's theorem regarding the product of complementary compounds. Of course the elements of ∇ and ∇' might be so arranged that any element of the one would contain those columns of $|\, a_{1n} \,|$ and $|\, b_{1n} \,|$ which are not contained in the corresponding element of the other, and be thus, in an extended sense, *complementary*. Picquet might also have noted that if $|\, b_{1n} \,|$ were the conjugate of $|\, a_{1n} \,|$, his mixed compound would then be equal to

$$|\, a_{1n} \,|^{(n)_q}.$$

Towards the close of the memoir (pp. 238–241) an important related result is given, namely, that *the ratio of any* h*-line minor of* ∇ *to the complementary of the corresponding minor of* ∇' *is equal to*

$$\frac{\left\{ |\, a_{1n} \,| \cdot |\, b_{1n} \,| \right\}^{h}}{|\, a_{1n} \,|^{(n-1)_{q-1}} \cdot |\, b_{1n} \,|^{(n-1)_q}}.$$

Two modes of proof are given, the nature of which will be guessed when it is recognized that the theorem is a generalization of Franke's of 1862, into which, as a matter of fact, it degenerates when $|\, b_{1n} \,|$ is such that $b_{rr} = 1$, $b_{rs} = 0$. In this again Picquet was forestalled by Reiss.

The ninth theorem (p. 216) is professedly Sylvester's of March 1851 (*Hist.*, ii. pp. 193–197) : the tenth is an alternative mode of stating the ninth : and the eleventh is essentially the same as one of D'Ovidio's of 1876 (§ iii. of the ' Nota '). This last, for the sake of uniformity, we may enunciate in our own way, namely, *If* M_h *be an* h*-line minor of* $|\, a_{1n} \,|$, M'_{n-h} *being its complementary, and there be formed all the minors of the* $(n-h-k)^{th}$ *order such that neither all their rows nor all their columns belong to* M'_{n-h}, *the determinant whose elements are these minors is equal to*

$$(M_h)^{(n-h-1)_k} \cdot |\, a_{1n} \,|^{(n-1)_{h+k}-(n-h-1)_k}.$$

At this stage considerable space (pp. 222–233) is devoted to examples illustrating the theorems thus far reached and leading to another theorem (no. 12) in which the m^{th} compound of $|\, a_{1n} \,|$ is made to produce a mixed m^{th} compound by replacements from the m^{th} compound of $|\, b_{1n} \,|$. There then follows Sylvester's theorem of August 1851 (*Hist.*, ii. pp. 61–62): and the memoir ends with a theorem regarding a special minor of the mixed n^{th} compound ∇ above referred to.

RUBINI, R. (1878).

[Formole di trasformazioni nella teorica dei determinanti. *Giornale di Mat.*, xvi. pp. 198–208, 344.]

Rubini's main theorem is that extreme case of Bazin's which had already been rediscovered by Zehfuss in 1862. His mode of establishing it is to multiply $|\, a_{1n} \,|$ by $|\, x_{1n} \,|$, equate the elements of the resulting product to the elements of $|\, b_{1n} \,|$ in order, solve for the x's in rows, and then substitute their values in $|\, x_{1n} \,|$. The outcome being a compound determinant equal to AB^{n-1}, the companion determinant equal to $A^{n-1}B$ is noted, and easy deductions made. Attention is also drawn to special cases; in particular, how by taking one of the given determinants equal to unity, say equal to

$$\begin{vmatrix} (k)_0 & (k+1)_0 & (k+2)_0 & \cdots \\ (k)_1 & (k+1)_1 & (k+2)_1 & \cdots \\ (k)_2 & (k+1)_2 & (k+2)_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n,$$

the adjugate of the other determinant can be obtained in any number of forms.

The really fresh result of the paper is contained in § 5, where he shows, merely by making a column of A identical with a column of B, that his supposed new compound determinant equal to AB^{n-1} has every one of its primary minors exactly divisible by B^{n-2}, and that as a consequence of dividing by this factor there is obtained as an expression for AB an aggregate of products of pairs of determinants,—in other words, Sylvester's expansion of the years 1839, 1851. From this, again, easy specializations and deductions are made.

SYLVESTER, J. J. (1879).

[Sur les déterminants composés. *Crelle's Journ.*, lxxxviii. pp. 49–67 ; or *Collected Math. Papers*, iii. pp. 456–473.]

At the outset (pp. 49–53) Sylvester rehearses his previous work on the subject, the appearance of it being slightly changed because of a fresh notation : for example, the theorem regarding the evaluation of the m^{th} compound takes the form

$$\sum a_1 a_2 \dots a_m \divideontimes \sum a_1 a_2 \dots a_m = \begin{pmatrix} a_1 a_2 & \dots & a_m \\ a_1 a_2 & \dots & a_m \end{pmatrix}^{(n-1)_{m-1}},$$

and the 'extended' m^{th} compound is written

$$[b_1 b_2 \dots b_p \sum a_1 a_2 \dots a_m] \divideontimes [\beta_1 \beta_2 \dots \beta_p \sum a_1 a_2 \dots a_m].$$

He also illustrates diagrammatically the degeneration of one of his theorems (*Hist.*, ii. pp. 197–200) into the ordinary multiplication-theorem, the product of two rows or two columns

$$(a_1, a_2, a_3, a_4)(a_1, a_2, a_3, a_4)$$

being representable in the form

$$- \begin{vmatrix} 1 & . & . & . & a_1 \\ . & 1 & . & . & a_2 \\ . & . & 1 & . & a_3 \\ . & . & . & 1 & a_4 \\ a_1 & a_2 & a_3 & a_4 & . \end{vmatrix}.$$

There then comes his new but improved generalization (pp. 53–55), the subject of which is already familiar to us from the writings of others, namely, the expression of a minor of the m^{th} compound of $|\, a_{1n} \,|$ in terms of minors of $|\, a_{1n} \,|$. Reiss, who had tackled the problem in 1867, is not referred to. In effect Sylvester says that every minor is so expressible if it be constructed so as to have each of its elements containing a rows and a columns chosen from a fixed rows and a fixed columns of the generating determinant, β rows and β columns from b other fixed rows and columns of the generating determinant, and so on, the number of lines in each element thus being

$$a + \beta + \dots + \xi,$$

the number of lines in the generating determinant

$$a + b + \dots + z,$$

and the number of lines in the constructed minor

$$C_{a,\alpha}+C_{b,\beta}+\ldots+C_{z,\zeta}.$$

For example, if the generating determinant be

$$\left|\begin{matrix}1&2&3&4&5&6\\1&2&3&4&5&6\end{matrix}\right|,$$

and if the submitted minor of its third compound be such that each of its elements contains two row-numbers and two column-numbers chosen from 1, 2, 3, and one row-number and one column-number chosen from 4, 5, 6, then the said submitted minor must be

$$\left\|\begin{matrix}|124|&|125|&|126|&|134|&|135|&|136|&|234|&|235|&|236|\\|124|&|125|&|126|&|134|&|135|&|136|&|234|&|235|&|236|\end{matrix}\right\|,$$

and according to Sylvester it is equal to

$$\left|\begin{matrix}123456\end{matrix}\right|^2\cdot\left|\begin{matrix}123\end{matrix}\right|^4\cdot\left|\begin{matrix}456\end{matrix}\right|.$$

Following on this illustration, or, rather, on the 'extensional' of it, there is given an alternative mode of stating the theorem together with full diagrammatic illustrations (pp. 56–65). The paper concludes with an afterthought involving a further generalization in which oblong arrays take the place of determinants.

MUIR, T. (1879).

(See under this heading in chap. i.)

HUNYADY, E. (1879).

[A másodfokú felületek elméletéhez. *Értekesézck a math. tudományok köréből* (Budapest), vii. 5 (36 pp.).]

[Beitrag zur Theorie der Fläche zweiten Grades. *Crelle's Journ.*, lxxxix. pp. 47–69.]

Hunyady's mode of discovery, as we have already seen, is the familiar one of seeking out two or more expressions for one and the same condition and then comparing the results. On this occasion the condition is that connected with the existence of ten points on a quadric surface. As the outcome of some research (pp. 47–48) he brings together a series of related problems in geometry, and

thence obtains by elimination six forms of determinants whose vanishing is the condition in question.

If the coordinates of the i^{th} of the ten points be denoted by

$$(x_i,\ y_i,\ z_i,\ w_i),$$

and the determinant $|\, x_i y_j z_k w_l \,|$ by

$$|\, ijkl \,|,$$

the eliminants reached are

$$\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & w_1^2 & x_1y_1 & x_1z_1 & \dots & z_1w_1 \\ x_2^2 & y_2^2 & z_2^2 & w_2^2 & x_2y_2 & x_2z_2 & \dots & z_2w_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0^2 & y_0^2 & z_0^2 & w_0^2 & x_0y_0 & x_0z_0 & \dots & z_0w_0 \end{vmatrix}, \quad \text{or D say,}$$

where 0 is written for 10 :

$$\begin{vmatrix} |1235|\cdot|1245| & |1235|\cdot|1345| & |1235|\cdot|2345| & |1245|\cdot|1345| & |1245|\cdot|2345| & |1345|\cdot|2345| \\ |1236|\cdot|1246| & |1236|\cdot|1346| & |1236|\cdot|2346| & |1246|\cdot|1346| & |1246|\cdot|2346| & |1346|\cdot|2346| \\ \dots & \dots & \dots & \dots & \dots & \dots \\ |1230|\cdot|1240| & |1230|\cdot|1340| & |1230|\cdot|2340| & |1240|\cdot|1340| & |1240|\cdot|2340| & |1340|\cdot|2340| \end{vmatrix} \equiv \text{E},$$

$$\begin{vmatrix} |1237|\cdot|4567| & |1347|\cdot|2567| & |1457|\cdot|2367| & |1257|\cdot|3467| \\ |1238|\cdot|4568| & |1348|\cdot|2568| & |1458|\cdot|2368| & |1258|\cdot|3468| \\ |1239|\cdot|4569| & |1349|\cdot|2569| & |1459|\cdot|2369| & |1259|\cdot|3469| \\ |1230|\cdot|4560| & |1340|\cdot|2560| & |1450|\cdot|2360| & |1250|\cdot|3460| \end{vmatrix} \equiv \text{F},$$

$$\begin{vmatrix} |1256|\cdot|2346| & |1236|\cdot|3456| & |1456|\cdot|2346| & |1256|\cdot|3456| & |1236|\cdot|1456| \\ |1257|\cdot|2347| & |1237|\cdot|3457| & |1457|\cdot|2347| & |1257|\cdot|3457| & |1237|\cdot|1457| \\ \dots & \dots & \dots & \dots & \dots \\ |1250|\cdot|2340| & |1230|\cdot|3450| & |1450|\cdot|2340| & |1250|\cdot|3450| & |1230|\cdot|1450| \end{vmatrix} \equiv \text{G},$$

and two others of still more complicated form.

As a first step towards the establishment of the relations between D, E, F, G, Hunyady proves the identity existing between $|\, x_1y_2z_3w_4 \,|^5$ and the ten-line determinant, Q say, whose rows in order consist of the terms of the expansions of

$$\begin{gathered} (x_1+y_1+z_1+w_1)^2 \\ \dots \\ (x_4+y_4+z_4+w_4)^2 \\ (x_1+y_1+z_1+w_1)(x_2+y_2+z_2+w_2) \\ \dots \\ (x_3+y_3+z_3+w_3)(x_4+y_4+z_4+w_4) \end{gathered}$$

respectively. His procedure is similar to that followed by Scholtz* in dealing with the corresponding theorem regarding $|x_1y_2z_3|^4$,— that is to say, he multiplies the ten-line determinant by

$$|y_2z_3w_4|\cdot|y_3z_4w_1|\cdot|y_4z_1w_2|\cdot|y_1z_2w_3|,$$

reduces the product to the form

$$-|1234|^5\begin{vmatrix} y_1y_2 & z_1z_2 & w_1w_2 & y_1z_2+y_2z_1 & y_1w_2+y_2w_1 & z_1w_2+z_2w_1 \\ y_1y_3 & z_1z_3 & w_1w_3 & y_1z_3+y_3z_1 & y_1w_3+y_3w_1 & z_1w_3+z_3w_1 \\ y_1y_4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_2y_3 & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_2y_4 & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_3y_4 & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

the six-line determinant of which he says is equal to

$$-|y_2z_3w_4|\cdot|y_3z_4w_1|\cdot|y_4z_1w_2|\cdot|y_1z_2w_3|.$$

He thus obtains

$$S = |1234|^5$$

as desired.

Taking then the determinant $\bar{S}$ derived from S by substituting for each element its cofactor in $|1234|$, Hunyady multiplies D by it, and obtains after reduction

$$D\bar{S} = |1234|^8\cdot E,$$

which, since $\bar{S}$ is equal to $|1234|^{3\times5}$, gives the relation between D and E to be

$$E = |1234|^7\cdot D.$$

Similarly, with the help of other lemmas, it is shown that

$$F = \{|1245|\cdot|1346|\cdot|2356| - |1256|\cdot|1345|\cdot|2346|\}\ D,$$
$$G = -|2345|\cdot|3451|\cdot|4512|\cdot|5123|\cdot|1234|\cdot D.$$

The fifth and sixth forms of eliminant are also satisfactorily accounted for, being shown, like the others, to be multiples of D, namely, the one equal to

$$-|7890|\cdot F,$$

and the other equal to

$$-|7890|\cdot|8906|\cdot|9067|\cdot|0678|\cdot|6789|\cdot G.$$

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*See above under *General Determinants.*

HUNYADY, E. (1879).

[Der Satz von Desargues über perspectivische Dreiecke. *Crelle's Journ.*, lxxxix. pp. 79–81.]

Here Hunyady's object is to establish the identity in coordinate geometry which is the equivalent of Desargues' theorem. To this end he first shows, in regard to the array

$$\begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6, \end{matrix}$$

that

$$\begin{vmatrix} |y_1z_4| & |z_1x_4| & |x_1y_4| \\ |y_2z_5| & |z_2x_5| & |x_2y_5| \\ |y_3z_6| & |z_3x_6| & |x_3y_6| \end{vmatrix} = \begin{vmatrix} |x_1y_2z_5| & |x_4y_2z_5| \\ |x_1y_3z_6| & |x_4y_3z_6| \end{vmatrix},$$

which is readily done by multiplying the determinant on the left rowwise by

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \\ x_2 & y_2 & z_2 \end{vmatrix}.$$

He then forms the array

$$\begin{matrix} \mathrm{X}_1 & \mathrm{X}_2 & \mathrm{X}_3 & \mathrm{X}_4 & \mathrm{X}_5 & \mathrm{X}_6 \\ \mathrm{Y}_1 & \mathrm{Y}_2 & \mathrm{Y}_3 & \mathrm{Y}_4 & \mathrm{Y}_5 & \mathrm{Y}_6 \\ \mathrm{Z}_1 & \mathrm{Z}_2 & \mathrm{Z}_3 & \mathrm{Z}_4 & \mathrm{Z}_5 & \mathrm{Z}_6, \end{matrix}$$

where $|\mathrm{X}_1\mathrm{Y}_2\mathrm{Z}_3|$, $|\mathrm{X}_4\mathrm{Y}_5\mathrm{Z}_6|$ are the adjugates of $|x_1y_2z_3|$, $|x_4y_5z_6|$ respectively, and applying to it the theorem just reached obtains of course

$$\begin{vmatrix} |\mathrm{Y}_1\mathrm{Z}_4| & |\mathrm{Z}_2\mathrm{X}_5| & |\mathrm{X}_3\mathrm{Y}_6| \end{vmatrix} = \begin{vmatrix} |\mathrm{X}_1\mathrm{Y}_2\mathrm{Z}_5| & |\mathrm{X}_4\mathrm{Y}_2\mathrm{Z}_5| \\ |\mathrm{X}_1\mathrm{Y}_3\mathrm{Z}_6| & |\mathrm{X}_4\mathrm{Y}_3\mathrm{Z}_6| \end{vmatrix}.$$

Here, however, each element of the first column on the right can, by multiplying by $|x_1y_2z_3|$, be shown to have $|x_1y_2z_3|$ for a factor, and similarly each element of the second column to have $|x_4y_5z_6|$ for a factor, the right-hand member thus becoming

$$|x_1y_2z_3| \cdot |x_4y_5z_6| \cdot \begin{vmatrix} |x_4y_3z_6| & |x_1y_3z_6| \\ |x_4y_2z_5| & |x_1y_2z_5| \end{vmatrix};$$

so that, on using again our initial identity, we have

$$\left|\,|\,Y_1Z_4\,|\;|\,Z_2X_5\,|\;|\,X_3Y_6\,|\,\right| = |\,x_1y_2z_3\,|\cdot|\,x_4y_5z_6\,|\cdot\left|\,|\,y_1z_4\,|\;|\,z_2x_5\,|\;|\,x_3y_6\,|\,\right|,$$

which is the result sought to be established.

As the said initial identity is not essentially different from one given by Scholtz in the second volume of *Müegyetemi Lapok*, it is important to note that Hunyady here states that the substance of the present paper had appeared previously in the *first* volume of the same publication.

PASCH, M. (1879).

[Ueber gewisse Determinanten, welche in der Lehre von den Kegelschnitten vorkommen. *Crelle's Journ.*, lxxxix. pp. 247–251.]

Save in one instance the determinants referred to in the title are those with which Hunyady, Mertens and Scholtz have already made us familiar. The demonstrations are Pasch's own, but present no feature of particular interest. The fresh result is

$$\left|\begin{array}{cccc} |\,x_2y_3z_5\,|\;|\,x_2y_3z_6\,| & |\,x_2y_3z_6\,|\;|\,x_2y_3z_4\,| & |\,x_2y_3z_4\,|\;|\,x_2y_3z_5\,| \\ |\,x_3y_1z_5\,|\;|\,x_3y_1z_6\,| & |\,x_3y_1z_6\,|\;|\,x_3y_1z_4\,| & |\,x_3y_1z_4\,|\;|\,x_3y_1z_5\,| \\ |\,x_1y_2z_5\,|\;|\,x_1y_2z_6\,| & |\,x_1y_2z_6\,|\;|\,x_1y_2z_4\,| & |\,x_1y_2z_4\,|\;|\,x_1y_2z_5\,| \end{array}\right|$$

$$= |\,x_1y_2z_3\,|^2\cdot|\,x_1^2\;\;y_2^2\;\;z_3^2\;\;y_4z_4\;\;z_5x_5\;\;x_6y_6\,|.$$

By expanding the left-hand member in terms of the elements of its first row and their complementaries, we obtain without difficulty

$$(235\cdot236\cdot314\cdot124\cdot165 - 236\cdot234\cdot315\cdot125\cdot164 + 234\cdot235\cdot316\cdot126\cdot154)\,123.$$

But of the three terms here within the brackets,

$$\text{the first} = 235\cdot236\cdot314(125\cdot416 - 126\cdot415),$$
$$\text{the second} = 236\cdot125\cdot146(235\cdot431 - 231\cdot435),$$
$$\text{the third} = 235\cdot126\cdot154(236\cdot314 - 231\cdot346),$$

and consequently their sum is

$$-(236\cdot125\cdot146\cdot435 - 235\cdot126\cdot154\cdot346)\,123.$$

We thus have for our compound determinant the equivalent

$$123^2(125\cdot146\cdot236\cdot345 - 126\cdot145\cdot235\cdot346),$$

which, with the help of a previous identity ($-D = P$) becomes the result desired.*

BORCHARDT, C. W. (1880).

[Remarque relative au mémoire de M. Sylvester sur les déterminants composés. *Crelle's Journ.*, lxxxix. pp. 82–85; or *Werke*, pp. 494 *et seqq.*]

After restating Sylvester's theorem of the preceding year, Borchardt intimates that in consequence of correspondence with Sylvester he had been requested by the latter to withdraw the theorem for the present. He then conclusively shows that the enunciation cannot be accepted as correct by taking the simplest possible case, namely, where

$$n = 4, \quad a = 2, \quad \alpha = 1,$$
$$b = 2, \quad \beta = 1,$$

and where, therefore, the subject under discussion is the resolvability of

$$\begin{vmatrix} \begin{vmatrix} 13 \\ 13 \end{vmatrix} & \begin{vmatrix} 14 \\ 14 \end{vmatrix} \\ \begin{vmatrix} 23 \\ 23 \end{vmatrix} & \begin{vmatrix} 24 \\ 24 \end{vmatrix} \end{vmatrix},$$

that is to say, of the central four-line minor of the second compound of $\begin{vmatrix} 1234 \\ 1234 \end{vmatrix}$. According to Sylvester this central four-line minor must be equal to

$$\begin{vmatrix} 1234 \\ 1234 \end{vmatrix} \cdot \begin{vmatrix} 12 \\ 12 \end{vmatrix} \cdot \begin{vmatrix} 34 \\ 34 \end{vmatrix},$$

and Borchardt formally proves that it is equal to

$$\begin{vmatrix} 1234 \\ 1234 \end{vmatrix} \cdot \begin{vmatrix} \begin{vmatrix} 12 \\ 12 \end{vmatrix} & \begin{vmatrix} 12 \\ 34 \end{vmatrix} \\ \begin{vmatrix} 34 \\ 12 \end{vmatrix} & \begin{vmatrix} 34 \\ 34 \end{vmatrix} \end{vmatrix},$$

as indeed is manifest from Franke's theorem of 1862, which Borchardt had established and had helped to make known.

* This is not Pasch's proof, which would have necessitated the introduction of additional notation and auxiliary matter not otherwise of use in the present connection.

CHAPTER VII.

RECURRENTS, FROM 1858 TO 1879.

No special form, save Alternants, has received so much attention during the period now reached as Recurrents; and, what is more striking, the signs of increase of interest taken in it are much more conspicuous than in the case even of Alternants, the number of writings being about five-fold what it was in the preceding period. Doubtless part of this is due to the fact that the form lies near the surface, so to say, in many parts of Algebra, so that examples are easily got at,—one for every recurrence-formula known or unknown. In this way not a small portion of the work spent on it seems somewhat futile, and we have consequently curtailed our reports as far as was possible without doing injustice. As stated at the close of our first chapter on the subject (*Hist.*, i. p. 218) it is the recurrence-formula, not the recurrent, that is of prime importance.

Two of the writings here dealt with belong to the period of the preceding volume; there also formal note should have been made that Continuants, whose history begins in 1853 and is treated separately, form a special class of Recurrents.

BRIOSCHI, F. (1858).

[Sulle funzioni Bernoulliane ed Euleriane. *Annali di Mat.*, i. pp. 260–263; or *Opere Mat.*, i. pp. 343–347.]

In this notice of no. 9 of Raabe's *Mathematische Mittheilungen* (160 pp. Zürich, 1858) Brioschi gives in passing the result

$$\begin{vmatrix} a_1 & 1 & . & . & \dots \\ a_2 & a_1 & 1 & . & \dots \\ a_3 & a_2 & a_1 & 1 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n = \sum (-1)^{n+\Sigma e} \cdot (\Sigma e)! \frac{a_1^{e_1} a_2^{e_2} \dots a_n^{e_n}}{e_1! e_2! \dots e_n!},$$

where $e_1, e_2, \ldots, e_n$, being neither negative nor fractional, satisfy the equation

$$e_1+2e_2+3e_3+\ldots+ne_n = n,$$

and where Σe stands for $e_1+e_2+\ldots+e_n$.

BRUNO, F. FAÀ DI (1859).

[THÉORIE GÉNÉRALE DE L'ÉLIMINATION . . . x+224 pp. Paris.]

Of the seven carefully formulated propositions regarding symmetric functions which are the subject of Bruno's opening chapter (pp. 1–22), the first and last involve a recurrent. The first, being already known (*Hist.*, ii. pp. 211–212), is only worth mentioning because of the fact that in a footnote (p. 2) its quasi-converse is deduced from the same set of equations, namely,

$$a_r = (-1)^r \frac{1}{r!} \begin{vmatrix} s_1 & 1 & . & . & . & . \\ s_2 & s_1 & 2 & . & . & . \\ s_3 & s_2 & s_1 & 3 & . & . \\ . & . & . & . & . & . \end{vmatrix}_r.$$

The other is taken from Brioschi,* by whom it was used for the purpose of expressing in terms of the a's any symmetric function ϕ of the roots of the equation

$$x^n+a_1x^{n-1}+\ldots+a_n \equiv (x-a_1)(x-a_2)\ldots(x-a_n) = 0.$$

It is

$$\phi = \begin{vmatrix} r\frac{\partial\phi}{\partial s_r} & 1 & . & . & . \\ (r-1)\frac{\partial\phi}{\partial s_{r-1}} & a_1 & 1 & . & . \\ . & . & . & . & . \\ 1\frac{\partial\phi}{\partial s_1} & a_{r-1} & a_{r-2} & . & 1 \\ . & ra_r & (r-1)a_{r-1} & . & a_1 \end{vmatrix}.$$

* BRIOSCHI, F. Sulle funzioni simmetriche delle radici di una equazione. *Annali di sci. mat. e fis.*, v. (1854), pp. 422–428.

The recurrence-formula on which it is founded, namely,

$$\frac{\partial \phi}{\partial a_r} + a_1 \frac{\partial \phi}{\partial a_{r+1}} + \ldots + a_{n-r} \frac{\partial \phi}{\partial a_n} = -r \frac{\partial \phi}{\partial s_r}$$

is obtained from

$$\frac{\partial \phi}{\partial s_r} = \frac{\partial \phi}{\partial a_1} \cdot \frac{\partial a_1}{\partial s_r} + \frac{\partial \phi}{\partial a_2} \cdot \frac{\partial a_2}{\partial s_r} + \ldots + \frac{\partial \phi}{\partial a_n} \cdot \frac{\partial a_n}{\partial s_r}$$

on substituting *

$$-\frac{1}{r} a_{h-r} \quad \text{for} \quad \frac{\partial a_h}{\partial s_r}.$$

It is evidently only helpful when $\partial\phi/\partial s_r$, $\partial\phi/\partial s_{r-1}$, ... are already known in terms of the a's, a contingency which would supervene if the ϕ's were being calculated for tabulation in ascending order of degree. Thus, in the illustrative case where $\phi = \sum a_1^4 a_2^3 a_3$, Brioschi has to assume as known

$$\frac{\partial \phi}{\partial s_3} = \sum a_1^4 a_2 = -a_1^3 a_2 + a_1^2 a_3 + \ldots$$

$$\frac{\partial \phi}{\partial s_4} = \sum a_1^3 a_2 - \sum a_1^4 = -a_1^4 + 5 a_1^2 a_2 + \ldots$$

. . . . . .

A one-page table of such results is given by Bruno (pp. 19–20), but, unfortunately, it is very inaccurate. It could readily have been checked with Hirsch's *Sammlung* of 1804 or Cayley's memoir of 1856–57 (*Philos. Transac. Roy. Soc. London*, cxlvii. pp. 489–499).

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* From the identities

$$s_1 + a_1 = 0, \qquad s_2 + a_1 s_1 + 2a_2 = 0, \qquad \ldots,$$

there is obtained by differentiation

$$\frac{\partial a_1}{\partial s_r} = -\frac{\partial s_1}{\partial s_r},$$

$$\frac{\partial a_2}{\partial s_r} = -a_1 \frac{\partial s_1}{\partial s_r} - \frac{1}{2} \frac{\partial s_2}{\partial s_r},$$

$$\frac{\partial a_3}{\partial s_r} = -a_2 \frac{\partial s_1}{\partial s_r} - \frac{1}{2} a_1 \frac{\partial s_2}{\partial s_r} - \frac{1}{3} \frac{\partial s_3}{\partial s_r},$$

. . . . . ,

and it is this, arrived at in a different manner, which is Brioschi's warrant for the substitution here used.

Attention is drawn to the fact that in the case of both propositions the final expansion is *isobaric*, the *weight* of each term being of course equal to the degree of the symmetric function concerned.

SYLVESTER, J. J. (1862).

[On the integral of the general equation in differences. *Philos. Magazine*, xxiv. pp. 436–441 ; or *Collected Math. Papers*, ii. pp. 318–322.]

The given equation being

$$a_x u_x + b_x u_{x-1} + c_x u_{x-2} + \ldots = 0,$$

with the conditions $u_0 = 1$, $u_{<0} = 0$, it is readily seen that

$$u_n = \mathrm{R}_n \div (-)^n a_1 a_2 \ldots a_n,$$

where R_n is a recurrent ; for example,

$$\mathrm{R}_4 = \begin{vmatrix} b_4 & c_4 & d_4 & e_4 \\ a_3 & b_3 & c_3 & d_3 \\ . & a_2 & b_2 & c_2 \\ . & . & a_1 & b_1 \end{vmatrix}.$$

Sylvester consequently is led to seek a rule for finding the final development of such a determinant. To this end he writes the determinant in the form

$$\begin{vmatrix} 4,3 & 4,2 & 4,1 & 4,0 \\ 3,3 & 3,2 & 3,1 & 3,0 \\ . & 2,2 & 2,1 & 2,0 \\ . & . & 1,1 & 1,0 \end{vmatrix},$$

takes all the descending sets of integers beginning with 4 and ending with 0, namely,

4 3 2 1 0
4 3 2 0
4 3 1 0
4 2 1 0
4 3 0
4 2 0
4 1 0
4 0,

and finally, from these sets, constructs the terms

$$\begin{aligned}
&4,3\times3,2\times2,1\times1,0\\
-&4,3\times3,2\times2,0\times(1,1)\\
-&4,3\times3,1\times1,0\times(2,2)\\
-&4,2\times2,1\times1,0\times(3,3)\\
&4,3\times3,0\times(1,1\times2,2)\\
&4,2\times2,0\times(1,1\times3,3)\\
&4,1\times1,0\times(2,2\times3,3)\\
-&4,0\times(1,1\times2,2\times3,3).
\end{aligned}$$

No justification of the ' rule ' is given.

TRUDI, N. (1862).

[Teoria dei Determinanti, e loro Applicazioni (pp. 122–128), ix+268 pp. Napoli.]

In the division of $a_0x^m+\ldots$ by $b_0x^n+\ldots$, where m and n are positive integers and $m \geqq n$, Trudi stops on reaching the remainder which is of the $(n-1)^{\text{th}}$ degree in x. Assuming the identity

$$\frac{a_0x^m+\ldots}{b_0x^n+\ldots} = q_0x^{m-n}+\ldots+\frac{c_0x^{n-1}+\ldots}{b_0x^n+\ldots},$$

$$\frac{A}{B} = Q+\frac{R}{B} \quad \text{say,}$$

he multiplies both sides by B, and, equating coefficients of like powers of x, obtains the necessary equations for determining the q's and the c's in terms of the a's and b's. The expressions for the q's are of course in accord with Faure's (1855); the expressions for the c's are

$$\frac{1}{b_0}\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}, \quad \frac{1}{b_0}\begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix}, \quad \frac{1}{b_0}\begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix}, \quad \ldots\ldots \text{ when } m = n,$$

$$\frac{1}{b_0^2}\begin{vmatrix} a_0 & a_1 & a_2 \\ . & b_0 & b_1 \\ b_0 & b_1 & b_2 \end{vmatrix}, \quad \frac{1}{b_0^2}\begin{vmatrix} a_0 & a_1 & a_3 \\ . & b_0 & b_2 \\ b_0 & b_1 & b_3 \end{vmatrix}, \quad \ldots\ldots \text{ when } m = n+1,$$

and so on. He also by summation obtains a single determinant equal to Q and another equal to R.

He does not, however, note that the results when once got are susceptible of the simplest possible verification. Thus, whatever the letters may denote, we have as manifest identities

$$A = \frac{1}{b_0}\begin{vmatrix} a_0 & a_1 \\ . & b_0 \end{vmatrix} \quad B - \frac{1}{b_0}\begin{vmatrix} a_0 & A \\ b_0 & B \end{vmatrix},$$

$$A = \frac{1}{b_0^2}\begin{vmatrix} a_0 & a_1 & . \\ . & b_0 & 1 \\ b_0 & b_1 & x \end{vmatrix} \quad B - \frac{1}{b_0^2}\begin{vmatrix} a_0 & a_1 & A \\ . & b_0 & B \\ b_0 & b_1 & Bx \end{vmatrix},$$

$$A = \frac{1}{b_0^3}\begin{vmatrix} a_0 & a_1 & a_2 & . \\ . & . & b_0 & 1 \\ . & b_0 & b_1 & x \\ b_0 & b_1 & b_2 & x^2 \end{vmatrix} \quad B + \frac{1}{b_0^3}\begin{vmatrix} a_0 & a_1 & a_2 & A \\ . & . & b_0 & B \\ , & b_0 & b_1 & Bx \\ b_0 & b_1 & b_2 & Bx^2 \end{vmatrix},$$

. . . . . .

and, A and B being defined as above, the first identity is to be used when $m = n$, the second when $m = n+1$, and so on. For example, when $m = n+2$,

$$Q = -\frac{1}{b_0^3}\begin{vmatrix} a_0 & a_1 & a_2 \\ . & . & b_0 \\ . & b_0 & b_1 \end{vmatrix} \quad x^2 + \frac{1}{b_0^3}\begin{vmatrix} a_0 & a_1 & a_2 \\ . & . & b_0 \\ b_0 & b_1 & b_2 \end{vmatrix} \quad x - \frac{1}{b_0^3}\begin{vmatrix} a_0 & a_1 & a_2 \\ . & b_0 & b_1 \\ b_0 & b_1 & b_2 \end{vmatrix}$$

$$= \frac{1}{b_0} \cdot a_0 x^2 - \frac{1}{b_0^2}\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} \quad x + \frac{1}{b_0^3}\begin{vmatrix} a_0 & b_0 & . \\ a_1 & b_1 & b_0 \\ a_2 & b_2 & b_1 \end{vmatrix};$$

and

$$R = \frac{1}{b_0^3}\begin{vmatrix} a_0 & a_1 & a_2 & A & -a_0x^{n+2} - a_1x^{n+1} - a_2x^n \\ . & . & b_0 & B & -b_0x^n \\ . & b_0 & b_1 & Bx & -b_0x^{n+1} - b_1x^n \\ b_0 & b_1 & b_2 & Bx^2 & -b_0x^{n+2} - b_1x^{n+1} - b_2x^n \end{vmatrix}$$

$$= \frac{1}{b_0^3}\begin{vmatrix} a_0 & a_1 & a_2 & a_3x^{n-1} + a_4x^{n-2} + \ldots \\ . & . & b_0 & b_1x^{n-1} + b_2x^{n-2} + \ldots \\ . & b_0 & b_1 & b_2x^{n-1} + b_3x^{n-2} + \ldots \\ b_0 & b_1 & b_2 & b_3x^{n-1} + b_4x^{n-2} + \ldots \end{vmatrix}$$

$$= \frac{1}{b_0^2} \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ . & . & b_0 & b_1 \\ . & b_0 & b_1 & b_2 \\ b_0 & b_1 & b_2 & b_3 \end{vmatrix} x^{n-1} + \ldots,$$

the coefficients of x^{n-1}, x^{n-2}, . . . agreeing with the expressions found for c_0, c_1, . . . in the early part of the exposition.

In a closing remark on the possibility of the unlimited continuance of the division he gives as an example,

$$\frac{a_0x+a_1}{b_0x^2+b_1x+b_2} = \frac{1}{b_0} \cdot a_0x^{-1} - \frac{1}{b_0^2} \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix} x^{-2} + \frac{1}{b_3^0} \begin{vmatrix} a_0 & b_0 & . \\ a_1 & b_1 & b_0 \\ . & b_2 & b_1 \end{vmatrix} x^{-3} - \ldots,$$

where the determinants are continuants.

TRUDI, N. (1862).

[Intorno ad alcune formole di sviluppo. *Rendic. dell' Accad.* . . . . (Napoli), pp. 135–143.]

Having obtained independently Faure's result of 1855, Trudi sets himself the task of finding an expression for the development of the related determinant, namely, the determinant of which

$$1/(a_0+a_1x+\ldots+a_mx^m)$$

is the generating function. Applying to the case where $f(u) = 1/u$ a general formula given by Fergola for the r^{th} differential coefficient of $f(u)$ with respect to x he establishes Brioschi's result of 1858 in a slightly extended form, namely,

$$\begin{vmatrix} a_1 & a_0 & . & . & \ldots \\ a_2 & a_1 & a_0 & . & \ldots \\ a_3 & a_2 & a_1 & a_0 & \ldots \\ . & . & . & . & . \end{vmatrix}_n = \sum (-1)^{n+\Sigma e} \cdot (\Sigma e)! \cdot a_0^{n-\Sigma e} \frac{a_1^{e_1} a_2^{e_2} \ldots a_n^{e_n}}{e_1!\, e_2! \ldots e_n!}.$$

He then uses this to arrive at a similar expression for the corresponding determinant in the case of the quotient

$$\frac{b_0x^n+b_1x^{n-1}+\ldots+b_n}{a_0x^m+a_1x^{m-1}+\ldots+a_m},$$

and finally making the numerator here the derivate of the denominator reaches Waring's expression for a sum of like powers of the roots of an equation.

ZEIPEL, V. v. (1862).

[Om en klass af determinanter. *Öfversigt . . . Vetenskaps-Akad. Förhandl.* (Stockholm), Årg. xix. pp. 439–445.]

The kind of determinant referred to is exemplified by

$$\begin{vmatrix} c\delta x & -x \\ c\delta^2 x & (c-1)\delta x \end{vmatrix} \quad \begin{vmatrix} c\delta x & -x & . \\ c\delta^2 x & (c-1)\delta x & -x \\ c\delta^3 x & (2c-1)\delta^2 x & (c-2)\delta x \end{vmatrix}, \quad \ldots\ldots,$$

where the first row is

$$c\delta x, \quad -x, \quad 0, \quad 0, \quad 0, \quad \ldots\ldots$$

and any other element a_{rs} is got from the formula

$$a_{rs} = \delta a_{r-1,s} + a_{r-1,s-1}.$$

We may with convenience denote them temporarily by

$$Z_2, \quad Z_3, \quad \ldots\ldots$$

Zeipel's first theorem calls to mind one of the earliest found properties of Wronskians, being to the effect that *the differential of such a determinant is got by substituting for the last row the row which would follow the last on continuing to apply the same law of formation*: for example,

$$\delta \begin{vmatrix} 2\delta x & -x & . \\ 2\delta^2 x & \delta x & -x \\ 2\delta^3 x & 3\delta^2 x & . \end{vmatrix} = \begin{vmatrix} 2\delta x & -x & . \\ 2\delta^2 x & \delta x & -x \\ 2\delta^4 x & 5\delta^3 x & 3\delta^2 x \end{vmatrix}.$$

The gradational style of proof is used, it being shown that, if it hold for the first r cases, it must hold also for the $(r+1)^{\text{th}}$. To this end Z_{r+1} is expressed in terms of the elements of the last row and their complementary minors, with the result

$$Z_{r+1} = x^r \cdot c\delta^{r+1}x + (cr_1 - r_0)x^{r-1}Z_1\delta^r x + (cr_2 - r_1)x^{r-2}Z_2\delta^{r-1}x + \ldots\ldots$$

differentiation is then performed, and in the subsequent condensation of terms use is made of the fact that according to hypothesis

$$\begin{aligned} Z_2 &= x\delta Z_1 + (c-1)Z_1\delta x, \\ Z_3 &= x\delta Z_2 + (c-2)Z_2\delta x, \\ &\ldots\ldots\ldots\ldots\ldots \\ Z_r &= x\delta Z_{r-1} + (c-r+1)Z_{r-1}\delta x. \end{aligned}$$

The final expression thus obtained for δZ_{r+1} is found to be equivalent to the determinant form aimed at.

The other matter dealt with is the so-called 'evaluation' of the Z's. The result of differentiating x^c being $cx^{c-1}\delta x$, we have

$$\delta x^c = x^{c-1}Z_1;$$

and from this, in the same way,

$$\begin{aligned}\delta^2 x^c &= x^{c-1}\delta Z_1 + Z_1(c-1)x^{c-2}\delta x \\ &= x^{c-2}\{x\delta Z_1 + (c-1)Z_1\delta x\} \\ &= x^{c-2}Z_2;\end{aligned}$$

and so forth. The result thus led up to is

$$Z_r = x^{r-c}\delta^r x^c.$$

FERGOLA, E. (1863).

[Questioni 5, 6, 7. *Giornale di Mat.*, i. pp. 63–64; i. pp. 221–222; iii. pp. 190–201; iii. pp. 94–99, 377–380.]

The interest of these problems attaches mainly to the Theory of Integers. What belongs to the theory of determinants is usefully segregated by Mola in his second paper, namely, the question of the final expansion of the recurrent

$$\begin{vmatrix} a_1 & 1 & . & . & \dots \\ a_2 & a_1 & 2 & . & \dots \\ a_3 & a_2 & a_1 & 3 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n.$$

The nature of the 'rule' which he reaches is evident from the result given by it in the case of the fourth order, namely,

$$4!\left\{\frac{a_1a_1a_1a_1}{1(1+1)(1+2)(1+3)} + \frac{a_2a_2}{2(2+2)} + \frac{a_1a_3}{1(1+3)} + \frac{a_3a_1}{3(3+1)} - \frac{a_1a_1a_2}{1(1+1)(1+3)} - \frac{a_1a_2a_1}{1(1+2)(1+3)} - \frac{a_2a_1a_1}{2(2+1)(2+2)} - \frac{a_4}{4}\right\},$$

i.e.

$$a_1^4 + 3a_2^2 + 8a_1a_3 - 6a_1^2a_2 - 6a_4,$$

where the sets of suffixes are permutations of the partitions of 4. The final form of expansion is that of an isobaric function, namely,

$$\sum(-1)^{n-\Sigma\epsilon}\cdot\frac{a_1^{\epsilon_1}a_2^{\epsilon_2}\cdots a_n^{\epsilon_n}}{1^{\epsilon_1}2^{\epsilon_2}\cdots n^{\epsilon_n}\cdot\epsilon_1!\,\epsilon_2!\cdots\epsilon_n!}$$

where, in forming the terms to be summed, each ϵ may be 0 or any positive integer subject to the condition

$$1\epsilon_1+2\epsilon_2+3\epsilon_3+\ldots+n\epsilon_n = n.$$

D'OVIDIO, E. (1863).

[Due teoremi di determinanti. *Giornale di Mat.*, i. pp. 135–139.]

D'Ovidio's first determinant is best viewed as being constructed by prefixing the row

$$\frac{1}{p},\quad 1,\quad 0,\quad 0,\ \ldots.$$

and the column

$$\frac{1}{p},\quad \frac{1}{1!(p+1)},\quad \frac{1}{2!(p+2)},\quad \ldots,\quad \frac{1}{(n-p)!\,n}$$

to the persymmetric recurrent

$$(-1)^{\frac{1}{2}(n-p)(n-p+1)}\cdot \mathbf{P}\left(\frac{1}{(n-p-1)!},\ \frac{1}{(n-p+2)!},\ \ldots,\ \frac{1}{2!},\ \frac{1}{1!},\ 1,\ 0,\ 0,\ \ldots\right).$$

Calling it D_{n-p+1} and performing the operation

$$\mathrm{col}_2 - p\,\mathrm{col}_1,$$

D'Ovidio at once obtains

$$\mathrm{D}_{n-p+1} = \frac{1}{p}\mathrm{D}_{n-p},$$

and thence finally

$$\mathrm{D}_{n-p+1} = \frac{1}{p(p+1)\ldots n}.$$

As a corollary he has of course $\mathrm{D}_n = \frac{1}{n!}$, where D_n is a persymmetric recurrent exactly like that to which the row and column were prefixed; for example,

$$\begin{vmatrix} \frac{1}{1!} & 1 & . & . \\ \frac{1}{2!} & \frac{1}{1!} & 1 & . \\ \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} & 1 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & \frac{1}{1!} \end{vmatrix} = \frac{1}{4!}.$$

His second is viewable as a generalization of another persymmetric recurrent, namely, of

$$(-1)^{\frac{1}{2}n(n-1)} \cdot \mathbf{P}(m_n,\ m_{n-1},\ \ldots\ .,\ m_1,\ 1,\ 0,\ 0,\ \ldots\ .),$$

where $m_n = m(m-1)\ \ldots\ .\ (m-n+1)/1\cdot 2\cdot\ \ldots\ .\ n$. The last row of this being

$$m_n,\ m_{n+1},\ \ldots\ .,\ m_1,$$

we have only got to multiply its first element by

$$1/(n+1)(n+2)\ \ldots\ .\ (n+p)$$

and all the other elements by

$$1/n(n+1)\ \ldots\ .\ (n+p-1)$$

to obtain the last row of the new determinant, D'_n say, every other row being formable from the row following it by lessening p by 1. Evaluation is then effected by performing the operation

$$\mathrm{col}_2 - \frac{p+1}{m}\,\mathrm{col}_1,$$

which leads to the result

$$\mathrm{D}'_n = (m+p+1)\,\frac{(n-1)!}{(p+1)!}\,\mathrm{D}'_{n-1},$$

and thence finally to

$$\mathrm{D}'_n = \frac{(m+p+1)(m+p+2)\ \ldots\ .\ (m+p+n-1)}{\dfrac{(p+1)!}{(n-1)!}\cdot\dfrac{(p+2)!}{(n-2)!}\cdot\ \ldots\ .\ \dfrac{(p+n-1)!}{1!}}\cdot\frac{m}{(p+n)!}.$$

As a corollary, obtained by putting $p = 0$, it is noted that the persymmetric recurrent used as a basis for construction on starting is equal to

$$m(m+1)\ \ldots\ .\ (m+n-1)/n!,\quad i.e.\ (m+n-1)_n,$$

the four-line case being

$$\begin{vmatrix} m_1 & 1 & . & . \\ m_2 & m_1 & 1 & . \\ m_3 & m_2 & m_1 & 1 \\ m_4 & m_3 & m_2 & m_1 \end{vmatrix} = \frac{m(m+1)(m+2)(m+3)}{1\cdot 2\cdot 3\cdot 4}.$$

We may note for ourselves in passing that the two corollaries, which D'Ovidio says he received from Trudi, are very readily obtained by using the multiplication-theorem, the multipliers for the 4th order being

$$\begin{vmatrix} 1 & -\frac{1}{1!} & \frac{1}{2!} & -\frac{1}{3!} \\ . & 1 & -\frac{1}{1!} & \frac{1}{2!} \\ . & . & 1 & -\frac{1}{1!} \\ . & . & . & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & -m_1 & (m+1)_2 & -(m+2)_3 \\ . & 1 & -m_1 & (m+1)_2 \\ . & . & 1 & -m_1 \\ . & . & . & 1 \end{vmatrix}$$

respectively, and the products

$$\begin{vmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ -\frac{1}{4!} & \frac{1}{3!} & -\frac{1}{2!} & \frac{1}{1!} \end{vmatrix}, \qquad \begin{vmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ -(m+3)_4 & (m+2)_3 & -(m+1)_2 & m_1 \end{vmatrix},$$

by reason of the vanishing of the $(r+1)$-termed expressions

$$\frac{1}{r!} - \frac{1}{(r-1)!}\cdot\frac{1}{1!} + \frac{1}{(r-2)!}\cdot\frac{1}{2!} - \ldots\ldots$$

$$m_r - m_{r-1}\cdot m_1 + m_{r-2}\cdot(m+1)_2 - \ldots\ldots$$

respectively.

TRUDI, N. (1864).

[Intorno ad un determinante più generale . . . . *Rendic. dell' Acad.* . . . . (Napoli), pp. 121–134; or *Giornale di Mat.*, ii. pp. 152–158, 180–186.]

The second part of Trudi's paper concerns the relations between the ordinary combination-functions Σa, $\Sigma a\beta$, $\Sigma a\beta\gamma$, ... and the aleph functions of the same variables. Writing the expression

$$(x-a)(x-\beta)(x-\gamma)(x-\delta)$$

in the form

$$\frac{|\, a^0\beta^1\gamma^2\delta^3x^4\,|}{|\, a^0\beta^1\gamma^2\delta^3\,|},$$

and treating the first four columns of the numerator in the manner set forth in the earlier part of his paper, he obtains

$$\begin{vmatrix} 1 & . & . & . & 1 \\ \aleph_1 & 1 & . & . & x \\ \aleph_2 & \aleph_1 & 1 & . & x^2 \\ \aleph_3 & \aleph_2 & \aleph_1 & 1 & x^3 \\ \aleph_4 & \aleph_3 & \aleph_2 & \aleph_1 & x^4 \end{vmatrix} = x^4 - \Sigma\alpha \cdot x^3 + \Sigma\alpha\beta \cdot x^2 - \ldots,$$

whence are got, by equating coefficients, the determinant expressions for $\Sigma\alpha$, $\Sigma\alpha\beta$, ... in terms of the alephs. The two other modes of expressing the same relations follow in the way one would expect.

HANKEL, H. (1865).

[Darstellung symmetrischer Functionen durch die Potenzsummen. *Crelle's Journ.*, lxvii. pp. 90–94.]

Hankel's theorem may be put in the form

$$\sum x_1^{\alpha} x_2^{\beta} \ldots . \dot{x}_n^{\nu} = \frac{1}{n!} \sum{}' (\Delta_n) = \frac{1}{(n-1)!} \sum{}'' (\Delta_n),$$

where

$$\Delta_n = \begin{vmatrix} s_\alpha & 1 & . & . & \ldots & . \\ s_{\alpha+\beta} & s_\beta & 2 & . & \ldots & . \\ s_{\alpha+\beta+\gamma} & s_{\beta+\gamma} & s_\gamma & 3 & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ s_{\alpha+\beta+\ldots+\nu} & s_{\beta+\gamma+\ldots+\nu} & s_{\gamma+\ldots+\nu} & s_{\delta+\ldots+\nu} & \ldots & s_\nu \end{vmatrix}$$

and the items comprised in $\Sigma'(\Delta_n)$ are obtained by permutation of $\alpha, \beta, \gamma, \ldots, \nu$, and those comprised in $\Sigma''(\Delta_n)$ by permutation of all except ν. For example,

$$\sum x_1^{\alpha} x_2^{\beta} x_3^{\gamma} = \frac{1}{2!} \sum{}'' \left(\begin{vmatrix} s_\alpha & 1 & . \\ s_{\alpha+\beta} & s_\beta & 2 \\ s_{\alpha+\beta+\gamma} & s_{\beta+\gamma} & s_\gamma \end{vmatrix} \right)$$

$$= \frac{1}{2!} \sum{}'' (s_\alpha s_\beta s_\gamma + 2 s_{\alpha+\beta+\gamma} - s_{\alpha+\beta} s_\gamma - 2 s_{\beta+\gamma} s_\alpha)$$

$$= \frac{1}{2!} \left[\begin{matrix} s_\alpha s_\beta s_\gamma + 2 s_{\alpha+\beta+\gamma} - s_{\alpha+\beta} s_\gamma - 2 s_{\beta+\gamma} s_\alpha \\ + s_\beta s_\alpha s_\gamma + 2 s_{\beta+\alpha+\gamma} - s_{\beta+\alpha} s_\gamma - 2 s_{\alpha+\gamma} s_\beta \end{matrix} \right]$$

$$= s_\alpha s_\beta s_\gamma + 2 s_{\alpha+\beta+\gamma} - s_{\alpha+\beta} s_\gamma - s_{\beta+\gamma} s_\alpha - s_{\alpha+\gamma} s_\beta,$$

and $$\sum x_1^\alpha x_2^\beta x_3^\gamma = \frac{1}{3!}\sum{}' (s_\alpha s_\beta s_\gamma + 2s_{\alpha+\beta+\gamma} - s_{\alpha+\beta}s_\gamma - 2s_{\beta+\gamma}s_\alpha)$$

$$= \frac{1}{3!}\left[\begin{array}{r} 6s_\alpha s_\beta s_\gamma + 12s_{\alpha+\beta+\gamma} - 2(s_{\alpha+\beta}s_\gamma + s_{\alpha+\gamma}s_\beta + s_{\beta+\gamma}s_\alpha) \\ -4(s_{\beta+\gamma}s_\alpha + s_{\beta+\alpha}s_\gamma + s_{\gamma+\alpha}s_\beta) \end{array}\right]$$

$$= s_\alpha s_\beta s_\gamma + \ldots\ldots, \text{ as before.}$$

The mode of proof is the gradational, the step being from one value of n to the next higher.

SIACCI, F. (1865).

[Sull' uso dei determinanti per rappresentare la somma delle potenze intere dei numeri naturali. *Annali di Mat.*, vii. pp. 19–24.]

Beginning with the determinant

$$\left|\begin{array}{cccccccc} z & 1 & . & . & \ldots & . & . \\ z^2 & 1 & 2 & . & \ldots & . & . \\ z^3 & 1 & 3 & 3 & \ldots & . & . \\ z^4 & 1 & 4 & 6 & \ldots & . & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ z^r & 1 & (r)_1 & (r)_2 & \ldots & (r)_{r-2} & (r)_{r-1} \\ z^{r+1} & 1 & (r+1)_1 & (r+1)_2 & \ldots & (r+1)_{r-2} & (r+1)_{r-1} \end{array}\right|,$$ or R(z) say,

Siacci performs the operation

$$\text{col}_1 + \text{col}_2 + z\,\text{col}_3 + z^2\text{col}_4 + \ldots\ldots + z^{r-1}\text{col}_{r+1},$$

and obtains at once

$$\text{R}(z) = \text{R}(z+1) - (r+1)!\,z^r,$$

whence $$\text{R}(z+1) = \text{R}(z+2) - (r+1)!\,(z+1)^r,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\text{R}(z+n) = \text{R}(z+n+1) - (r+1)!\,(z+n)^r,$$

and, therefore, by addition and substitution of 0 for z,

$$\text{R}(n+1) = (r+1)!\,(1^r + 2^r + \ldots + n^r)$$

as desired.

Siacci, however, goes on to point out that by performing on R(z) the operation

$$\text{col}_1 + \text{col}_2 - z\,\text{col}_3 + z^2\text{col}_4 - z^3\text{col}_5 + \ldots\ldots$$

there is obtained

$$R(-z) = R(-z+1) + (-1)^r(r+1)!\, z^r,$$

and thence, in the same way as before,

$$(-1)^r R(-n) = (r+1)!\,(1^r+2^r+ \ldots +n^r).$$

He thus has two expressions for $1^r+2^r+ \ldots +n^r$, namely,

$$\frac{R(n+1)}{(r+1)!} \quad \text{and} \quad (-1)^r \frac{R(-n)}{(r+1)!},$$

and the curious deduction

$$R(n+1) = (-1)^r R(-n),$$

which it is interesting to verify by taking the ordinary expression for the sum of the 3rd or 4th powers.

The rest of the paper is occupied with deducing from $R(n)$ the well-known non-determinant form of expression involving Bernoulli's numbers, these last appearing as persymmetric recurrents, namely,

$$B_n = (-1)^{n+1} \cdot (2n)! \begin{vmatrix} \frac{1}{2!} & 1 & . & . & \ldots \\ \frac{1}{3!} & \frac{1}{2!} & 1 & . & \ldots \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \ldots \\ . & . & . & . & . \end{vmatrix}_{2n}.$$

SALMON, G. (1866).

[Lessons Introductory to the Modern Higher Algebra, 2nd edition. viii+296 pp. Dublin.]

A table is given (pp. 285–292) of the simple symmetric functions of $\alpha, \beta, \gamma, \ldots$ from the 1st degree to the 10th inclusive in terms of $\Sigma\alpha, \Sigma\alpha\beta, \Sigma\alpha\beta\gamma, \ldots$, the results being taken from Cayley's memoir of 1856 and checked with Meyer Hirsch's table of 1804.

The connection of such a table with recurrents has more than once appeared (see above, p. 210, . . .): it should be recalled, however, that a no less close connection exists with alternants.

WOLSTENHOLME, J. (1867).

[A Book of Mathematical Problems, ... 344 pp. London and Cambridge.]

Problem 922 concerns the determinant whose first row is

$$1-x_1, \quad x_1(1-x_2), \quad x_1x_2(1-x_3), \quad \ldots, \quad x_1x_2\ldots x_n,$$

and whose $(r+1)^{\text{th}}$ row is in every case got from the r^{th} by differentiating with respect to x_r. The value of the determinant is readily seen to be 1.

SARDI, C. (1867).

[Nota sui numeri primi. *Giornale di Mat.*, v. pp. 371–376.]

Defining $k_{n,p}$ by the identical equation

$$(x+1)(x+2)\ldots(x+n) \equiv x^n+k_{n,1}x^{n-1}+k_{n,2}x^{n-2}+\ldots.$$

Sardi establishes the relations

$$\left.\begin{aligned} k_{n,1}-(n+1)_2 &= 0, \\ 2k_{n,2}-(n)_2k_{n,1}-(n+1)_3 &= 0, \\ 3k_{n,3}-(n-1)_2k_{n,2}-(n)_3k_{n,1}-(n+1)_4 &= 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &, \end{aligned}\right\}$$

and thence obtains

$$k_{n,p} = \frac{1}{p!}\begin{vmatrix} (n+1)_2 & -1 & . & \ldots & . & . \\ (n+1)_3 & (n)_2 & -2 & \ldots & . & . \\ (n+1)_4 & (n)_3 & (n-1)_2 & \ldots & . & . \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & & \\ (n+1)_p & (n)_{p-1} & (n-1)_{p-2} & \ldots & (n-p+3)_2 & -(p-1) \\ (n+1)_{p+1} & (n)_p & (n-1)_{p-1} & \ldots & (n-p+3)_2 & (n-p+2)_2 \end{vmatrix},$$

from which his results regarding primes are deduced.

It must be noted, however, that almost a century earlier the said set of relations was ready to hand for the construction of a recurrent, having been reached by Lagrange in 1771.*

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* Lagrange, J. L. Démonstration d'un théorème nouveau concernant les nombres premiers. *Nouv. Mém. . . . Acad. des Sci.* (Berlin), An. 1771, p. 125; or *Œuvres complètes*, iii. pp. 422–438.

DIETRICH, M. (1868).

[Ueber den Zusammenhang gewisser Determinanten mit Bruchfunctionen. *Crelle's Journ.*, lxix. pp. 190–196.]

Viewing $1/(1+a_1x+a_2x^2+\ldots.)$ as the generating function of the persymmetric recurrent

$$\begin{vmatrix} a_1 & 1 & . & \ldots. & . \\ a_2 & a_1 & 1 & \ldots. & . \\ . & . & . & . & . \\ a_n & a_{n-1} & a_{n-2} & \ldots. & a_1 \end{vmatrix},$$

Dietrich seeks the generating functions of two recurrents A_n and B_n which differ from this in having b's in place of a's in certain columns, namely, A_n in the even-numbered columns and B_n in the odd-numbered columns. From an expansion of A_n and B_n it readily follows that

$$A_n - a_1B_{n-1} + a_2A_{n-2} - a_3B_{n-3} + \ldots. = 0,$$
$$B_n - b_1A_{n-1} + b_2B_{n-2} - b_3A_{n-3} + \ldots. = 0;$$

so that on putting

$$a(x) \equiv 1 + a_1x + a_2x^2 + \ldots,$$
$$b(x) \equiv 1 + b_1x + b_2x^2 + \ldots,$$

and

$$A(x) \equiv 1 - B_1x + A_2x^2 - B_3x^3 + \ldots,$$
$$B(x) \equiv 1 - A_1x + B_2x^2 - A_3x^3 + \ldots,$$

he obtains

$$a(x)\cdot A(x) = 1 - (B_1 - a_1)\cdot x + 0\cdot x^2 - \ldots,$$
$$b(x)\cdot A(x) = 1 + 0\cdot x + (A_2 - b_1B_1 + b_2)x^2 + \ldots,$$
$$a(x)\cdot B(x) = 1 + 0\cdot x + (B_2 - a_1A_1 + a_2)x^2 + \ldots,$$
$$b(x)\cdot B(x) = 1 - (A_1 - b_1)\cdot x + 0\cdot x^2 - \ldots,$$

and consequently

$$\left.\begin{aligned} a(x)\cdot A(x) + a(-x)\cdot A(-x) &= 2, \\ b(x)\cdot A(x) - b(-x)\cdot A(-x) &= 0, \end{aligned}\right\}$$
$$\left.\begin{aligned} a(x)\cdot B(x) - a(-x)\cdot B(-x) &= 0, \\ b(x)\cdot B(x) + b(-x)\cdot B(-x) &= 2, \end{aligned}\right\}$$

whence

$$A(x) = \frac{2b(-x)}{a(x)\cdot b(-x)+a(-x)\cdot b(x)},$$

$$B(x) = \frac{2a(-x)}{a(x)\cdot b(-x)+a(-x)\cdot b(x)}.$$

In like manner he next finds the three generating functions of three similarly related recurrents, and so generally.

POLIGNAC, PRINCE C. DE (1868).

[On a problem in combinations. *Proceed. London Math. Soc.*, ii. pp. 69–74.]

The problem is to find the number of combinations of r things taken from the $2p+q$ things

$$a_1,\ a_2,\ \ldots,\ a_p,\ b_1,\ b_2,\ \ldots,\ b_p,\ c_1,\ c_2,\ \ldots,\ c_q,$$

subject to the condition that no b can be taken along with the corresponding a. The section (§ iii.) of the paper in which recurrents are used is not only of little interest for the student of determinants, but is quite superfluous in view of the satisfactory solution obtained in the preceding section (p. 71), namely,

$$2^r C_{p,r} + 2^{r-1} C_{p,r-1} C_{q,1} + 2^{r-2} C_{p,r-2} C_{q,2} + \ldots + C_{q,r},$$

which (p. 74) is the coefficient of x^r in the expansion of

$$(1+2x)^p (1+x)^q.$$

SARDI, C. (1869).

[Sulle somme dei divisori dei numeri. *Giornale di Mat.*, vii. pp. 112–116.]

[Su talune serie, ed applicazione all' aritmetica. *Giornale di Mat.*, vii. pp. 257–312.]

The expression obtained in the first of these papers for the sum of the divisors of any integer n is

$$(-1)^n \begin{vmatrix} A_1 & 1 & . & . & \ldots. \\ 2A_2 & A_1 & 1 & . & \ldots. \\ 3A_3 & A_2 & A_1 & 1 & \ldots. \\ 4A_4 & A_3 & A_2 & A_1 & \ldots. \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n,$$

where $A_{\frac{1}{2}p(p+1)} = (-1)^p(2p+1)$, and the other A's vanish. Another is given in the second paper, which contains in addition a number of other recurrents connected with the Theory of Integers.

EUGENIO, V. (1870).

[Considerazioni intorno a taluni determinanti particolari. *Giornale di Mat.*, vii. pp. 285–290.]

The determinant

$$\begin{vmatrix} x_1 & 1 & . & . & \dots \\ 2x_2 & x_1 & 1 & . & \dots \\ 3x_3 & x_2 & x_1 & 1 & \dots \\ 4x_4 & x_3 & x_2 & x_1 & \dots \\ . & . & . & . & . \end{vmatrix}_n, \quad \text{or N say,}$$

we are already familiar with, knowing that it equals

$$\alpha^n+\beta^n+\gamma^n+\dots,$$

when $x_1 = \Sigma\alpha$, $x_2 = \Sigma\alpha\beta$, $x_3 = \Sigma\alpha\beta\gamma$, . . . (*Hist.*, ii. p. 211). Eugenio considers two other special cases, obtaining

$$\mathrm{N} = (-1)^{n-1}(a^n+b^n), \quad \text{when} \quad x_r = (a^{r+1}-b^{r+1})/(a-b),$$
$$\mathrm{N} = (-1)^{n-1}a, \quad \text{when} \quad x_r = (a+r-1)_r;$$

and thence making deductions regarding the roots of certain special equations.

HESS, E. (1872).

[Zur Theorie der Vertauschung der unabhängigen Variabeln. *Zeitschrift f. Math. u. Phys.*, xvii. pp. 1–12.]

The subject here is essentially the same as that of Bruno's paper of 1855 (*Hist.*, ii. p. 214), namely, the successive differentiation of a function of a function. The aims, however, of the writers are different. Bruno expresses the successive differential-coefficients of $\phi\{\psi(x)\}$ with respect to x in the form of determinants in which appear successive differential-coefficients of ψ with respect to x and successive differential-coefficients of ϕ with respect to ψ. Hess, on the other hand, expresses the last of these sets of differential-coefficients in terms of the first two sets. For example, denoting the three sets by

$$\mathrm{F}_1, \mathrm{F}_2, \mathrm{F}_3, \dots; \quad \psi_1, \psi_2, \psi_3, \dots; \quad \phi_1, \phi_2, \phi_3, \dots$$

respectively, we have from him

$$\phi_3 = \frac{1}{2\,!\,\psi_1^5}\begin{vmatrix} 2\psi_1 & . & F_1 \\ 2\psi_2 & \psi_1 & F_2 \\ 2\psi_3 & 3\psi_2 & F_3 \end{vmatrix}, \quad \phi_4 = \frac{1}{3\,!\,\psi_1^7}\begin{vmatrix} 3\psi_1 & . & . & F_1 \\ 3\psi_2 & 2\psi_1 & . & F_2 \\ 3\psi_3 & 5\psi_2 & \psi_1 & F_3 \\ 3\psi_4 & 8\psi_3 & 6\psi_2 & F_4 \end{vmatrix},$$

where the coefficients of the ψ's in the diagonal decrease by 1, and the coefficients of any other ψ, say that in the $(r, s)^{\text{th}}$ place, is the sum of the coefficients in the places $(r-1, s-1)$ and $(r-1, s)$.

SIACCI, F. (1872).

[Quistione 10. *Giornale di Mat.*, x. p. 360; or *Annali di Mat.*, (2) v. p. 304. Solution by F. Tirelli in *Giornale di Mat.*, xii. pp. 75–78.]

The determinants occurring here are two n-line variants of Spottiswoode's recurrent

$$\begin{vmatrix} a_0 & -1 & . & . & \ldots & . & . \\ a_1 & x & -1 & . & \ldots & . & . \\ a_2 & . & x & -1 & \ldots & . & . \\ a_3 & . & . & x & \ldots & . & . \\ . & . & . & . & \ldots & . & . \\ a_{n-1} & . & . & . & \ldots & x & -1 \\ a_n & . & . & . & \ldots & . & x \end{vmatrix}$$

for $a_0x^n + a_1x^{n-1} + \ldots + a_n$, or $F(x)$ say, the one variant P being got from this by the initial operation

$$\text{col}_2 + \frac{1}{a_0}\text{col}_1$$

and being equal to $F(x)/a_0$, and the other Q by the initial operation

$$\text{col}_{n+1} - \frac{x}{a_n}\text{col}_1$$

and being equal to $F(x)/a_n$ after a change of sign of all the elements. Denoting the complementary minor of the $(r, s)^{\text{th}}$ element in the one by $P_{r,s}$ and in the other by $Q_{r,s}$ Siacci affirms (1) that

$$a_0xP_{r,s} + a_nQ_{r,s} = F(x) \quad \text{or} \quad 0$$

according as r and s are equal or unequal: and (2) that if each element of P be increased by y times the corresponding element of Q, the resulting determinant, R say, is equal to $F(x) \cdot F(y)/a_0 a_n$.

The former proposition the solver finds the more troublesome to prove: the latter he establishes by noting that R is symmetric with respect to x and y, and by performing on it the operation which evaluates P and Q, namely,

$$\text{row}_n + x\,\text{row}_{n-1} + x^2\,\text{row}_{n-2} + \ldots .$$

He does not, however, seem to observe that R can also be obtained by multiplying $Q_{x=y}$ by P in row-by-column fashion.

GAMBARDELLI, F. (1873).

[Sui coefficienti delle facoltà analitiche, iv. Appendice. Sullo svilippo delle funzioni isobariche. *Giornale di Mat.*, xi. pp. 86–89.]

Gambardelli's appendix, with its elementary account of isobaric functions and its illustrative examples, resembles Bruno's paper of 1856. It is his fifth example that explicitly concerns us, this being Brioschi's of 1858. He might, however, with equal appropriateness and merely by equating two known expressions (Waring's and Brioschi's) have given us as his second example

$$\begin{vmatrix} a_1 & 1 & . & . & \ldots \\ 2a_2 & a_1 & 1 & . & \ldots \\ 3a_3 & a_2 & a_1 & 1 & \ldots \\ 4a_4 & a_3 & a_2 & a_1 & \ldots \\ . & . & . & . & . \end{vmatrix}_n = n \cdot \sum (-1)^{n+\Sigma e} \cdot (\Sigma e - 1)! \frac{a_1^{e_1} a_2^{e_2} \ldots a_n^{e_n}}{e_1!\, e_2! \ldots e_n!},$$

and with no great additional trouble (see Glaisher 1879, infra)

$$\begin{vmatrix} a_1 & -1 & . & . & \ldots \\ 2a_2 & a_1 & -2 & . & \ldots \\ 3a_3 & 2a_2 & a_1 & -3 & \ldots \\ 4a_4 & 3a_3 & 2a_2 & a_1 & \ldots \\ . & . & . & . & . \end{vmatrix}_n = n! \sum \frac{a_1^{e_1} a_2^{e_2} \ldots a_n^{e_n}}{e_1!\, e_2! \ldots e_n!}$$

as his third example. In this way inquiry might have been suggested as to the existence of a relation between isobaric functions and a certain form of recurrent.

BRUNO, F. FAÀ DI (1873).

[Sur les fonctions symétriques. *Comptes Rendus . . . . Acad. des Sci.* (Paris), lxxvi. pp. 163–168.]

Bruno here compensates for the above-mentioned inaccuracies in his *Théorie de l'Élimination* by suggesting an improvement in the tabulation of the symmetric functions and by reprinting as illustrations the first eight tables. Table VI. appears as follows:

	a_6	a_5a_1	a_4a_2	a_3^2	$a_4a_1^2$	$a_3a_2a_1$	$a_3a_1^3$	a_2^3	$a_2^2a_1^2$	$a_2a_1^4$	a_1^6
$\Sigma\alpha^6 =$	−6	6	6	3	−6	−12	6	−2	9	−6	1
$\Sigma\alpha^5\beta =$	6	−1	−6	−3	1	7	−1	2	−4	1	
$\Sigma\alpha^4\beta^2 =$	6	−6	2	−3	2	4	−2	−2	1		
$\Sigma\alpha^3\beta^3 =$	3	−3	−3	3	3	−3	0	1			
$\Sigma\alpha^4\beta\gamma =$	−6	1	2	3	−1	−3	1				
$\Sigma\alpha^3\beta^2\gamma =$	−12	7	4	−3	−3	1					
$\Sigma\alpha^3\beta\gamma\delta =$	6	−1	−2	0	1						
$\Sigma\alpha^2\beta^2\gamma^2 =$	−2	2	−2	1							
$\Sigma\alpha^2\beta^2\gamma\delta =$	9	−4	1								
$\Sigma\alpha^2\beta\gamma\delta\epsilon =$	−6	1									
$\Sigma\alpha\beta\gamma\delta\epsilon\zeta =$	1										

It differs from Cayley's in having Cayley's 4th and 5th columns interchanged, and also the 7th and 8th rows, thus exhibiting perfect axisymmetry and ensuring that in every case a partition and its conjugate shall be equidistant from the middle of the series: for example,

1 1 1 1 and 1 1
1 1 1 1
1
1,

i.e. 4+2 and 2+2+1+1,

in the third place from the beginning and the third place from the end respectively.

GÜNTHER, S. (1873, 1876).

[Darstellung der Näherungswerthe von Kettenbrüchen in independenter Form (pp. 41–46). iv+129 pp. Erlangen.]

[Ueber aufsteigende Kettenbrüche. *Zeitschrift f. Math. u. Phys.*, xxi. pp. 178–191.]

Günther points out that the numerator of the single fraction

$$\frac{a_1b_2b_3\ldots b_n + a_2b_3b_4\ldots b_n + a_3b_4\ldots b_n + \ldots + a_{n-1}b_n + a_n}{b_1b_2\ldots b_n},$$

which is the equivalent of the so-called 'ascending' continued fraction

$$\cfrac{a_1 + \cfrac{a_2 + \cfrac{a_3 + \cdot}{b_3}}{b_2}}{b_1} \qquad \cdot + \frac{a_n}{b_n},$$

or of

$$\frac{a_1}{b_1} + \frac{a_2}{b_1b_2} + \frac{a_3}{b_1b_2b_3} + \ldots + \frac{a_n}{b_1b_2\ldots b_n},$$

is expressible as a determinant, namely,

$$\begin{vmatrix} a_1 & -1 & . & . & \ldots & . & . \\ a_2 & b_2 & -1 & . & \ldots & . & . \\ a_3 & . & b_3 & -1 & \ldots & . & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1} & . & . & . & \ldots & b_{n-1} & -1 \\ a_n & . & . & . & \ldots & . & b_n \end{vmatrix},$$

the form of which recalls Allegret's of 1857. He also transforms a multiple of this into a continuant by performing the operations

$$\begin{aligned} a_{n-1}\cdot\text{row}_n &- a_n \cdot\text{row}_{n-1}, \\ a_{n-2}\cdot\text{row}_{n-1} &- a_{n-1}\cdot\text{row}_{n-2}, \\ \ldots\ldots & \ldots\ldots \end{aligned}$$

and thus shows that the given continued fraction is equal to the 'descending' continued fraction

$$\frac{a_1}{b_1} - \frac{a_2b_1}{a_1b_2+a_2} - \frac{a_1a_3b_2}{a_2b_3+a_3} - \ldots,$$

a result previously reached by Schlömilch * through the form

$$\frac{a_1}{b_1} + \frac{a_2}{b_1 b_2} + \ldots .$$

STUDNIČKA, F. J. (1874).

[Ueber die independente Darstellung der n-ten Derivation von gebrochenen Functionen einer Veränderlichen. *Sitzungsb. . . . böhm. Ges. d. Wiss.* (Prag.), Jahrg. 1874, pp. 1–2.]

The expression obtained for the n^{th} derivative of u/v is not essentially different from Spottiswoode's of 1853, but is much less simple, being in fact Spottiswoode's expression for the $(n-1)^{\text{th}}$ derivative of $(vu' - uv')/v^2$.

WEIHRAUCH, K. (1874).

[Zur Determinantenlehre. *Zeitschrift f. Math. u. Phys.*, xix. pp. 354–360.]

When dealing with the number of terms in the final expansion of a zero-axial determinant (see below in chap. xxi.), Weihrauch proves in two ways (pp. 357–358) that

$$\begin{vmatrix} (m)_1 & 1 & . & . & \ldots & . \\ (m)_2 & (m-1)_1 & 1 & . & \ldots & . \\ (m)_3 & (m-1)_2 & (m-2)_1 & 1 & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (m)_{n-1} & (m-1)_{n-2} & (m-2)_{n-3} & (m-3)_{n-4} & \ldots & 1 \\ (m)_n & (m-1)_{n-1} & (m-2)_{n-2} & (m-3)_{n-3} & \ldots & (m-n+1)_1 \end{vmatrix} = (m)_n .$$

His initial form of expression for the said number of terms is also a recurrent.

We may note in passing that Weihrauch's result is readily

* See *Zeitschrift f. Math. u. Phys.*, iii. (1858), *Literaturz.*, pp. 63–64; and for the subject generally Günther's *Vermischte Untersuchungen zur Geschichte d. math. Wiss.* (1876), pp. 93–135.

established by using the multiplication-theorem, the multiplier and product being, in the case of the 4th order,

$$\begin{vmatrix} 1 & -(m)_1 & (m)_2 & -(m)_3 \\ . & 1 & -(m-1)_1 & (m-1)_2 \\ . & . & 1 & -(m-2)_1 \\ . & . & . & 1 \end{vmatrix}, \quad \begin{vmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ -(m)_4 & (m-1)_3 & -(m-2)_2 & (m-3)_1 \end{vmatrix},$$

respectively.

NÄGELSBACH, H. (1874).

[Zur independenten Darstellung der Bernoullischen Zahlen. *Zeitschrift f. Math. u. Phys.*, xix. pp. 219–233.]

From recurrence-formulae Nägelsbach deduces four different ways of expressing a Bernoulli number by means of a determinant. The two simplest of the results are

$$B_n = \begin{vmatrix} 1 & 3_2 & . & . & \dots \\ 2 & 5_2 & 5_4 & . & \dots \\ 3 & 7_2 & 7_4 & 7_6 & \dots \\ 4 & 9_2 & 9_4 & 9_6 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n \frac{1}{3 \cdot 5 \cdot 7 \dots (2n+1) \cdot 2^{2n-1}}$$

and

$$B_n = \begin{vmatrix} 1 & 3_1 & . & . & \dots \\ 1 & 5_1 & 5_3 & . & \dots \\ 1 & 7_1 & 7_3 & 7_5 & \dots \\ 1 & 9_1 & 9_3 & 9_5 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n \frac{2^{n-1}}{(2n+1)!}$$

where r_s stands for $r(r-1) \dots (r-s+1)/s!$.

HAMMOND, J. (1875).

[On the relation between Bernoulli's numbers and the binomial coefficients. *Proceed. London Math. Soc.*, vii. pp. 9–14.]

Hammond's fundamental result is

$$\frac{\psi(x)}{\phi(x)} = \frac{\psi}{\phi} - \left(\frac{1}{\phi}\right)^2 \begin{vmatrix} \psi & \phi \\ \psi' & \phi' \end{vmatrix} x + \left(\frac{1}{\phi}\right)^3 \begin{vmatrix} \psi & \phi & . \\ \psi' & \phi' & \phi \\ \psi'' & \phi'' & 2\phi' \end{vmatrix} \frac{x^2}{2!} - \dots,$$

where the values of $\psi(x)$, $\psi'(x)$, ... when $x = 0$, are denoted by ψ, ψ', ... It is an immediate consequence of Spottiswoode's of 1853 (*Hist.*, ii. p. 210) in regard to the n^{th} differential-quotient of $\psi(x) \div \phi(x)$. From consideration of the four special cases

$$(a)\ \psi(x) = x, \qquad \phi(x) = e^x - 1,$$
$$(b)\ \psi(x) = e^x - 1, \qquad \phi(x) = e^x + 1,$$
$$(c)\ \psi(x) = \sin x, \qquad \phi(x) = \cos x,$$
$$(d)\ \psi(x) = x, \qquad \phi(x) = \sin x,$$

he obtains four expressions for the n^{th} Bernoulli's number B_n, namely,

$$\frac{(-1)^{n-1}}{(2n+1)!}\begin{vmatrix} 1 & 2 & . & . & \dots \\ 1 & 3 & 3 & . & \dots \\ 1 & 4 & 6 & 4 & \dots \\ . & . & . & . & . \end{vmatrix}_{2n}, \qquad \frac{1}{(2^{2n}-1)\cdot 2^n \cdot (n-1)!}\begin{vmatrix} 1 & 2 & . & . & \dots \\ 1 & 4 & 4 & . & \dots \\ 1 & 6 & 20 & 6 & \dots \\ . & . & . & . & . \end{vmatrix}_{n},$$

$$\frac{2n}{2^{2n}(2^{2n}-1)}\begin{vmatrix} 1 & 1 & . & . & \dots \\ 1 & 3 & 1 & . & \dots \\ 1 & 5 & 10 & 1 & \dots \\ 1 & 7 & 35 & 21 & \dots \\ . & . & . & . & . \end{vmatrix}_{n}, \qquad \frac{2^{n-1}\cdot n!}{(2n+1)!\,(2^{2n-1}-1)}\begin{vmatrix} 1 & 3 & . & . & \dots \\ 1 & 10 & 5 & . & \dots \\ 1 & 21 & 35 & 7 & \dots \\ . & . & . & . & . \end{vmatrix}_{n}.$$

He also considers the case where $\psi(x) = 1$ and $\phi(x) = \cos x$, and obtains for the n^{th} Eulerian number E_n the expression

$$\begin{vmatrix} 1 & 1 & . & . & \dots \\ 1 & 6 & 1 & . & \dots \\ 1 & 15 & 15 & 1 & \dots \\ 1 & 28 & 70 & 28 & \dots \\ . & . & . & . & . \end{vmatrix}.$$

It is important to note, however, that all these are with equal appropriateness viewable as cases of Faure's expression of 1855 for the quotient of two power-series,

$$\frac{a_0x^m + a_1x^{m-1} + \dots}{b_0x^n + b_1x^{n-1} + \dots} = \frac{a_0}{b_0}x^{m-n} - \frac{|a_0b_1|}{b_0^2}x^{m-n-1} + \frac{1}{b_0^3}\begin{vmatrix} a_0 & b_0 & . \\ a_1 & b_1 & b_0 \\ a_2 & b_2 & b_1 \end{vmatrix}x^{m-n-2} - \dots.$$

GLAISHER, J. W. L. (1876).

[Expressions for Laplace's coefficients, Bernoullian and Eulerian numbers, etc., as determinants. *Messenger of Math.*, vi. pp. 49–63.]

This paper seems consequent on Hammond's, and implicitly recognises the fact that such a general result as Hammond's or Faure's enables any one to multiply instances at will. Thus, from consideration of the quotient $x/(e^x-1)$ Glaisher obtains Siacci's expression for the n^{th} Bernoulli number with the concomitant result,

$$\begin{vmatrix} \frac{1}{2!} & 1 & . & . & \dots \\ \frac{1}{3!} & \frac{1}{2!} & 1 & . & \dots \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \dots \\ . & . & . & . & . \end{vmatrix}_{2n-1} = 0;$$

also, by partitioning the original set of equations into two, he obtains from the one Scherk's implied result (*Hist.*, i. p. 479),

$$\mathrm{B}_n = \tfrac{1}{2}\cdot(2n)! \begin{vmatrix} \frac{1}{3!} & 1 & . & . & \dots \\ \frac{3}{5!} & \frac{1}{3!} & 1 & . & \dots \\ \frac{5}{7!} & \frac{1}{5!} & \frac{1}{3!} & 1 & \dots \\ \frac{7}{9!} & \frac{1}{7!} & \frac{1}{5!} & \frac{1}{3!} & \dots \\ . & . & . & . & . \end{vmatrix}_{n'}$$

and from the other

$$\mathrm{B}_n = 2^n\cdot(2n)! \begin{vmatrix} \frac{1}{4!} & 1 & . & . & \dots \\ \frac{2}{6!} & \frac{1}{4!} & 1 & . & \dots \\ \frac{3}{8!} & \frac{1}{6!} & \frac{1}{4!} & 1 & \dots \\ \frac{4}{10!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & \dots \\ . & . & . & . & . \end{vmatrix}_{n'}$$

Similarly, having reached Hammond's first recurrent with the concomitant result,

$$\begin{vmatrix} 1 & 2 & . & . & \ldots. \\ 1 & 3 & 3 & . & \ldots. \\ 1 & 4 & 6 & 4 & \ldots. \\ . & . & . & . & . \end{vmatrix}_{2n-1} = 0,$$

he obtains, by the like partitionment,

$$B_n = \frac{1}{2\cdot 3\cdot 5 \ldots (2n+1)} \begin{vmatrix} 1 & 2 & . & . & \ldots. \\ 1 & 3 & 3 & . & \ldots. \\ 1 & 5 & 10 & 5 & \ldots. \\ 1 & 7 & 21 & 35 & \ldots. \\ . & . & . & . & . \end{vmatrix}_{n+1}$$

and

$$B_n = \frac{2^n}{(2n+2)!} \begin{vmatrix} 1 & 2 & . & . & \ldots. \\ 1 & 4 & 6 & . & \ldots. \\ 1 & 6 & 15 & 15 & \ldots. \\ 1 & 8 & 28 & 70 & \ldots. \\ . & . & . & . & . \end{vmatrix}_{n+1}.$$

To these expressions for B_n have also to be added

$$\frac{(2n)!}{2^{2n-1}(2^{2n}-1)} \begin{vmatrix} \frac{1}{3!} & 1 & . & \ldots. \\ \frac{2}{5!} & \frac{1}{2!} & 1 & \ldots. \\ \frac{3}{7!} & \frac{1}{4!} & \frac{1}{2!} & \ldots. \\ . & . & . & . \end{vmatrix}_{n-1}, \qquad \frac{(2n)!}{2^{2n-1}} \begin{vmatrix} \frac{1}{3!} & 1 & . & \ldots. \\ \frac{2}{5!} & \frac{1}{3!} & 1 & \ldots. \\ \frac{3}{7!} & \frac{1}{5!} & \frac{1}{3!} & \ldots. \\ . & . & . & . \end{vmatrix}_{n},$$

the one originating in the quotient $\tan x/x$ and the other in $x/\tan x$. The latter, however, is readily obtained from Nägelsbach's first expression of 1874 by dividing the rows in order by 3!, 5!, 7!, ... and multiplying the columns in order by 0!, 2!, 4!, ... We may note also that a like process of transformation applied to the other expression given under Nägelsbach leads to

$$B_n = \begin{vmatrix} \frac{1}{3!} & \frac{1}{2!} & . & . & .... \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & . & .... \\ \frac{1}{7!} & \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & .... \\ \frac{1}{9!} & \frac{1}{8!} & \frac{1}{6!} & \frac{1}{4!} & .... \\ . & . & . & . & . \end{vmatrix}_n \quad 2^{n-1} \cdot (2n-1)!$$

The other quotients considered are

$$1/\cos x, \quad x/\log(1+x), \quad 1/e^x, \quad 1/(1-2hx+h^2)^{\frac{1}{2}},$$

the first leading to Scherk's implied result,

$$E_n = (2n)! \begin{vmatrix} \frac{1}{2!} & 1 & . & .... \\ \frac{1}{4!} & \frac{1}{2!} & 1 & .... \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & .... \\ . & . & . & . \end{vmatrix}_n,$$

the second to

$$\int_0^1 x(x-1)\dots(x-n+1)\,dx = n! \begin{vmatrix} \frac{1}{2} & 1 & . & .... \\ \frac{1}{3} & \frac{1}{2} & 1 & .... \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & .... \\ . & . & . & . \end{vmatrix}_n,$$

the third to Trudi's of 1862, and the fourth to

$$\begin{vmatrix} 1 & 1 & . & . & . & .... \\ . & x & 1 & . & . & .... \\ \frac{1}{2}(x^2-1) & x^2 & 2x & 1 & . & .... \\ . & x^3 & 3x^2 & 3x & 1 & .... \\ \frac{1}{2}\cdot\frac{3}{4}(x^2-1)^2 & x^4 & 4x^3 & 6x^2 & 4x & .... \\ . & . & . & . & . & . \end{vmatrix}_{n+1}$$

as an expression for the so-called n^{th} 'Laplace's' coefficient, that is to say, for the coefficient of h^n in the expansion of the quotient in question.

In a different and less satisfactory way the result, usually attributed * to Janni,

$$a_0 - na_1 + \tfrac{1}{2}n(n-1)a_2 - \ldots = \begin{vmatrix} a_0 & 1 & . & . & . & \ldots \\ a_1 & 1 & 1 & . & . & \ldots \\ a_2 & 1 & 2 & 1 & . & \ldots \\ a_3 & 1 & 3 & 3 & 1 & \ldots \\ . & . & . & . & . & . \end{vmatrix}_{n+1}$$

is obtained, the left-hand member of which becomes

$$(1-a)^n, \quad (-1)^n \Delta^n 0^m,$$

when for a_r we substitute

$$a^r, \quad r^m$$

respectively.

Lastly, by differentiating $\exp e^x$ and its equivalent power-series, in which the coefficient of x^n is $e^{1+\Delta 0^n}/n!$, a set of equations is reached giving

$$e^{\Delta 0^n} = \begin{vmatrix} 1 & -1 & . & . & . & \ldots \\ 1 & 1 & -2 & . & . & \ldots \\ \frac{1}{2!} & 1 & 1 & -3 & . & \ldots \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 1 & -4 & \ldots \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 1 & \ldots \\ . & . & . & . & . & . \end{vmatrix}_{n}.$$

Hammond being the only previous writer referred to, and Faure's general result being in substance arrived at, it might have been well to indicate the relationship that is established by merely substituting

$$\psi + x\psi' + \tfrac{1}{2}x^2\psi'' + \ldots, \qquad \phi + x\phi' + \tfrac{1}{2}x^2\phi'' + \ldots,$$

for the former's $\psi(x)$, $\phi(x)$.

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* See below in chap. xx. under Janni (1876).

BRUNO, F. FAÀ DI (1876): GLAISHER, J. W. L. (1876).

[Théorie des Formes Binaires (§§ 89, 90). xvi+320 pp. Turin.]

[On certain determinants. *Report . . . British Assoc.*, xlvi. (Glasgow), pp. 13–14.]

To the tables of symmetric functions previously noted by us may now be added the table of 'weight eleven,' giving the fifty-six functions Σa^{11}, $\Sigma a^{10}\beta$, $\Sigma a^{9}\beta^{2}$, $\Sigma a^{9}\beta\gamma$, ... in terms of Σa, $\Sigma a\beta$, $\Sigma a\beta\gamma$, ... This Bruno reprints from a paper of his of the preceding year (*Nachrichten . . . Ges. d. Wiss.* (Göttingen), 1875, pp. 390–393).

In regard to the other paper, it is enough to say that the recurrents appearing in it are connected with the Theory of Integers and resemble those of Sardi (1869).

LUCAS, E. (1876, 1877).

[Théorie nouvelle des nombres de Bernoulli et d'Euler. *Comptes Rendus . . . . Acad. des Sci.* (Paris), lxxxiii. pp. 539–541; in full in *Annali di Mat.*, viii. pp. 56–79.]

[Sur les rapports qui existent entre le triangle arithmétique de Pascal et les nombres de Bernoulli. *Nouv. Annales de Math.*, xv. pp. 497–499.]

[Sur les sommes des puissances semblables des nombres entiers. *Nouv. Annales de Math.*, xvi. pp. 18–26.]

Lucas' main purpose is to point out a fecund source of recurrence-formulae for the sum S_r of the r^{th} powers of the first n integers. Having shown that by putting n, $n-1$, $n-2$, . . . , 1 for n in the identity

$$(n+1)^r - n^r = (r)_1 n^{r-1} + (r)_2 n^{r-2} + \ldots .$$

and adding there is obtained

$$(n+1)^r - 1 = (r)_1 S_{r-1} + (r)_2 S_{r-2} + \ldots = (S+1)^r_{\downarrow} - S_r, \text{ say},$$

and thence Siacci's recurrent for $(r+1)!\,S_r$, followed by a recurrent for the n^{th} Bernoulli's number, he proceeds to say that instead of $(n+1)^r - n^r$ we might take

$$f(n+1) - f(n) = a_0 n^r + a_1 n^{r-1} + \ldots + a_r,$$

and thence obtain

$$f(n+1)-f(1) = a_0S_r+a_1S_{r-1}+\ldots+a_rS_0.$$

As a matter of fact, however, the forms of $f(n)$ which he finds useful are not very varied, being, in addition to n^r,

$$(n-1)^r, \quad (n-\tfrac{1}{2})^r, \quad n(n+1)\ldots(n+r-1),$$

the second of which leads to the recurrence-formula

$$(2n+1)^r-1 = (2S+1)^r_{\downarrow}-(2S-1)^r_{\downarrow},$$

and thence to

$$2^{2r+1}\cdot 1\cdot 3\cdot 5\ldots(2r+1)S_{2r}$$

$$= (-1)^{\frac{1}{2}r(r+1)}\cdot \begin{vmatrix} (2n+1)^1 & 1 & . & . & \ldots & . \\ (2n+1)^3 & 1 & (3)_2 & . & \ldots & . \\ (2n+1)^5 & 1 & (5)_2 & (5)_4 & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (2n+1)^{2r-1} & 1 & (2r-1)_2 & (2r-1)_4 & \ldots & (2r-1)_{2r-2} \\ (2n+1)^{2r+1} & 1 & (2r+1)_2 & (2r+1)_4 & \ldots & (2r+1)_{2r-2} \end{vmatrix}_{r+1}.$$

Since with Lucas the Bernoulli numbers, $\beta_0, \beta_1, \beta_2, \ldots$ are

$$1, \ -\tfrac{1}{2}, \ \tfrac{1}{6}, \ 0, \ -\tfrac{1}{30}, \ 0, \ \tfrac{1}{42}, \ 0, \ -\tfrac{1}{30}, \ 0, \ \ldots,$$

our B_r being thus equal to $(-1)^{r-1}\beta_{2r}$, he is enabled to write Bernoulli's relation in the form

$$r(S_{r-1}-n^{r-1}) = (\beta+n)^r_{\downarrow}-\beta_r,$$

and thus obtain such recurrence-formulae as

$$(\beta+1)^r_{\downarrow}-\beta_r = 0,$$
$$\beta_r-(\beta-1)^r_{\downarrow} = r\cdot(-1)^{r-1},$$
$$\ldots\ldots\ldots$$

STUDNIČKA, F. J. (1877).

[Ueber die independente Darstellung der n-ten Derivation einer Potenz, deren Basis und Exponent verschiedene Funktionen einer Variablen bilden. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), pp. 368–373.]

[Weitere Beiträge zur Differentialrechnung. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), pp. 393–399.]

In the first of these papers the n^{th} differential-quotient of u^v is obtained in the form

$$u^v\cdot\Delta_n,$$

where

$$\Delta_n = \begin{vmatrix} \phi' & -1 & . & . & . \\ \phi'' & \phi' & -1 & . & . \\ \phi''' & 2\phi' & \phi' & . & . \\ . & . & . & . & . \\ \phi^{(n)} & (n-1)_1\phi^{(n-1)} & (n-1)_2\phi^{(n-2)} & . & \phi' \end{vmatrix}$$

and $\phi = v \log u$.

The second paper is occupied with special cases of the corresponding expression for the n^{th} differential-quotient of u/v.

PELNAŘ, M. (1877): STUDNIČKA, F. J. (1878): MORENO, G. (1878).

[O souctu m-tých mocnin čisel řady přirozené. *Časopis pro pěstování math. a fys.*, vi. pp. 279–280.]

[Ueber eine neue Formel der Combinatorik. *Zitzungsb. . . . Ges. d. Wiss.* (Prag), pp. 295–298.]

[Dimostrazione di un teorema di Eisenstein (Nota *). *Giornale di Mat.*, xvi. pp. 175–176; or *Nouv. Corresp. Math.*, iv. p. 334; vi. pp. 79–81.]

The result of the first paper is Siacci's of 1865; that of the second paper is Sardi's of 1867; and that of the third is Weihrauch's of 1874.

WOLSTENHOLME, J. (1878).

[Mathematical Problems . . . . 2nd edition. x+480 pp. London.]

Two fresh results are given. The first is

$$\begin{vmatrix} a & 1 & . & . & . \\ a & a & 1 & . & . \\ 1 & a & a & 1 & . \\ . & 1 & a & a & . \\ . & . & 1 & a & . \\ . & . & . & . & . \end{vmatrix}_n = \frac{v_{n+1} - v_n - 1}{a-3},$$

where

$$v_n = (a-1)^n \quad (n-1)_1(a-1)^{n-2} + (n-2)_2(a-1)^{n-4} - \ldots .$$

The value is also stated to be

$$\{(p^{n+2}-p^{n+1}-p)-(q^{n+2}-q^{n+1}-q)\} \div (p-q)(p+q-2),$$

where p, q are the roots of the equation $x^2-(a-1)x+1 = 0$, and to be

$$\frac{1}{a-3}\left\{1+(-1)^n \frac{\sin(n+1)\theta+\sin(n+2)\theta}{\sin\theta}\right\},$$

where $2\cos\theta = 1-a$.

Next, the value of the n-line persymmetric recurrent, whose repeating elements in the successive diagonals are

$$1,\ a_1,\ a_2,\ \ldots\ldots,\ a_r,\ 0,\ 0,\ \ldots\ldots$$

respectively, is given equal to

$$\frac{x_1^{n+r-1}}{f'(x_1)} + \frac{x_2^{n+r-1}}{f'(x_2)} + \ldots\ldots + \frac{x_r^{n+r-1}}{f'(x_r)},$$

where $f(x) = x^r-a_1x^{r-1}+a_2x^{r-2}-\ldots\ldots$, and $x_1, x_2, \ldots\ldots, x_r$ are the roots of the equation $f(x) = 0$. It is also noted that when each a becomes 1, the determinant vanishes, save when

$$n \equiv 0 \quad \text{or} \quad 1 \pmod{r},$$

the value then being $(-1)^{\frac{n}{r}}$ or $(-1)^{\frac{n-1}{r}}$ respectively. The former statement recalls Brioschi's general result of 1855 (*Hist.*, ii. pp. 212, . . .), the determinant being equal to the n^{th} aleph function of $x_1, \ldots, x_r$. The latter cannot be right; for, if $U_{n,r}$ stands for the n-line persymmetric recurrent with r consecutive units in its first column, it is clear that

$$U_{1,1} = 1, \quad \text{and} \quad U_{n,n} = 0 \quad \text{when } n>1,$$

the vanishing being due to the identity of the last two rows. Also, we have

$$\begin{aligned}
U_{n,n-1} &= U_{n-1,n-1}-U_{n-2,n-2}+\ldots\ldots+(-1)^{n-2}U_{1,1} = (-1)^n,\\
U_{n,n-2} &= U_{n-1,n-2}-U_{n-2,n-2}+\ldots\ldots+(-1)^{n-3}U_{2,2} = (-1)^{n-1},\\
U_{n,n-3} &= U_{n-1,n-3}-U_{n-2,n-3}+\ldots\ldots+(-1)^{n-4}U_{3,3}\\
&= U_{n-1,n-3}-U_{n-2,n-3} \quad \text{if } n>4,\\
&= (-1)^{n-2}-(-1)^{n-2} = 0, \quad \text{if } n>4.
\end{aligned}$$

. . . . . .

$$U_{n,1} = 1.$$

JANNI, V. (1878).

[Sopra una formola di Waring. *Rendic. . . . Accad. delle Sci. . . .* (Napoli), Anno xvii. pp. 27–31.]

The determinant form (*Hist.*, ii. p. 211) for s_r in terms of the coefficients of the equation

$$x^n+a_1x^{n-1}+a_2x^{n-2}+ \ . \ . \ . \ +a_n = 0$$

is taken, and expanded in terms of the elements of the first column and their complementary minors, the result being written

$$(-1)^r s_r = a_1\mathrm{D}_{r-1}-2a_2\mathrm{D}_{r-2}+3a_3\mathrm{D}_{r-3}- \ . \ . \ . \ .$$

He then expands the D's as Trudi did in 1862; and making substitution for them deduces Waring's formula, as Trudi also did.

Another contribution is noted in chap. xx., under Janni, 1876.

GLAISHER, J. W. L. (1878).

[On a class of determinants. *Messenger of Math.*, vii. pp. 160–165.]

The results here given by Glaisher are fairly evident consequences of Faure's proposition of 1855, namely, that if

$$\frac{1+b_1x+b_2x^2+ \ . \ . \ .}{1+a_1x+a_2x^2+ \ . \ . \ .} \equiv 1-\mathrm{P}_1x+\mathrm{P}_2x^2-\mathrm{P}_3x^3+ \ . \ . \ . \ .$$

then

$$\mathrm{P}_n = \begin{vmatrix} a_1-b_1 & 1 & . & . & . \ . \ . \ . \\ a_2-b_2 & a_1 & 1 & . & . \ . \ . \ . \\ a_3-b_3 & a_2 & a_1 & 1 & . \ . \ . \ . \\ . \ . \ . & . \ . \ . & . \ . \ . & . \ . \ . & . \ . \ . \ . \end{vmatrix}_n.$$

First he says that if

$$(1+a_1x+a_2x^2+ \ . \ . \ .)^{-1} \equiv 1-\mathrm{A}_1x+\mathrm{A}_2x^2-\mathrm{A}_3x^3+ \ . \ . \ . \ ,$$

it follows that A_n is the same function of the a's as a_n is of the A's, and that if in addition

$$(1+b_1x+b_2x^2+ \ . \ . \ .)^{-1} \equiv 1-\mathrm{B}_1x+\mathrm{B}_2x^2-\mathrm{B}_3x^3+ \ . \ . \ . \ ,$$

then we have

$$\begin{vmatrix} a_1-b_1 & 1 & . & . & . \ . \ . \\ a_2-b_2 & a_1 & 1 & . & . \ . \ . \\ a_3-b_3 & a_2 & a_1 & 1 & . \ . \ . \\ . \ . \ . & . \ . \ . & . \ . \ . & . \ . \ . & . \ . \ . \end{vmatrix}_n = (-1)^n \begin{vmatrix} \mathrm{B}_1-\mathrm{A}_1 & 1 & . & . & . \ . \ . \\ \mathrm{B}_2-\mathrm{A}_2 & \mathrm{B}_1 & 1 & . & . \ . \ . \\ \mathrm{B}_3-\mathrm{A}_3 & \mathrm{B}_2 & \mathrm{B}_1 & 1 & . \ . \ . \\ . \ . \ . & . \ . \ . & . \ . \ . & . \ . \ . & . \ . \ . \end{vmatrix}_n,$$

$$\begin{vmatrix} a_1-b & 1 & . & . & \dots \\ a_2-b^2 & a_1 & 1 & . & \dots \\ a_3-b^3 & a_2 & a_1 & 1 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n = \begin{vmatrix} a_1-b & 1 & . & . & \dots \\ a_2-ba_1 & a_1-b & 1 & . & \dots \\ a_3-ba_2 & a_2-ba_1 & a_1-b & 1 & \dots \\ . & . & . & . & \dots \end{vmatrix}_{n},$$

and other identities of like character.

GLAISHER, J. W. L. (1879).

[On a class of determinants, with a note on partitions. *Messenger of Math.*, viii. pp. 158–167.]

The differential-quotient of $\log(1+a_1x+a_2x^2+\dots)$ being the quotient of two power-series, Glaisher obtains with the help of Faure's result

$$(-)^{n-1}\frac{1}{n}\begin{vmatrix} a_1 & 1 & . & . & \dots \\ 2a_2 & a_1 & 1 & . & \dots \\ 3a_3 & a_2 & a_1 & 1 & \dots \\ 4a_4 & a_3 & a_2 & a_1 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n = \text{coefficient of } x^n \text{ in } \log(1+a_1x+a_2x^2+\dots),$$

and in similar fashion

$$\frac{1}{n}\begin{vmatrix} a_1 & -1 & . & . & \dots \\ 2a_2 & a_1 & -2 & . & \dots \\ 3a_3 & 2a_2 & a_1 & -3 & \dots \\ 4a_4 & 3a_3 & 2a_2 & a_1 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n = \text{coefficient of } x^n \text{ in } \exp(a_1x+a_2x^2+\dots).$$

If we call the former coefficient b_n we have

$$\exp(b_1x+b_2x^2+\dots) = 1+a_1x+a_2x^2+\dots,$$

and therefore

$$a_n = \frac{1}{n!}\begin{vmatrix} b_1 & -1 & . & . & \dots \\ 2b_2 & b_1 & -2 & . & \dots \\ 3b_3 & 2b_2 & b_1 & -3 & \dots \\ 4b_4 & 3b_3 & 2b_2 & b_1 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n$$

By putting n for a_n in the first result and noting that the function expanded in powers of x is then

$$-\log(1-x)-\log(1-x^2)+\log(1+x^3),$$

it is found that

$$\begin{vmatrix} 1^2 & 1 & . & . & \dots \\ 2^2 & 1 & 1 & . & \dots \\ 3^2 & 2 & 1 & 1 & \dots \\ 4^2 & 3 & 2 & 1 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n = 0,\ 1,\ -3,\ 4,\ -3,\ 1$$

when to the modulus 6

$$n \equiv 0,\ 1,\ 2,\ 3,\ 4,\ 5.$$

Similarly, by putting $n+1$ for a_n and noting that the function is then $-2\log(1-x)$, it follows that

$$\begin{vmatrix} 1 & 1 & . & . & \dots \\ (3)_1 & 2 & 1 & . & \dots \\ (4)_2 & 3 & 2 & 1 & \dots \\ (5)_3 & 4 & 3 & 2 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n = (-1)^{n-1};$$

and by putting $(n+1)^2$ for a_n, the function then being

$$\log\{(1+x)/(1-x)^3\},$$

we have

$$\begin{vmatrix} 1^3 & 1^2 & . & . & \dots \\ 2^3 & 2^2 & 1^2 & . & \dots \\ 3^3 & 3^2 & 2^2 & 1^2 & \dots \\ 4^3 & 4^2 & 3^2 & 2^2 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n = \begin{cases} -4 \text{ when } n \text{ is even,} \\ \ \ 2 \text{ when } n \text{ is odd.} \end{cases}$$

The note on partitions is of still smaller interest from the point of view of determinants, and closes with a caution, that might have been amplified, against using any of the recurrents of the paper for the practical purpose of finding the coefficients of the logarithmic expressions in question. The caution, indeed, also applies to his papers of 1876 and 1878, the three being considered by him as one and the paragraphs numbered accordingly.

STUDNIČKA, F. J. (1879).

[Notiz über einige Determinanten, in welchen Binomialcoefficienten als Elemente auftreten. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), pp. 292–295.]

One of the results is Trudi's (see above under d'Ovidio), and the other is

$$\begin{vmatrix} (m)_2 & (m)_1 & 1 & . & \dots \\ (m)_3 & (m)_2 & (m)_1 & 1 & \dots \\ (m)_4 & (m)_3 & (m)_2 & (m)_1 & \dots \\ . & . & . & . & . \end{vmatrix}_n = \begin{vmatrix} (m+n-1)_n & (m+n-2)_{n-1} \\ (m+n)_{n+1} & (m+n-1)_n \end{vmatrix},$$

where the given persymmetric determinant is not itself a recurrent but a primary minor of such a determinant, and a simple case of Zeipel's third form of 1865 (see chap. xx.).

It is difficult, however, to see why Studnička did not proceed with his evaluation. Performing on his two-line determinant the operations

$$\mathrm{col}_1 - \mathrm{col}_2, \quad \mathrm{row}_2 - \mathrm{row}_1,$$

we obtain

$$\begin{vmatrix} (m+n-2)_n & (m+n-2)_{n-1} \\ (m+n-2)_{n+1} & (m+n-2)_n \end{vmatrix},$$

and using on this the identity $(h)_k = h!/k!(h-k)!$ we readily reach

$$\frac{(m+n-1)_n \cdot (m+n-1)_{n+1}}{m+n-1}.$$

TANNER, H. W. L. (1879).

[Question 6040. *Educ. Times*, xxxii. p. 243; *Math. from Educ. Times*, xxxvi. p. 83.]

Tanner's result,—in a less awkward form than his,—is

$$\begin{vmatrix} (b)^{n/a} & (b)^{1/b} & . & . & \dots & . \\ (2b)^{n/a} & (2b)^{1/b} & (2b)^{2/b} & . & \dots & . \\ (3b)^{n/a} & (3b)^{1/b} & (3b)^{2/b} & (3b)^{3/b} & \dots & . \\ . & . & . & . & . & . \\ (nb)^{n/a} & (nb)^{1/b} & (nb)^{2/b} & (nb)^{3/b} & \dots & (nb)^{(n-1)/b} \end{vmatrix}$$

$$= (-)^{n-1} \cdot b^{\frac{1}{2}n(n+1)} \cdot 1!\,2!\,3! \dots n!,$$

where

$$b^{n/a} = b(b-a)(b-2a) \ldots (b-\overline{n-1}\cdot a).$$

To the element in the $(n, 1)^{\text{th}}$ place Tanner has $-(nb)^{a/b}$ annexed, the right-hand member being then 0.

CROCCHI, L. (1879).

[Una relazione fra le funzioni simmetriche semplici e le funzioni simmetriche complete. *Giornale di Mat.*, xviii. pp. 377–380.]

The relation is Brioschi's of 1857 connecting the s's and alephs (*Hist.*, ii. pp. 216–217). Crocchi, however, notes in addition that $s_1, s_3, \ldots$ *are the same functions of the* c's *as they* (*the* s's) *are of the alephs, and that* $s_2, s_4, \ldots$ *are the same functions of the* c's *as* $-s_2, -s_4, \ldots$ *are of the alephs*; for example,

$$s_1 = c_1, \qquad = \aleph_1,$$

$$s_2 = \begin{vmatrix} c_1 & 1 \\ 2c_2 & c_1 \end{vmatrix} = -\begin{vmatrix} \aleph_1 & 1 \\ 2\aleph_2 & \aleph_1 \end{vmatrix},$$

$$s_3 = \begin{vmatrix} c_1 & 1 & . \\ 2c_2 & c_1 & 1 \\ 3c_3 & c_2 & c_1 \end{vmatrix} = \begin{vmatrix} \aleph_1 & 1 & . \\ 2\aleph_2 & \aleph_1 & 1 \\ 3\aleph_3 & \aleph_2 & \aleph_1 \end{vmatrix},$$

. . . . . .

He might readily have gone further and affirmed that *if we have got the expressions for the* c's *in terms of the* s's *we have only to change the* 1, 2, 3, . . . *of the minor diagonal into* $-1, -2, -3, \ldots$ *in order to get the expressions for the alephs in terms of the* s's; for example,

$$c_1 = s_1, \qquad \aleph_1 = s_1,$$

$$c_2 = \frac{1}{2!}\begin{vmatrix} s_1 & 1 \\ s_2 & s_1 \end{vmatrix}, \qquad \aleph_2 = \frac{1}{2!}\begin{vmatrix} s_1 & -1 \\ s_2 & s_1 \end{vmatrix},$$

$$c_3 = \frac{1}{3!}\begin{vmatrix} s_1 & 1 & . \\ s_2 & s_1 & 2 \\ s_3 & s_2 & s_1 \end{vmatrix}, \qquad \aleph_3 = \frac{1}{3!}\begin{vmatrix} s_1 & -1 & . \\ s_2 & s_1 & -2 \\ s_3 & s_2 & s_1 \end{vmatrix},$$

. . . . . .

and that *the alephs are the same functions of the* c*'s as the* c*'s are of the alephs*; for example,

$$\aleph_1 = c_1, \qquad c_1 = \aleph_1,$$

$$\aleph_2 = \begin{vmatrix} c_1 & 1 \\ c_2 & c_1 \end{vmatrix}, \qquad c_2 = \begin{vmatrix} \aleph_1 & 1 \\ \aleph_2 & \aleph_1 \end{vmatrix},$$

$$\aleph_3 = \begin{vmatrix} c_1 & 1 & . \\ c_2 & c_1 & 1 \\ c_3 & c_2 & c_1 \end{vmatrix}, \qquad c_3 = \begin{vmatrix} \aleph_1 & 1 & . \\ \aleph_2 & \aleph_1 & 1 \\ \aleph_3 & \aleph_2 & \aleph_1 \end{vmatrix},$$

. . . . . .

As has already been pointed out, however (*Hist.*, ii. p. 218), all these results are of less importance than the three sets of equations from which they are derived, namely,

$$\left.\begin{aligned} s_1 - c_1 &= 0 \\ s_2 - s_1c_1 + 2c_2 &= 0 \\ &\cdots\cdots \end{aligned}\right\} \qquad \left.\begin{aligned} \aleph_1 - c_1 &= 0 \\ \aleph_2 - \aleph_1c_1 + c_2 &= 0 \\ &\cdots\cdots \end{aligned}\right\} \qquad \left.\begin{aligned} s_1 - \aleph_1 &= 0 \\ s_2 + s_1\aleph_1 - 2\aleph_2 &= 0 \\ &\cdots\cdots \end{aligned}\right\}$$

They are but illustrations of the general proposition that the set of recurrence-formulae

$$\left.\begin{aligned} a_1P_1 + a_2Q_1 &= 0 \\ b_1P_2 + b_2P_1Q_1 + b_3Q_2 &= 0 \\ c_1P_3 + c_2P_2Q_1 + c_3P_1Q_2 + c_4Q_3 &= 0 \\ &\cdots\cdots \end{aligned}\right\}$$

gives P_r as a recurrent in terms of Q_r, Q_{r-1}, . . . , and at the same time Q_r as a recurrent in terms of P_r, P_{r-1}, ...; for example,

$$P_3 = -\begin{vmatrix} a_2Q_1 & a_1 & . \\ b_3Q_2 & b_2Q_1 & b_1 \\ c_4Q_3 & c_3Q_2 & c_2Q_1 \end{vmatrix} \div a_1b_1c_1, \qquad Q_3 = -\begin{vmatrix} a_1P_1 & a_2 & . \\ b_1P_2 & b_2P_1 & b_3 \\ c_1P_3 & c_2P_2 & c_3P_1 \end{vmatrix} \div a_2b_3c_4.$$

CHAPTER VIII.

WRONSKIANS, FROM 1862 TO 1874.

SYLVESTER, J. J. (1862).

[Sur une classe nouvelle d'équations différentielles et . . . . *Comptes Rendus . . . Acad. des Sci.* (Paris), liv. pp. 129–132 ; or *Collected Math. Papers*, ii. pp. 308–312.]

Taking the persymmetric Wronskian

$$\begin{vmatrix} y & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_5 \\ y_3 & y_4 & y_5 & y_6 \end{vmatrix},$$

where y_r stands for $d^r y/dx^r$, Sylvester denotes it by

$$\mathrm{D}_4 y,$$

and of course readily finds that

$$\frac{d}{dx}(\mathrm{D}_3 y) = \begin{vmatrix} y & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_3 & y_4 & y_5 \end{vmatrix}, \qquad \frac{d^2}{dx^2}(\mathrm{D}_3 y) = \begin{vmatrix} y & y_1 & y_3 \\ y_1 & y_2 & y_4 \\ y_3 & y_4 & y_6 \end{vmatrix}.$$

He then points out that on this account

$$\begin{vmatrix} \mathrm{D}_3 y & \dfrac{d}{dx}(\mathrm{D}_3 y) \\ \dfrac{d}{dx}(\mathrm{D}_3 y) & \dfrac{d^2}{dx^2}(\mathrm{D}_3 y) \end{vmatrix}, \quad i.e.\ \mathrm{D}_2(\mathrm{D}_3 y),$$

is a minor of the adjugate of $\mathrm{D}_4 y$, and that therefore

$$\mathrm{D}_2(\mathrm{D}_3 y) = \mathrm{D}_2 y \cdot \mathrm{D}_4 y,$$

the generalization being

$$\mathrm{D}_2(\mathrm{D}_m y) = \mathrm{D}_{m-1} y \cdot \mathrm{D}_{m+1} y.$$

FUCHS, L. (1865).

[Zur Theorie der linearen Differentialgleichungen mit veränderlichen Coefficienten. *Crelle's Journ.*, lxvi. pp. 121–160.]

The second section of this (pp. 126–131) deals with the Wronskian, but not in any fresh connection.

ZMURKO, W. (1871).

[Dowód na twierdzenie Hesse'go o wyznaczniku funkcyjnym. *Pamietnik Towarzystwa Nauk Ścislych w Paryżu*, i. pp. 89–92.]

The theorem here proved and spoken of as Hesse's is the first of the two theorems of Hesse's which had been previously reached by Christoffel (*Hist.*, ii. p. 228). Zmurko had already dealt with the subject in 1866 (see above, p. 16).

CAYLEY, A. (1872).

[On Wronski's theorem. *Quart. Journ. of Math.*, xii. pp. 221–228; or *Collected Math. Papers*, ix. pp. 96–102.]

Cayley first shows that in Wronski's own form of development the "loi suprême" is not more general than

$$\mathrm{F}(x) = \mathrm{F} - \frac{\lambda}{|1}\frac{1}{\phi'} f\mathrm{F}' + \frac{\lambda^2}{1\cdot 2}\frac{1}{\phi'^3}\begin{vmatrix} \phi' & f^2\mathrm{F}' \\ \phi'' & (f^2\mathrm{F}')' \end{vmatrix}\frac{1}{|1}$$

$$- \frac{\lambda^3}{1\cdot 2\cdot 3}\frac{1}{\phi'^6}\begin{vmatrix} \phi' & (\phi^2)' & f^3\mathrm{F}' \\ \phi'' & (\phi^2)'' & (f^3\mathrm{F}')' \\ \phi''' & (\phi^2)''' & (f^3\mathrm{F}')'' \end{vmatrix}\frac{1}{1\cdot 1\cdot 2}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots,$$

where F, f, F′, ... stand for $\mathrm{F}(a)$, $f(a)$, $\mathrm{F}'(a)$, . . . , and a is a root of the equation $\phi(x) = 0$, and therefore also of $\phi(x)+\lambda f(x) = 0$. He then obtains, by the use of Lagrange's theorem, the alternative form

$$\mathrm{F}(x) = \mathrm{F} - \frac{\lambda}{1}\mathrm{F}'\frac{f}{\psi} + \frac{\lambda^2}{\cdot 2}\left\{\mathrm{F}'\left(\frac{f}{\psi}\right)^2\right\}' - \frac{\lambda^3}{1\cdot 2\cdot 3}\left\{\mathrm{F}'\left(\frac{f}{\psi}\right)^3\right\}'' + \ldots,$$

where the conditions are the same save that $(x-a)\psi(x)$ is put for $\phi(x)$. As the first development of $F(x)$ must be identical with the second, he is thus face to face with unverified evaluations of the Wronskians in the former of the two, and he sets himself the task of supplying the want. The procedure followed, however, is not at all attractive.

It is impossible not to note the irony involved in Cayley's use of Lagrange when we recall the object of Wronski's *Réfutation*.

FROBENIUS, G. (1873).

[Ueber den Begriff der Irreductibilität in der Theorie der linearen Differentialgleichungen. *Crelle's Journ.*, lxxvi. pp. 236–270.]

[Ueber die Determinante mehrerer Functionen einer Variabeln. *Crelle's Journ.*, lxxvii. pp. 245–257.]

For the discussion of the subject of the first of these papers, the theorem regarding the vanishing of a Wronskian is found by Frobenius to be a requisite, and he consequently devotes § 1 (pp. 238–240) to the consideration of it. It would seem that he was thus led to the investigation of Wronskians in general, and apart from their relation to differential equations, for, after a brief interval, the second paper, a short methodically arranged exposition, was written.

Immediately following the definition comes Christoffel's theorem of 1857, namely,

$$W(yy_1,\ yy_2,\ \ldots,\ yy_n) = y^n \cdot W(y_1,\ y_2,\ \ldots,\ y_n),$$

the proof consisting in writing for y^n in the right-hand member the determinant

$$\begin{vmatrix} y & . & . & \ldots & . \\ y' & y & . & \ldots & . \\ y'' & 2y' & y & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ y^{(n-1)} & (n-1)y^{(n-2)} & \tfrac{1}{2}(n-1)(n-2)y^{(n-3)} & \ldots & y \end{vmatrix},$$

and then performing the multiplication in row-by-column fashion. From this identity, by substituting $1/y_1$ for y and by noting that

$$\frac{d}{dx}\frac{y_h}{y_1} = \frac{W(y_1, y_h)}{y_1^2},$$

there comes the next result, namely,

$$W(y_1, \ldots, y_n) = \frac{1}{y_1^{n-2}} W\{W(y_1, y_2), W(y_1, y_3), \ldots, W(y_1, y_n)\},$$

a condensation-theorem, we may note, similar to Chio's for general determinants. (*Hist.*, ii. p. 80.)

An interruption here takes place in order to establish the theorem of the previous paper, namely, that *if* $W(y_1, \ldots, y_n)$ *vanish identically the* y*'s must be connected by a linear homogeneous relation with constant coefficients.* The method of proof is gradational, the case for λ functions being made dependent on the case for $\lambda - 1$. Thus, it being given that

$$W(y_1, \ldots, y_\lambda) = 0,$$

it follows from the identity just established that

$$W\{W(y_1, y_2), \quad W(y_1, y_3), \ldots, W(y_1, y_\lambda)\} = 0,$$

and therefore by hypothesis that

$$c_2 \cdot W(y_1, y_2) + c_3 \cdot W(y_1, y_3) + \ldots + c_\lambda \cdot W(y_1, y_\lambda) = 0.$$

This, however, through division by y_1^2 can be written

$$c_2 \cdot \frac{\partial}{\partial x}\frac{y_2}{y_1} + c_3 \cdot \frac{\partial}{\partial x}\frac{y_3}{y_1} + \ldots + c_\lambda \cdot \frac{\partial}{\partial x}\frac{y_\lambda}{y_1} = 0,$$

whence by integration followed by multiplication by y_1, we have

$$c_1 y_1 + c_2 y_2 + \ldots + c_\lambda y_\lambda = 0$$

as desired.

Returning to the condensation-theorem, Frobenius by repeated use of it shows that

$$W(y_1, \ldots, y_n) = \frac{W\{W(y_1, y_2, y_3), W(y_1, y_2, y_4), \ldots, W(y_1, y_2, y_n)\}}{\{W(y_1, y_2)\}^{n-3}},$$

and thence proceeds to establish generally that

$$\mathrm{W}(u_1, u_2, \ldots, u_\mu, v_1, v_2, \ldots, v_\nu) = \frac{\mathrm{W}\{\mathrm{W}(u_1, \ldots, u_\mu, v_1), \mathrm{W}(u_1, \ldots, u_\mu, v_2), \ldots, \mathrm{W}(u_1, \ldots, u_\mu, v_\nu)\}}{\{\mathrm{W}(u_1, \ldots, u_\mu)\}^{\nu-1}}.$$

This, we may pause to remark, is viewable even in the simplest case as the resolution of a special compound Wronskian into factors, and is thus comparable with the series of identities given at the close of Christoffel's paper (*Hist.*, ii. p. 227).

The case where ν is 2 is considered of special interest, and thereon the remainder of the investigation is based. It is first stated in the form

$$\mathrm{W}(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n, y_k, y) = \frac{\mathrm{W}\{\mathrm{W}(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n, y_k), \mathrm{W}(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n, y)\}}{\mathrm{W}(y_1, \ldots, y_n)},$$

but this is readily seen to be the same as

$$\frac{\mathrm{W}(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)}{\mathrm{W}(y_1, \ldots, y_n)} \cdot \frac{\mathrm{W}(y, y_1, \ldots, y_n)}{\mathrm{W}(y_1, \ldots, y_n)} = -\frac{\partial}{\partial x} \frac{\mathrm{W}(y, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)}{\mathrm{W}(y_1, \ldots, y_n)};$$

and the three important quotients here being denoted—not merely for shortness' sake—by

$$(-1)^{n+k} z_k, \quad (-1)^n \mathrm{P}(y), \quad (-1)^{k-1} \mathrm{P}(y_1, z_k),$$

we are free to put it in the form

$$z_k \cdot \mathrm{P}(y) = \frac{\partial}{\partial x} \mathrm{P}(y, z_k).$$

A study of the z's then follows (§§ 2, 3, pp. 248–252). Developing the determinant

$$\begin{vmatrix} y_1 & y_2 & \ldots & y_n \\ y_{1,1} & y_{2,1} & \ldots & y_{n,1} \\ \ldots & \ldots & \ldots & \ldots \\ y_{1,n-2} & y_{2,n-2} & \ldots & y_{n,n-2} \\ y_{1,\lambda} & y_{2,\lambda} & \ldots & y_{n,\lambda} \end{vmatrix},$$

where $y_{r,s}$ stands for $\partial^s y_r / \partial x^s$, and which therefore is $\mathrm{W}(y_1, \ldots, y_n)$, when $\lambda = n-1$, in terms of the elements of the last row and their

complementaries, we obtain by taking λ in succession equal to $0, 1, 2, \ldots, n-1$ the set of equations

$$\left.\begin{aligned} y_1 z_1 \quad &+ y_2 z_2 \quad + \ldots + y_n z_n \quad = 0, \\ y_{1,1} z_1 \quad &+ y_{2,1} z_2 \quad + \ldots + y_{n,1} z_n \quad = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ y_{1,n-2} z_1 &+ y_{2,n-2} z_2 + \ldots + y_{n,n-2} z_n = 0, \\ y_{1,n-1} z_1 &+ y_{2,n-1} z_2 + \ldots + y_{n,n-1} z_n = 1, \end{aligned}\right\},$$

in which, as we may note for ourselves, the definition of the z's is involved. If, for shortness' sake, we put

$$s_{\alpha,\beta} \quad \text{for} \quad y_{1,\alpha} z_{1,\beta} + y_{2,\alpha} z_{2,\beta} + \ldots + y_{n,\alpha} z_{n,\beta},$$

we thus know that

$$s_{0,0} = s_{1,0} = s_{2,0} = \ldots = s_{n-2,0} = 0,$$

and

$$s_{n-1,0} = 1.$$

But by differentiating $s_{\lambda-1,0}$ once, $s_{\lambda-2,0}$ twice, and so on, it is readily shown that more is known in regard to $s_{\alpha,\beta}$, that, indeed,

$$s_{\alpha,\beta} = \begin{cases} 0 \\ (-1)^{\beta} \end{cases} \quad \text{if} \quad \alpha+\beta \begin{cases} < n-1 \\ = n-1. \end{cases}$$

Putting therefore $\alpha = 0$, $\beta = 0, 1, 2, \ldots, n-1$, we have

$$\left.\begin{aligned} z_1 y_1 \quad &+ z_2 y_2 \quad + \ldots + z_n y_n \quad = 0, \\ z_{1,1} y_1 \quad &+ z_{2,1} y_2 \quad + \ldots + z_{n,1} y_n \quad = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ z_{1,n-2} y_1 &+ z_{2,n-2} y_2 + \ldots + z_{n,n-2} y_n = 0, \\ z_{1,n-1} y_1 &+ z_{2,n-1} y_2 + \ldots + z_{n,n-1} y_n = (-1)^{n-1}, \end{aligned}\right\}$$

and are face to face with the interesting fact that what we have reached is almost exactly producible from the previous set of equations by the interchange of y and z. Important conclusions follow of course with ease. In the first place, the determinant of the new set being $W(z_1, \ldots, z_n)$ and not being zero, it is known that the z's are not connected by a linear homogeneous relation with constant coefficients. Next, it is seen that the solution of the set is

$$y_k = (-1)^{k-1} \frac{W(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)}{W(z_1, \ldots, z_n)},$$

and a comparison of this with the definition of z_k makes it reasonable that the z's be termed the *adjunct* functions of the y's. Less expected is the identity

$$W(y_1, \ldots, y_n) \cdot W(z_{k+1}, \ldots, z_n) = W(y_1, \ldots, y_k),$$

which is obtained by simply using the multiplication-theorem, and which, when $k = 0$, becomes

$$W(y_1, \ldots, y_n) \cdot W(z_1, \ldots, z_n) = 1,$$

or, in words, *the Wronskian of a set of functions and the Wronskian of the adjunct functions are algebraical reciprocals.*

The subjects of §§ 4, 5 (pp. 252–255) are the quotients $P(y, z_k)$, $P(y)$, etc., and in the results, as might be expected, reciprocity is still a feature.

The paper concludes (§ 6) with an application to differential equations.

TRANSON, A. (1874).

[Réflexions sur l'événement scientifique d'une formule publiée par Wronski en 1812, et démontrée par M. Cayley en 1873. *Nouv. Annales de Math.*, xiii. pp. 161–174.]

[Loi des séries de Wronski : sa phoronomie. *Nouv. Annales de Math.*, (2) xiii. pp. 305–318.]

The 'formula' referred to is Wronski's "loi suprême," which Cayley had dealt with two years before. Although in telling sympathetically the story of Wronski's poorly appreciated labours Transon draws particular attention to the use of determinants, he merely restates (§ 4, pp. 313–315) the fundamental property of them enunciated in 1815 (*Hist.*, i. pp. 475–476), and gives the simplest illustration possible.

CASORATI, F. (1874).

[Sui determinanti di funzioni. *Mem. . . . Istituto Lombardo*, xiii. pp. 181–187.]

When dealing with other forms of determinants whose elements are differential-coefficients, Casorati formulates two theorems regarding Wronskians. The first, however, is that more than once

already referred to as being due to Christoffel, and the second is a mere companion to it in which the introduced function appears as a divisor instead of a multiplier; for example,

$$W\left(\frac{u}{f}, \frac{v}{f}, \frac{w}{f}\right) = \frac{1}{f^3} W(u, v, w).$$

PASCH, M. (1874).

[Note über die Determinanten, welche aus Functionen und deren Differentialen gebildet werden. *Crelle's Journ.*, lxxx. pp. 177–182.]

Pasch's addition to the theory is that *if the leading minor of the array*

$$\begin{vmatrix} y_1 & \frac{\partial y_1}{\partial x} & \frac{\partial^2 y_1}{\partial x^2} & \cdots \\ y_2 & \frac{\partial y_2}{\partial x} & \frac{\partial^2 y_2}{\partial x^2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_n & \frac{\partial y_n}{\partial x} & \frac{\partial^2 y_n}{\partial x^2} & \cdots \end{vmatrix}$$

—that is to say, the Wronskian of $y_1, y_2, \ldots, y_n$*—vanishes identically, every other* n*-line minor of the array vanishes also.*

Much more important is his acceptance from Rosanes (1872) of the conception of a determinant of more general form than the Wronskian, namely, the determinant

$$\sum(\pm y_1 \cdot \delta y_2 \cdot \delta^2 y_3 \ldots \delta^{n-1} y_n), \quad \text{or say} \quad \mathfrak{W}(y_1, y_2, \ldots, y_n),$$

where the y's are functions of $x_1, x_2, \ldots$, and where

$$\begin{aligned} \delta y &\text{ stands for } u_1 \frac{\partial y}{\partial x_1} + u_2 \frac{\partial y}{\partial x_2} + \ldots, \\ \delta^2 y &\quad\text{,,}\quad \delta(\delta y), \\ \delta^3 y &\quad\text{,,}\quad \delta(\delta^2 y), \\ &\cdots\cdots\cdots\cdots, \end{aligned}$$

and the u's are any magnitudes whatever. To this new determinant $\mathfrak{W}$ the above theorem regarding W is extended, and an application is made to binary quantics.

Connected with this application the interesting fact is brought forward that *when* $y_1, y_2, \ldots, y_n$ *are binary quantics of one and the same degree, Rosanes' determinant* (1872) *may be utilized instead of the Wronskian, the relation between the two being*

$$\begin{vmatrix} \frac{\partial^{n-1}u_1}{\partial x^{n-1}} & \frac{\partial^{n-1}u_1}{\partial x^{n-2}\partial y} & \cdots & \frac{\partial^{n-1}u_1}{\partial y^{n-1}} \\ \frac{\partial^{n-1}u_2}{\partial x^{n-1}} & \frac{\partial^{n-1}u_2}{\partial x^{n-2}\partial y} & \cdots & \frac{\partial^{n-1}u_2}{\partial y^{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^{n-1}u_n}{\partial x^{n-1}} & \frac{\partial^{n-1}u_n}{\partial x^{n-2}\partial y} & \cdots & \frac{\partial^{n-1}u_n}{\partial y^{n-1}} \end{vmatrix}$$

$$= \frac{m(m-1)^1(m-2)^2 \ldots (m-n+1)^{n-1}}{y^{\frac{1}{2}n(n-1)}} \cdot W(u_1, u_2, \ldots, u_n),$$

where the u*'s are binary* m*-thics.* By way of proof it is only necessary to multiply the determinant on the left by $y^{\frac{1}{2}n(n-1)}$ in the form

$$\begin{vmatrix} y^{n-1} & (n-1)_1 xy^{n-2} & (n-1)_2 x^2 y^{n-3} & \cdots & x^{n-1} \\ . & y^{n-2} & (n-2)_1 xy^{n-3} & \cdots & x^{n-2} \\ . & . & y^{n-3} & \cdots & x^{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & . & \cdots & y^0 \end{vmatrix}.$$

CHAPTER IX.

JACOBIANS, FROM 1862 TO 1877.

BRIOSCHI, F. (1863): HERMITE, C. (1863).

[Sur la théorie des formes cubiques à trois indéterminées. *Comptes Rendus . . . Acad. des. Sci.* (Paris), lvi. pp. 304–307; or *Opere Mat.*, iv. pp. 331–334.]

[Extrait d'une lettre de M. Hermite à M. Brioschi. *Crelle's Journ.*, lxiii. pp. 30–32; or *Œuvres*, ii. pp. 289-292.]

Brioschi points out that u being a ternary cubic, H its Hessian and K its sextic covariant, the Jacobian of u, H, K is also a covariant. To this Hermite adds that the new covariant contains the discriminant as a factor, and that the variable cofactor is resolvable into three cubics.

MALMSTEN, C. J. (1862): BERTRAND, J. (1864, 1870).

[Mémoire sur l'intégration des équations différentielles. *Journ.* (*de Liouville*) *de Math.*, (2) vii. pp. 257–374.]

[Déterminant d'un système de fonctions. *Traité de Calc. Diff. et de Calc. Int.*, i. pp. 61–72.]

[Changement de variables dans les intégrales multiples. *Traité de Calc. Diff. et de Calc. Int.*, ii. pp. 468–469.]

These are really second-editions, the first being a free translation of Malmsten's memoir of 1858, and the others being reproduced from Bertrand's memoir of 1851.

Bertrand adds seven interesting results to serve as exercises. One of them is an extension of the theorem regarding the vanishing of a Jacobian, namely, *The necessary and sufficient condition that* n

functions of n+h *variables shall be connected by an equation independent of the said variables is that the Jacobian of the functions with respect to any* n *of the variables shall vanish.* Two of them bear on the theorem concerning the condensation of the Jacobian into one product (*Hist.*, i. p. 391). The first is familiar, namely, he alters

$$\left.\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta\sin\psi \\ z &= \rho\sin\theta\cos\psi \end{aligned}\right\} \text{ into } \left\{\begin{aligned} z &= \sqrt{\rho^2-x^2-y^2} \\ x &= \rho\cos\theta \\ y &= \rho\sin\theta\sin\psi, \end{aligned}\right.$$

and so obtains

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\psi)} = \frac{\partial z}{\partial\rho}\cdot\frac{\partial x}{\partial\theta}\cdot\frac{\partial y}{\partial\psi} = \frac{\rho}{z}\cdot(-\rho\sin\theta)\cdot\rho\sin\theta\cos\psi = -\rho^2\sin\theta.$$

In the second the data are

$$\left.\begin{aligned} x_1 &= \cos\phi_1 \\ x_2 &= \sin\phi_1\cos\phi_2 \\ x_3 &= \sin\phi_1\sin\phi_2\cos\phi_3 \\ &\cdots\cdots\cdots\cdots \\ x_n &= \sin\phi_1\sin\phi_2\ldots\sin\phi_{n-1}\cos\phi_n \end{aligned}\right\},$$

and the result

$$\frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(\phi_1, \phi_2, \ldots, \phi_n)} = \sin^n\phi_1\cdot\sin^{n-1}\phi_2\cdot\sin^{n-2}\phi_3\ldots\sin^1\phi_n.$$

What may be viewed as an illustration of another theorem of Jacobi's (*Hist.*, i. p. 389) is the result

$$\frac{\partial(x_1x_n^{-1}, x_2x_n^{-1}, \ldots, x_{n-1}x_n^{-1})}{\partial(x_1, x_2, \ldots, x_n)} = \frac{1}{x_n^{n+1}},$$

where $x_1^2+x_2^2+\ldots+x_n^2 = 1$.

WOLSTENHOLME, J. (1863).

[Question 4892. *Educ. Times*, xxviii. p. 252; or *Math. from Educ. Times*, xxvi. pp. 104–105.]

The theorem here dealt with is that of § 15 of Jacobi's memoir of 1841 (*Hist.*, i. pp. 378–380). Any fresh interest is due to the example given in illustration of the simplest case of the theorem, namely, the Jacobian of

$a^2+b^2+c^2$, $ax+by+cz$, $x^2+y^2+z^2$, $bz-cy$, $cx-az$, $ay-bx$;

or, say,

$$P,\quad Q,\quad R,\quad A,\quad B,\quad C,$$

with respect to a, b, c, x, y, z. This, because of the well-known relation

$$\begin{vmatrix} P & Q \\ Q & R \end{vmatrix} = A^2+B^2+C^2,$$

must vanish; and it is thus suggested to ask for an independent proof of the identity

$$\begin{vmatrix} a & x & . & . & -z & y \\ b & y & . & z & . & -x \\ c & z & . & -y & x & . \\ . & a & x & . & c & -b \\ . & b & y & -c & . & a \\ . & c & z & b & -a & . \end{vmatrix} = 0.$$

Such a proof is not given.*

BAEHR, G. F. W. (1864).

[Note sur le changement des variables dans les intégrales multiples. *Archiv d. Math. u. Phys.*, xli. pp. 453–495.]

The relevant theorem regarding the Jacobian is dealt with on pp. 474–479.

COMBESCURE, E. (1867).

[Sur les déterminants fonctionnels, et .... *Annales sci. de l'École Norm. Sup.* (1), iv. pp. 93–131.]

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* One, however, will be found in the fact that the columns of the determinant are connected by the equation

$$R\,col_1 - Q\,col_2 + P\,col_3 - A\,col_4 - B\,col_5 - C\,col_6 = 0.$$

But it is also interesting to note that, if we express the determinant in terms of the minors formed from the first three rows and the minors complementary to these, we find that Q is a factor of each term of the development, and that the cofactor vanishes because it is equal to

$$\begin{vmatrix} P & Q \\ Q & R \end{vmatrix} - (A^2+B^2+C^2).$$

CLIFFORD, W. K. (1864).

[On Jacobians and polar opposites. *O.C. and D. Messenger of Math.*, ii. pp. 229–239; or *Math. Papers*, pp. 23–33.]

In accordance with Cayley's usage of 1843, the statement

$$\left\| \begin{matrix} \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \\ \frac{\partial L}{\partial x} & \frac{\partial L}{\partial y} & \frac{\partial L}{\partial z} \end{matrix} \right\| = 0$$

means that all the principal minors of the array vanish: and in view of Donkin's notation for Jacobians a natural abridgment of this would be

$$\frac{\partial(U, L)}{\partial(x, y, z)} = 0,$$

—an abridgment actually found in a later paper of Cayley's (*Quart. Journ. of Math.*, xiv. 1876, p. 295). Clifford, however, extends to the array the name 'Jacobian' and views it merely as a locus. Thus $U = 0$ being the trilinear equation of a conic, and $L = 0$ that of a straight line, the array, called 'the Jacobian of U and L,' represents the pole of L with respect to U. The paper is otherwise wholly geometrical.

BOOLE, G. (1865).

[A Treatise on Differential Equations. 2nd edition, pp. 24–25. Supplementary vol., pp. 56–59.]

In establishing the simplest case of the theorem regarding the vanishing of a Jacobian Boole had closely followed Jacobi (*Hist.*, i. pp. 365–366). Later, when generalizing, the datum now being

$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = 0,$$

he keeps the old path only up to the point of showing that u_n can be expressed as a function of $u_1, u_2, \ldots, u_{n-1}, x_n$. He then, without any apparent advantage, introduces differentials, showing that the equation

$$\frac{\partial u_n}{\partial u_1}\partial u_1 + \frac{\partial u_n}{\partial u_2}\partial u_2 + \ldots + \frac{\partial u_n}{\partial u_{n-1}}\partial u_{n-1} + \frac{\partial u_n}{\partial x_n}\partial x_n = 0$$

must be a consequence of the equations

$$\partial u_1 = 0, \quad \partial u_2 = 0, \ \ldots, \ \partial u_{n-1} = 0,$$

and that therefore

$$\frac{\partial u_n}{\partial x_n} = 0,$$

from which, as before, the required conclusion can be drawn.

LIPSCHITZ, R. (1866).

[Ueber gewisse Beziehungen zwischen räumlichen Gebilden. *Crelle's Journ.*, lxvi. pp. 267–284.]

Incidentally Lipschitz shows (pp. 279–280) that if

$$\begin{aligned} \phi_1 &= x_1 f(x_1, x_2, \ldots, x_n) \\ \phi_2 &= x_2 f(x_1, x_2, \ldots, x_n) \\ &\cdots\cdots\cdots \\ \phi_n &= x_n f(x_1, x_2, \ldots, x_n), \end{aligned}$$

then

$$\frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(x_1, x_2, \ldots, x_n)} = f^{n-1}\left(f + x_1\frac{\partial f}{\partial x_1} + \ldots + x_n\frac{\partial f}{\partial x_n}\right).$$

The proof is hardly needed.

NATANI, L. (1867).

[Substitution, lineare. *Hoffmann's Math. Wörterbuch*, vi. pp. 605–685.]

Seemingly, as an extension of Hesse's theorem of 1844 (*Hist.*, ii. pp. 235–237), Natani gives the rather loosely worded proposition that *if there be* n *homogeneous equations, all but one of the same degree, the partial differential-quotients of the Jacobian are proportional to those of the function of unique degree.*

We prefer to substitute the following:—*If* $u_1, u_2, \ldots, u_n$ *be homogeneous functions of* $x_1, x_2, \ldots, x_n$, *the first* $n-1$ *of them being of the* p^{th} *degree and the last of the* q^{th} *degree, then*

$$\left.\begin{aligned} x_r\frac{\partial J}{\partial x_s} &= p\sum u_t\frac{\partial[t,r]}{\partial x_s} + \quad 0 \quad + (q-p)\frac{\partial}{\partial x_s}(u_n[nr]) \\ x_r\frac{\partial J}{\partial x_r} &= p\sum u_t\frac{\partial[t,r]}{\partial x_r} + (p-1)J + (q-p)\frac{\partial}{\partial x_r}(u_n[nr]) \end{aligned}\right\},$$

where J *stands for the Jacobian of the* u*'s with respect to the* x*'s, and*

$$\left|[11]\ [22]\ \ldots\ [nn]\right|$$

for the adjugate of the Jacobian. Thus, when $n = 3$ we have

$$x_1\mathrm{J} = \begin{vmatrix} pu_1 & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ pu_2 & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ qu_3 & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}, \quad x_2\mathrm{J} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & pu_1 & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & pu_2 & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & qu_3 & \frac{\partial u_3}{\partial x_3} \end{vmatrix}, \quad x_3\mathrm{J} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & pu_1 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & pu_2 \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & qu_3 \end{vmatrix}$$

or

$$x_r\mathrm{J} = pu_1[1r] + pu_2[2r] + qu_3[3r],$$

and differentiating with respect to x_s there results

$$\begin{aligned} x_r\frac{\partial \mathrm{J}}{\partial x_s} &= \left(pu_1\frac{\partial[1r]}{\partial x_s} + p[1r]\frac{\partial u_1}{\partial x_s}\right) \\ &\quad + \left(pu_2\frac{\partial[2r]}{\partial x_s} + p[2r]\frac{\partial u_2}{\partial x_s}\right) \\ &\quad + \left(qu_3\frac{\partial[3r]}{\partial x_s} + q[3r]\frac{\partial u_3}{\partial x_s}\right), \\ &= p\sum u_t\frac{\partial[tr]}{\partial x_s} + p\sum[tr]\frac{\partial u_t}{\partial u_s} \\ &\quad + (q-p)u_3\frac{\partial[3r]}{\partial x_s} + (q-p)[3r]\frac{\partial u_3}{\partial x_s}, \\ &= p\sum u_t\frac{\partial[tr]}{\partial x_s} + 0 + (q-p)\frac{\partial}{\partial x_s}(u_3[3r]). \end{aligned}$$

Similarly, by differentiating with respect to r we have

$$x_r\frac{\partial \mathrm{J}}{\partial x_s} = p\sum u_t\frac{\partial[tr]}{\partial x_r} + (p-1)\mathrm{J} + (q-p)\frac{\partial}{\partial x_r}(u_3[3r]).$$

If, now, for a set of values of the x's the u's vanish, these equalities become

$$\left.\begin{aligned} x_r\frac{\partial \mathrm{J}}{\partial x_s} &= (q-p)[3r]\frac{\partial u_3}{\partial x_s}, \\ x_r\frac{\partial \mathrm{J}}{\partial x_r} &= (q-p)[3r]\frac{\partial u_3}{\partial x_r}, \end{aligned}\right\}$$

and therefore for the said set of values

$$\frac{\partial \mathrm{J}}{\partial x_s} : \frac{\partial \mathrm{J}}{\partial x_r} = \frac{\partial u_3}{\partial x_s} : \frac{\partial u_3}{\partial x_r},$$

as Natani asserts.

NEUMANN, C. (1868).

[Zur Theorie der Functionaldeterminanten. *Math. Annalen*, i. pp. 208–209.]

This is merely an unexpected verification of Jacobi's 'fundamental lemma' of 1844 (*Hist.*, ii. pp. 230–234).

CLEBSCH, A. (1868).

[Ueber eine Eigenschaft von Functionaldeterminanten. *Crelle's Journ.*, lxix. pp. 355–358.]

[Note zu dem Aufsatze "Ueber eine Eigenschaft . . . ." *Crelle's Journ.*, lxx. pp. 175–181.]

The property in question is to the effect that *If* $u_1, u_2, u_3, \ldots$ *be* $n+1$ *homogeneous integral functions of the* r^{th} *degree in the* n *variables* $x_1, x_2, x_3, \ldots$; $v_1, v_2, \ldots, v_{n+1}$ *be the set of Jacobians formed from the* u'*s*; *and* $w_1, w_2, \ldots, w_{n+1}$ *the set formed in like manner from the* v'*s*; *then*

$$\frac{u_1}{w_1} = \frac{u_2}{w_2} = \frac{u_3}{w_3} = \ldots .$$

By way of proof it is shown that, whatever values the a's and b's may have, the last row of the determinant

$$\begin{vmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \ldots & \frac{\partial v_1}{\partial x_n} & a_1 & b_1 \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \ldots & \frac{\partial v_2}{\partial x_n} & a_2 & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{\partial v_{n+1}}{\partial x_1} & \frac{\partial v_{n+1}}{\partial x_2} & \ldots & \frac{\partial v_{n+1}}{\partial x_n} & a_{n+1} & b_{n+1} \\ \cdot & \cdot & \ldots & \cdot & \sum a_h u_h & \sum b_h u_h \end{vmatrix}$$

can be made to vanish : that therefore

$$\sum b_h u_h \cdot \sum a_k w_k - \sum a_h u_h \cdot \sum b_k w_k = 0 ;$$

and that therefore, "by putting the coefficients of the products $a_k b_k$ separately equal to zero," there results

$$u_h w_k - u_k w_h = 0$$

as desired.

The problem of finding the common multiplier connecting any w with the corresponding u is only touched on in the first paper, but is fully solved in the second.

GILBERT, PH. (1869).

[Sur une propriété des déterminants fonctionnels et son application au développement des fonctions implicites. *Nouv. Mém. de l'Acad. roy. de Belgique*, xxxviii. pp. 1–12.]

The real character of the property in question is somewhat obscured by being discussed along with its application to the problem of extending Lagrange's theorem for the expansion of $\mathrm{F}(u)$ in a series of ascending powers of x when

$$u = a + xf(u).$$

It may be freed of unessentials and conveniently formulated as follows: *If the differential-coefficients of* $\mathrm{u}_1, \mathrm{u}_2, \ldots, \mathrm{u}_n$ *with respect to* x *be proportional to the differential-coefficients with respect to* ξ, *the common ratio being* ϕ, *and the* u's *be also functions of* $a_1, a_2, \ldots, a_n$, *then*

$$\frac{\partial}{\partial \mathrm{x}}\left\{\frac{\partial(\mathrm{u}_1, \ldots, \mathrm{u}_n)}{\partial(a_1, \ldots, a_n)}\right\} = \frac{\partial}{\partial \xi}\left\{\phi \frac{\partial(\mathrm{u}_1, \ldots, \mathrm{u}_n)}{\partial(a_1, \ldots, a_n)}\right\} - \frac{\partial(\mathrm{u}_1, \ldots \mathrm{u}_n, \phi)}{\partial(a_1, \ldots, a_n, \xi)}.$$

All that is given by way of proof is a verification for the first three cases. For example, when n is 1, the right-hand member

$$= \frac{\partial}{\partial \xi}\left\{\phi \frac{\partial u_1}{\partial a_1}\right\} - \frac{\partial(u_1, \phi)}{\partial(a_1, \xi)}$$

$$= \phi \frac{\partial^2 u_1}{\partial \xi \partial a_1} + \frac{\partial \phi}{\partial \xi}\frac{\partial u_1}{\partial a_1} - \frac{\partial u_1}{\partial a_1}\frac{\partial \phi}{\partial \xi} + \frac{\partial u_1}{\partial \xi}\frac{\partial \phi}{\partial a_1}$$

$$= \phi \frac{\partial^2 u_1}{\partial \xi \partial a_1} + \frac{\partial u_1}{\partial \xi}\frac{\partial \phi}{\partial a} = \frac{\partial}{\partial a_1}\left(\phi \frac{\partial u_1}{\partial \xi}\right)$$

$$= \frac{\partial}{\partial a_1}\frac{\partial u_1}{\partial x} = \text{the left-hand member.}$$

The case where n is 2 is made dependent on this, there being then the *two* given equations

$$\frac{\partial u_1}{\partial x} = \phi \frac{\partial u_1}{\partial \xi}, \qquad \frac{\partial u_2}{\partial x} = \phi \frac{\partial u_2}{\partial \xi};$$

and the case where n is 3 is similarly made to rest on the cases $n = 1$, $n = 2$. The work of verification, however, increases seriously with each step.

IMSCHENETSKY, V. G. (1869).

[Sur l'intégration des équations aux dérivées partielles du premier ordre. *Archiv d. Math. u. Phys.*, l. pp. 278–474.]

The third section of the first chapter (pp. 284–287) of Imschenetsky's memoir is devoted to a full and carefully expressed proof of the theorem regarding the vanishing of the Jacobian. What is of interest in it is the mode in which the Jacobian is shown by the translator J. Hoüel to be expressible as a single product. The functions of $x_1, \ldots, x_n$ being $u_1, \ldots, u_n$ there is first deduced by elimination

$$\begin{aligned}
u_1 &= \phi_1(x_1, \ldots, x_n),\\
u_2 &= \phi_2(u_1, x_2, \ldots, x_n),\\
u_3 &= \phi_3(u_1, u_2, x_3, \ldots, x_n),\\
&\cdots\cdots\cdots\cdots\\
u_n &= \phi_n(u_1, u_2, \ldots, u_{n-1}, x_n),
\end{aligned}$$

and then unity in the form

$$\begin{vmatrix}
1 & . & . & \ldots \\
-\dfrac{\partial\phi_2}{\partial u_1} & 1 & . & \ldots \\
-\dfrac{\partial\phi_3}{\partial u_1} & -\dfrac{\partial\phi_3}{\partial u_2} & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{vmatrix}_n$$

is multiplied in row-by-column fashion by

$$\partial(u_1, \ldots, u_n)/\partial(x_1, \ldots, x_n),$$

the result obtained being

$$\begin{vmatrix}
\dfrac{\partial\phi_1}{\partial x_1} & \dfrac{\partial\phi_1}{\partial x_2} & \dfrac{\partial\phi_1}{\partial x_3} & \ldots \\
. & \dfrac{\partial\phi_2}{\partial x_2} & \dfrac{\partial\phi_2}{\partial x_3} & \ldots \\
. & . & \dfrac{\partial\phi_3}{\partial x_3} & \ldots \\
\ldots & \ldots & \ldots & \ldots
\end{vmatrix}_n$$

by reason of the equality *

$$\frac{\partial \phi_r}{\partial u_1}\frac{\partial u_1}{\partial x_s} + \ldots . + \frac{\partial \phi_r}{\partial u_{r-1}}\frac{\partial u_{r-1}}{\partial x_s} + \frac{\partial \phi_r}{\partial x_s} = \frac{\partial u_r}{\partial x_s}.$$

A comparison of this with Baltzer's procedure of 1857 and 1864 will be found instructive. In his edition of 1870 a change was made.

KRONECKER, L. (1869).

[Bemerkungen zur Determinanten-Theorie. (Auszug aus Briefen an Herrn Baltzer.) *Crelle's Journ.*, lxxii. pp. 152–175; or *Werke*, i. pp. 237–269.]

Kronecker's second note (pp. 153–158) contains suggestions for the improvement of § 12 of Baltzer's third edition. He prefers (§ 1) to establish at an early stage Jacobi's theorem regarding the Jacobian of the y's with respect to the x's when

$$\mathrm{F}_1(y_1, \ldots, y_n, x_1, \ldots, x_n) = 0, \quad \ldots, \quad \mathrm{F}_n(y_1, \ldots, y_n, x_1, \ldots, x_n) = 0$$

and thence to deduce the otherwise known results (a) where F_i does not involve $x_1, x_2, \ldots, x_{i-1}$; ($b$) where

$$\mathrm{F}_i \equiv -y_i + f_i(x_1, x_2, \ldots, x_n);$$

(c) when the given equations are such that

$$y_i \equiv \phi_i(y_1, y_2, \ldots, y_{i-1}, x_i, \ldots);$$

these results being

(a) $$(-1)^n \frac{\partial \mathrm{F}_1}{\partial x_1} \ldots \frac{\partial \mathrm{F}_n}{\partial x_n} \div \begin{vmatrix} \frac{\partial \mathrm{F}_1}{\partial y_1} & \ldots & \frac{\partial \mathrm{F}_n}{\partial y_n} \end{vmatrix},$$

(b) $$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_n}{\partial x_n} \end{vmatrix},$$

(c) $$\begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \ldots & \frac{\partial \phi_n}{\partial x_n} \end{vmatrix}.$$

The rest of the paper (§§ 2, 3) is devoted to a critical examination of the nature of the relation which exists among the y's when their Jacobian with respect to the x's vanishes.

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* In differentiating, with respect to x_s, the members of

$$u_r = \phi_r(u_1, u_2, \ldots, u_{r-1}, x_r, x_{r+1}, \ldots, x_n),$$

the term $\partial\phi_r/\partial x_s$ occurs only when $r \gtreqless s$.

TRZASKA, W. (1871).

[O pewnem zastosowaniu wyznaczników funkcyjnych. *Pamiętnik Towarzystwa Nauk Scislych w Paryżu*, i. pp. 113–121.*]

The theorem here dealt with is an extension of one noted by Bertrand in 1864. In order to suggest the mode of proof it may be conveniently formulated as follows: *The necessary and sufficient condition that the functions* $u_1, u_2, \ldots, u_n$ *of the independent variables* $x_1, x_2, \ldots, x_m$ *shall satisfy* p *relations independent of these variables is that the leading minor of the* $(n-p)^{th}$ *order in the* n-*by*-m *array*

$$\begin{array}{cccccccc}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_{n-p}} & \frac{\partial u_1}{\partial x_{n-p+1}} & \cdots & \frac{\partial u_1}{\partial x_m} \\
\frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_{n-p}} & \frac{\partial u_2}{\partial x_{n-p+1}} & \cdots & \frac{\partial u_2}{\partial x_m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial u_{n-p}}{\partial x_1} & \frac{\partial u_{n-p}}{\partial x_2} & \cdots & \frac{\partial u_{n-p}}{\partial x_{n-p}} & \frac{\partial u_{n-p}}{\partial x_{n-p+1}} & \cdots & \frac{\partial u_{n-p}}{\partial x_m} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_{n-p}} & \frac{\partial u_n}{\partial x_{n-p+1}} & \cdots & \frac{\partial u_n}{\partial x_m}
\end{array}$$

shall not vanish, but that all the minors of the $(n-p+1)^{th}$ *order obtained by bordering the said leading minor shall vanish.*

On putting $p = 1$ and $m = n+h$ we have the case given by Bertrand, and on further putting $h = 0$ we reach the fundamental case dealt with by Jacobi.

MINCHIN G. M. (1871).

[Elementary demonstration of a fundamental theorem. *Quart. Journ. of Math.*, xii. pp. 172–175.]

The theorem in question is that of which the conclusion is

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} \cdot \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1\ y_2, \ldots, y_n)} = 1;$$

and the demonstration is based on the fact that the Jacobian is the determinant of the set of equations which give the increments

* Or Baraniecki's text-book, chap. xii. §118.

accruing to the functions as the result of assigning arbitrary increments to the independent variables. The procedure is far from direct, and has no special claim to be called 'elementary.'

ROSANES, J. (1872).

(See under this heading in chap. xxii.)

SPOTTISWOODE, W. (1872).

[Remarks on some recent generalizations of Algebra. *Proceed. London Math. Soc.*, iv. pp. 147–164.]

One of Spottiswoode's 'remarks' is (pp. 162–163) that

$$\frac{\partial(y_1, \ldots, y_n)}{\partial(x_1, \ldots, x_n)} = \left(\frac{\partial y}{\partial x}\right)^n$$

if

$$y = \epsilon_1 y_1 + \epsilon_2 y_2 + \ldots + \epsilon_n y_n$$

and

$$x = \epsilon_1 x_1 + \epsilon_2 x_2 + \ldots + \epsilon_n x_n,$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are a set of 'algebraic keys' or 'alternating units.' The reason for this is that we have two expressions for $\partial y/\partial x_r$, one from each of the assumed equalities, namely,

$$\epsilon_1 \frac{\partial y_1}{\partial x_r} + \epsilon_2 \frac{\partial y_2}{\partial x_r} + \ldots + \epsilon_n \frac{\partial y_n}{\partial x_r} \quad \text{and} \quad \frac{\partial y}{\partial x} \epsilon_r.$$

We have only then to give r in succession the values $1, 2, \ldots, n$ and multiply.

By way of application a theorem in Jacobians is got from the identity

$$\left(\frac{\partial y}{\partial x}\right)^n \left(\frac{\partial x}{\partial y}\right)^n = 1.$$

FALK, M. (1874, 1876).

[Om funktionaldeterminanter. *Tidskrift f. Mat. och Fys.*, v. pp. 193–206.]

[Bearbetning af några teorier angående differentialeqvationer. Dissert. 57 pp. Helsingfors.]

The first paper is an elementary exposition embodying three of the main properties: it may be viewed as a first edition of § 4

(pp. 45–59) of the "Lärobok" of 1876. The second paper opens with a section bearing the same title and dealing still more briefly with the same matters.

CASORATI, F. (1874).

[Sui determinanti di funzioni. *Mem. . . . Istituto Lombardo . . .* xiii. pp. 181–187.]

This paper is avowedly suggested by a part of Jacobi's of 1833, Casorati's purpose being to establish additional theorems regarding the two forms of determinant—the Jacobian and pre-Jacobian—dealt with by his predecessor, and to formulate analogous theorems for the Wronskian and Hessian. It is thus a study of the effect produced on such determinants by multiplying or dividing all the functions involved in them by another function of the same variables.

To Jacobi's theorem

$$\mathrm{J}(f^{-1}u, f^{-1}v, f^{-1}w) = \frac{1}{f^{3+1}}\mathrm{K}(f, u, v, w),$$

or

$$\frac{\partial(f^{-1}u, f^{-1}v, f^{-1}w)}{\partial(x, y, z)} = \frac{1}{f^{3+1}}\begin{vmatrix} f & u & v & w \\ f_1 & u_1 & v_1 & w_1 \\ f_2 & u_2 & v_2 & w_2 \\ f_3 & u_3 & v_3 & w_3 \end{vmatrix},$$

where the suffixes 1, 2, 3 denote differentiation with respect to x, y, z respectively, there is given the companion

$$\frac{\partial(fu, fv, fw)}{\partial(x, y, z)} = f^{3-1}\begin{vmatrix} f & -u & -v & -w \\ f_1 & u_1 & v_1 & w_1 \\ f_2 & u_2 & v_2 & w_2 \\ f_3 & u_3 & v_3 & w_3 \end{vmatrix};$$

and to Jacobi's

$$\mathrm{K}(f^{-1}u, f^{-1}v, f^{-1}w) = f^{-3}\mathrm{K}(u, v, w)$$

the companion

$$\mathrm{K}(fu, fv, fw) = f^{3}\mathrm{K}(u, v, w).$$

Attention is then drawn to the fact that Jacobi's first theorem involves the proposition that any pre-Jacobian may be expressed

by means of a Jacobian; and that thus, on using the well-known double theorem regarding the vanishing of a Jacobian, we obtain a corresponding theorem regarding the pre-Jacobian, namely, *If the pre-Jacobian of a set of functions vanishes, the functions are connected by a homogeneous relation: and if a set of functions be connected by a homogeneous relation, their pre-Jacobian with respect to a common set of variables vanishes.*

Of course this pair of theorems can also be established independently, and Casorati does so establish them. For example, it being given that

$$\mathrm{F}(u, v, w) = 0,$$

and that F is a homogeneous relation, he has at once from Euler

$$u\frac{\partial \mathrm{F}}{\partial u} + v\frac{\partial \mathrm{F}}{\partial v} + w\frac{\partial \mathrm{F}}{\partial w} = 0,$$

and by simple differentiation

$$\frac{\partial \mathrm{F}}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \mathrm{F}}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial \mathrm{F}}{\partial w}\cdot\frac{\partial w}{\partial x} = 0,$$

$$\frac{\partial \mathrm{F}}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \mathrm{F}}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial \mathrm{F}}{\partial w}\cdot\frac{\partial w}{\partial y} = 0,$$

so that on eliminating $\partial \mathrm{F}/\partial u$, $\partial \mathrm{F}/\partial v$, $\partial \mathrm{F}/\partial w$ there results

$$\begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = 0.$$

On this, however, we may note for ourselves that if u, v, w be functions of more than two variables, say of x, y, z, we have

$$\left\| \begin{matrix} u & u_1 & u_2 & u_3 \\ v & v_1 & v_2 & v_3 \\ w & w_1 & w_2 & w_3 \end{matrix} \right\| = 0,$$

one of the group of vanishing three-line determinants being the Jacobian, as is proper.

CROCCHI, L. (1878).

(See under this heading in chap. v.)

NANSON, E. J. (1877).

[Elementary proof of a theorem in functional determinants. *Messenger of Math.*, vii. pp. 120–121.]

This is an attempt to generalize the mode of proof by Boole (and much earlier by Jacobi) to establish a connection between u and v when $\partial(u, v)/d(x, y) = 0$. It seemingly avoids the method of 'induction,' but it does so by being made to rest on the contrapositive proposition.*

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* The case immediately following Boole's is where u, v, w are given functions of x, y, z and $\partial(u, v, w)/\partial(x, y, z) = 0$. Proceeding exactly as he does, we may consider x, y eliminated, and the resulting equation expressed in the form

$$w = f(u, v, z).$$

From this we obtain

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial y} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial z} &= \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial f}{\partial z} \end{aligned}\right\},$$

and these three equations enable us to reduce the last row of the vanishing Jacobian to

$$0, \quad 0, \quad \frac{\partial f}{\partial z};$$

so that we have

$$\frac{\partial f}{\partial z}\cdot\frac{\partial(u, v)}{\partial(x, y)} = 0.$$

Now it cannot be that the second factor here is 0, for, if it were, u and v would by the previous case be independent: hence

$$\frac{\partial f}{\partial z} = 0,$$

and therefore f is independent of z,—that is to say, w is a function of u an v only.

CHAPTER X.

SKEW DETERMINANTS AND PFAFFIANS, FROM 1861 TO 1880.

THE corresponding chapter in the preceding volume did not end with 1860, but brought the record up to 1865. There ought therefore to have been included in it a note of Sylvester's of 1861; the omission is herewith rectified. To the same longer period belongs Clebsch's memoir, "Ueber das Pfaff'sche Problem" (*Crelle's Journ.*, lx. pp. 193–251, lxi. pp. 146–179). In this case, however, the omission was intentional, as Clebsch, unlike Frobenius the author of an equally long memoir with the same title, does not devote any special attention to the determinants which he uses.

SYLVESTER, J. J. (1861).

[Generalization of a theorem of Cauchy on arrangements. *Philos. Magazine*, xxii. pp. 378–382; or *Collected Math. Papers*, ii. pp. 290–293.]

In the third footnote Sylvester incidentally states that 'polar' would in his opinion be a better name than 'skew,' and he defines a 'Pfaffian as a "sum of quantities typifiable completely, both as to sign and magnitude, by a duadic syntheme of $2n$ elements."

NATANI, L. (1867).

[Substitution, lineare. *Hoffmann's Math. Wörterbuch*, vi. pp. 605–685.]

Natani's § 5 (pp. 618–621) concerns "symmetrische und symmetrale Determinanten," the latter adjective (*symmetral*) being

introduced as an equivalent for 'zero-axial skew.' When the skew determinant has not a zero diagonal, he calls it 'improperly symmetral' (*uneigentlich symmetral*).

VELTMANN, W. (1871): MERTENS, F. (1876).

[Beiträge zur Theorie der Determinanten. *Zeitschrift f. Math. u. Phys.*, xvi. pp. 516–525.]

[Ueber die Determinanten, deren correspondirende Elemente a_{pq} and a_{qp} entgegengesetzt gleich sind. *Crelle's Journ.*, lxxxii. pp. 207–211.]

The two parts of the fundamental proposition regarding zero-axial skew determinants are here established from a direct consideration of the terms of the final development of a general determinant as set forth by Cauchy in 1841. This, as we have seen, had already been done by Cayley himself in 1860: Veltmann and Mertens, however, arrange their matter with greater care and in much fuller detail.

CLIFFORD, W. K. (1873 ?).

[On Pfaffians. *Math. Papers*, pp. 535–537.]

In § 1 of this posthumously printed note Clifford defines a Pfaffian by means of what, following Hankel, he calls 'alternate units.'* Such a definition was of course to be expected in view of the analogous definition of a determinant involved in the work of Grassmann (1844) and Cauchy (1847). Forming a linear function of the binary products of four alternate units, ι_1, ι_2, ι_3, ι_4, say the function

$$a_{12}\iota_1\iota_2+a_{13}\iota_1\iota_3+a_{14}\iota_1\iota_4+a_{23}\iota_2\iota_3+a_{24}\iota_2\iota_4+a_{34}\iota_3\iota_4,$$

and raising it to the second power, we obtain

$$2(a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23})\iota_1\iota_2\iota_3\iota_4,$$

in which the Pfaffian of the second degree appears as a factor. It is the general result including this which Clifford uses, his

* Cauchy's 'clefs algébriques' (1847): Sylvester's 'polar umbrae' (1862): Hankel's 'alternirende Einheiten' (1867).

definition being in effect that *a Pfaffian of the* n^{th} *order is the cofactor of* $n!\iota_1\iota_2 \ldots \iota_n$ *in the expansion of the* n^{th} *power of a linear function of the binary products of the* 2n *alternate units* $\iota_1, \iota_2, \ldots, \iota_{2n}$.

In § 2 he obtains a more complicated result, which might in like fashion be used as the definition of a zero-axial skew determinant.

HOVESTADT, . (1873).

[Ueber eine mit dem Problem der kleinen Schwingungen verwandte Aufgabe. Dissert. Bonn.]

According to Lipschitz * this deals with the properties of a skew determinant with univarial diagonal.

CUNNINGHAM, A. (1874).

[An investigation of the number of constituents, elements and minors of a determinant. *Quart. Journ. of Sci.*, (2), iv. pp. 212–228.]

Cunningham devotes §§ 11, 12 to the number of terms in skew and zero-axial skew determinants; but, as in the case of axi-symmetric determinants, he makes an oversight, and his results are correct only as far as the seventh order. He concerns himself also (§ 15) with the number of k-line minors in an n-line zero-axial skew determinant.

GÜNTHER, S. (1875, 1877).

[Lehrbuch der Determinanten-Theorie, ... viii+236 pp. Zweite Aufl. xii+209 pp. Erlangen.]

In his exposition of the elementary properties of skew determinants (pp. 95–106) Günther draws attention to a still more specialized form, namely, a determinant of the $2m^{th}$ order, which, besides being zero-axial skew, is symmetric with respect to the secondary diagonal,

* *Sitzungsberichte* . . . . *Akad. d. Wiss.* (Berlin), 1890, p. 508. The dissertation I have not seen.

and which can be shown to be equal to the square * of a determinant of the m^{th} order ; for example,

$$\begin{vmatrix} . & a & b & c & d & e \\ -a & . & f & g & h & d \\ -b & -f & . & k & g & c \\ -c & -g & -k & . & f & b \\ -d & -h & -g & -f & . & a \\ -e & -d & -c & -b & -a & . \end{vmatrix} = \begin{vmatrix} e & a+d & b+c \\ -a+d & h & f+g \\ -b+c & -f+g & k \end{vmatrix}^2.$$

He does not, however, note that, by changing the signs of the last m rows, the determinant becomes centro-symmetric, and that therefore, by Zehfuss' theorem of 1862, it is equal to

$$(-1)^3 \begin{vmatrix} e & a+d & b+c \\ -a+d & h & f+g \\ -b+c & -f+g & k \end{vmatrix} \cdot \begin{vmatrix} -e & a-d & b-c \\ -a-d & -h & f-g \\ -b-c & -f-g & -k \end{vmatrix},$$

where either determinant can be made the conjugate of the other by changing the signs of all its rows or of all its columns. Nor does he note the deduction regarding an axisymmetric Pfaffian.

FROBENIUS, G. (1876).

[Ueber das Pfaff'sche Problem. *Crelle's Journ.*, lxxxii. pp. 230–315.]

In the section (§ 4) dealing with general determinants the theorem last reached by Frobenius, as we have seen, concerned a determinant having an m-row array and an m-column array, each with a non-vanishing m-line minor and yet having no non-vanishing minors of the $(m+1)^{\text{th}}$ order. Applying this theorem to determinants of special form he obtained at once three corollaries, namely,

(1) *If in an axisymmetric* † *or zero-axial skew determinant there be an* m*-line minor that does not vanish while all the* (m+1)*-line minors do vanish, then the coaxial minor of the oblong array to which the non-vanishing minor belongs must also be different from* 0.

(2) *If in a zero-axial skew determinant* m *is the highest order of non-vanishing minors, then* m *is even, and among the said non-vanishing minors there must be coaxial minors.*

* To the square Günther prefixes $(-1)^m$ in the first edition and $\pm$ in the second.

† See Hesse's *Vorles. . . . Geometrie des Raumes*, 3 Ausg. (1876), pp. 453–454.

(3) *If in a zero-axial skew determinant all the minors of the* $2r^{th}$ *order vanish, so also must all the minors of the* $(2r-1)^{th}$ *order.*

Holding these results to be of special importance in connection with the main subject of his paper, he devotes a section (§ 5, pp. 242–245) to the independent consideration of them. As a basis to start with, it is affirmed that *if a zero-axial skew determinant of even order vanishes, so likewise do all its primary minors.* The proof consists in noting that in *any* vanishing determinant $|\ a_{1n}\ |$ we have

$$A_{\alpha\alpha}A_{\beta\beta} = A_{\alpha\beta}A_{\beta\alpha},$$

and that as regards the special kind here in question we have further

$$A_{\alpha\alpha} = 0 = A_{\beta\beta}, \quad A_{\alpha\beta} = -A_{\beta\alpha},$$

the result thus being at once

$$A_{\alpha\beta} = 0,$$

as desired. Following on this comes the theorem: *If in a zero-axial skew determinant* there be a coaxial minor of the* $2r^{th}$ *order that does not vanish, while all those coaxial minors formable from it by appending two additional rows and columns do vanish, then all the minors of the* $(2r+1)^{th}$ *order vanish also.* This is reached by applying the preceding theorem to the given vanishing coaxial minors of the $(2r+2)^{th}$ order, the result being proof of the vanishing of a sufficient number of minors of the $(2r+1)^{th}$ order to enable us to assert the vanishing of them all. Following on this again comes a direct deduction from it, obtained by noting that since all the minors of the $(2r+1)^{th}$ order vanish, so also must all those of the $(2r+2)^{th}$ order, and that therefore, by attending only to those of the latter that are coaxial, we can affirm the vanishing of all the Pfaffians of the $(r+1)^{th}$ order. There is thus obtained the analogue in Pfaffians to the well-known theorem in determinants which has just been used at the close of the preceding proof. It may be formulated as follows: *If in any Pfaffian there be a minor of the* r^{th} *order that does not vanish, while all those minors formable from it by appending two additional frame-lines do vanish, then all the other*

* As a rule Frobenius uses 'schief' not for 'gauche' but for 'gauche symétrique': on p. 241 the word 'alternirend' is also applied to a skew symmetric array.

minors of the $(r+1)^{th}$ *order vanish also.* Thus, using the oldest notation for Pfaffians, we know herefrom that with the data

$$\begin{array}{ccc} & (1234) \neq 0, & \\ (123456) = 0, & (123457) = 0, & (123458) = 0, \\ & (123467) = 0, & (123468) = 0, \\ & & (123478) = 0, \end{array}$$

we can assert that all the other six-line minors of (12345678) vanish also.

TANNER, H. W. L. (1878).

[A theorem relating to Pfaffians. *Messenger of Math.*, viii. pp. 56–60.]

In Tanner's words, this important theorem is that '*any relation between Pfaffians is true of their complementaries, provided the relation be made homogeneous.*' Unfortunately his proof (pp. 58–59) is little more than a verification of particular instances. A note is added by Cayley, who suggests that the true form of enunciation should be: *If the elements of the principal Pfaffian be replaced by their cofactors, then each minor Pfaffian is converted into its complementary.* Thus, the principal Pfaffian being [123456], so that the cofactors of 12, 13, ... are [3456], [4562], ... , Tanner says that if the relation

$$12{\cdot}34 - 13{\cdot}24 + 14{\cdot}23 = [1234]$$

hold, then also must the relation

$$[3456][5612] - [4562][5613] + [5623][4561] = 56{\cdot}[123456].$$

Cayley, on the other hand, says in effect that the second is a case of the first, namely, the case in which [3456], [5612], ... are substituted for 12, 34, ...

Whatever view we take, it is clear that Cayley's remark suggests a mode of proving Tanner's theorem, the only requirement in the given example being to show that the compound Pfaffian which [1234] becomes as a result of the substitution is equal to $56 \cdot [123456]$.

It is important to note that Cayley incidentally states that the new theorem holds for determinants also, thus forecasting the *Law of Complementaries.**

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* See above, p. 66 at top.

TANNER, H. W. L. (1878).

[On certain functions allied to Pfaffians. *Quart. Journ. of Math.*, xvi. pp. 34–45.]

The functions in question are of two kinds, namely, the ultra-symbolic determinants

$$y_1,\ \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\ y_1 & y_2 \end{vmatrix},\ \begin{vmatrix} y_1 & y_2 & y_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ y_1 & y_2 & y_3 \end{vmatrix},\ \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \\ y_1 & y_2 & y_3 & y_4 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix}, \ldots,$$

which Tanner denotes by

$$[1],\ [12],\ [123],\ [1234], \ldots,$$

and their companions

$$\frac{\partial}{\partial x_1},\ \begin{vmatrix} y_1 & y_2 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{vmatrix},\ \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ y_1 & y_2 & y_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \end{vmatrix},\ \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \\ y_1 & y_2 & y_3 & y_4 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \end{vmatrix}, \ldots,$$

which he denotes by

$$\{1\},\ \{12\},\ \{123\},\ \{1234\}, \ldots.$$

Of course it is understood that in each term of the final development of any one of the determinants, the elements are taken at the outset in the order of the rows to which they belong, and that each of the $\partial/\partial x$'s of any row operates only on the y's of the row following; thus,

$$\frac{\partial y_4}{\partial x_3}\frac{\partial y_2}{\partial x_1} \quad \text{and} \quad -\frac{\partial y_1}{\partial x_2}\frac{\partial}{\partial x_3}$$

are terms of [1234] and {123} respectively. The determinant [123] had already presented itself to Bellavitis in 1857 (*Hist.*, ii. p. 96).

The two kinds of functions are not without a certain relationship, as may be seen on expanding any one of them in terms of the elements of the last row and their cofactors; for example,

$$[1234] = \{123\}y_4 - \{124\}y_3 + \{134\}y_2 - \{234\}y_1,$$

$$\{1234\} = [123]\frac{\partial}{\partial x_4} - [124]\frac{\partial}{\partial x_3} + [134]\frac{\partial}{\partial x_2} - [234]\frac{\partial}{\partial x_1}.$$

The relation which those of the first kind bear to Pfaffians is considerably closer, it being understood that by Pfaffians in the title of the paper is meant a particular type of Pfaffian, namely, Pfaffians whose elements are of the form

$$\frac{\partial y_s}{\partial x_r} - \frac{\partial y_r}{\partial x_s}, \quad \text{or say } \overline{rs}.$$

Thus, expanding [1234] in terms of the minors of its first two rows and their complementaries, we obtain

$$\overline{12}\cdot\overline{34} - \overline{13}\cdot\overline{24} + \overline{14}\cdot\overline{23} + \overline{23}\cdot\overline{14} - \overline{24}\cdot\overline{13} + \overline{34}\cdot\overline{12},$$

$$i.e. \quad 2\{\overline{12}\cdot\overline{34} - \overline{13}\cdot\overline{24} + \overline{14}\cdot\overline{23}\};$$

so that, using a different notation for Pfaffians, we have the identity

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \\ y_1 & y_2 & y_3 & y_4 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = 2! \begin{vmatrix} \frac{\partial y_2}{\partial x_1} - \frac{\partial y_1}{\partial x_2} & \frac{\partial y_3}{\partial x_1} - \frac{\partial y_1}{\partial x_3} & \frac{\partial y_4}{\partial x_1} - \frac{\partial y_1}{\partial x_4} \\ & \frac{\partial y_3}{\partial x_2} - \frac{\partial y_2}{\partial x_3} & \frac{\partial y_4}{\partial x_2} - \frac{\partial y_2}{\partial x_4} \\ & & \frac{\partial y_4}{\partial x_3} - \frac{\partial y_3}{\partial x_4} \end{vmatrix},$$

there being for [123456], [12345678], ... similar Pfaffian expressions with the coefficients 3!, 4!, ... respectively.

As may be expected from this relationship, Tanner readily discovers analogues to the simpler properties of Pfaffians; and, in view of his previous paper of the same year, it is equally natural to find him formulating the 'complementaries' of these analogues. He also

arrives at formulae in which functions of both the forms { }, [ ] appear; for example,

$$[23]'\{1\}'-[13]'\{2\}'+[12]'\cdot\{3\}',$$

where $[23]'$, $\{1\}'$, ... stand for the complementaries of $[23]$, $\{1\}$, ...

We may note that Tanner might have expressed the odd-ordered functions

$$[123],\quad [12345],\ \ldots .$$

also as Pfaffians, namely,

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ & \overline{12} & \overline{13} \\ & & \overline{23} \end{vmatrix}, \qquad 2!\begin{vmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \\ & \overline{12} & \overline{13} & \overline{14} & \overline{15} \\ & & \overline{23} & \overline{24} & \overline{25} \\ & & & \overline{34} & \overline{35} \\ & & & & \overline{45} \end{vmatrix}.$$

ROBERTS, S. (1879).

[Note on certain determinants connected with algebraical expressions having the same terms as their component factors. *Messenger of Math.*, viii. pp. 138–140.]

Starting with Lagrange's identity

$$(x^2+ax_1^2+bx_2^2+abx_3^2)(\xi^2+a\xi_1^2+b\xi_2^2+ab\xi_3^2) = \mathrm{P}^2+a\mathrm{P}_1^2+b\mathrm{P}_2^2+ab\mathrm{P}_3^2,$$

where

$$\begin{aligned} \mathrm{P} &= x\xi +ax_1\xi_1+bx_2\xi_2+abx_3\xi_3, \\ \mathrm{P}_1 &= -x\xi_1+ x_1\xi -bx_2\xi_3+ bx_3\xi_2, \\ \mathrm{P}_2 &= -x\xi_2+ax_1\xi_3+ x_2\xi - ax_3\xi_1, \\ \mathrm{P}_3 &= -x\xi_3- x_1\xi_2+ x_2\xi_1+ x_3\xi, \end{aligned}$$

and considering the consequences of

$$\xi^2+a\xi_1^2+b\xi_2^2+ab\xi_3^2$$

becoming 0, Roberts deduces the result

$$\begin{vmatrix} \xi & a\xi_1 & b\xi_2 & ab\xi_3 \\ -\xi_1 & \xi & -b\xi_3 & b\xi_2 \\ -\xi_2 & a\xi_3 & \xi & -a\xi_1 \\ -\xi_3 & -\xi_2 & \xi_1 & \xi \end{vmatrix} = (\xi^2+a\xi_1^2+b\xi_2^2+ab\xi_3^2)^2;$$

and in similar fashion

$$\begin{vmatrix} \xi & a\xi_1 & ab\xi_2 & b\xi_3 & ac\xi_4 & c\xi_5 & bc\xi_6 & abc\xi_7 \\ -\xi_1 & \xi & b\xi_3 & -b\xi_2 & c\xi_5 & -c\xi_4 & -bc\xi_7 & bc\xi_6 \\ -\xi_2 & -\xi_3 & \xi & \xi_1 & c\xi_6 & c\xi_7 & -c\xi_4 & -c\xi_5 \\ -\xi_3 & a\xi_2 & -a\xi_1 & \xi & ac\xi_7 & -c\xi_6 & c\xi_5 & -ac\xi_4 \\ -\xi_4 & -\xi_5 & -b\xi_6 & -b\xi_7 & \xi & \xi_1 & b\xi_2 & b\xi_3 \\ -\xi_5 & a\xi_4 & -ab\xi_7 & b\xi_6 & -a\xi_1 & \xi & -b\xi_3 & ab\xi_2 \\ -\xi_6 & a\xi_7 & a\xi_4 & -\xi_5 & -a\xi_2 & \xi_3 & \xi & -a\xi_1 \\ -\xi_7 & -\xi_6 & \xi_5 & \xi_4 & -\xi_3 & -\xi_2 & \xi_1 & \xi \end{vmatrix}$$

$$= (\xi^2+a\xi_1^2+ab\xi_2^2+b\xi_3^2+ac\xi_4^2+c\xi_5^2+bc\xi_6^2+abc\xi_7^2)^4.$$

Of these two interesting determinants, R_4 and R_8 say, the former is seen to include Souillart's of 1860, and the latter Sylvester's of 1867.

Multiplying R_4 in row-by-column fashion by the determinant got from R_4 on changing the signs of ξ_1, ξ_2, ξ_3 he obtains a determinant having

$$\xi^2+a\xi_1^2+b\xi_2^2+ab\xi_3^2$$

in the places 11, 22, 33, 44, and zeros in all the other places: further, he asserts that all the primary minors have $\xi^2+a\xi_1^2+b\xi_2^2+ab\xi_3^2$ for a factor, and that quite similar properties are possessed by R_8.

It may be added that as the result of an investigation* made in 1879 Roberts convinced himself of the non-existence of an R_{16}.

SYLVESTER, J. J. (1879).

[Note on determinants and duadic disynthemes. *American Journ. of Math.*, ii. pp. 89–90, 214–222.]

[Sur une propriété arithmétique d'une série de nombres entiers. *Comptes Rendus . . . . Acad. des Sci.* (Paris), lxxxviii. pp. 1297–1298.]

The greater part of the space here is devoted to the number of terms in (the 'denumerant' of) skew and zero-axial skew determinants. In the latter case the difference-equation is given as

$$u_n = (n-1)^2u_{n-2} - \tfrac{1}{2}(n-1)(n-2)(n-3)u_{n-4},$$

* See *Quart. Journ. of Math.*, xvi. pp. 159–170. The paper also contains an interesting sketch of the history of the problem of representing the product of two sums of 2^n squares as a sum of 2^n squares.

and thence the values of u_2, u_4, u_6, u_8, ... as

$$1, \quad 6, \quad 120, \quad 5250, \ldots\ldots$$

In the former case the values appear of course as sums of multiples of u_2, u_4, ... ; for example, for the 7th order the value is

$$1 + C_{7,5}\cdot 1 + C_{7,3}\cdot 6 + C_{7,1}\cdot 120;$$

and the first seven values are

$$1, \ 2, \ 4, \ 13, \ 41, \ 226, \ 1072, \ldots\ldots$$

SYLVESTER, J. J. (1879).

[Sur la valeur moyenne des coefficients dans le développement d'un déterminant gauche ou symétrique. . . . *Comptes Rendus* . . . . *Acad. des Sci.* (Paris), lxxxix. pp. 24–26, 497–498; or *Collected Math. Papers*, iii. pp. 253–255, 257.]

In this is contained an improved form of one of the results just chronicled, namely, that the number of distinct terms in a zero-axial skew determinant of order $2n$ is

$$1\cdot 3\cdot 5 \ldots\ldots (2n-1)\cdot v_n,$$

where v_n is determined from the equations

$$v_n = (2n-1)v_{n-1} - (n-1)v_{n-2}, \quad v_0 = 1, \quad v_1 = 1,$$

or from the equation (corrected)

$$v_n = \left\{1 + n + 1\cdot 5C_{n,2} + 1\cdot 5\cdot 9C_{n,3} + \ldots\ldots\right\}\frac{1}{2^n}.$$

A similar statement is made in regard to 'déterminants doublement gauches,' that is to say, determinants which are skew with respect to both diagonals and have both diagonals full of zeros. In such a determinant of the order $4n$ Sylvester says the number of distinct terms is

$$2\cdot 4\cdot 6 \ldots\ldots (4n-2)\cdot w_n,$$

where w_n is determined from the equations

$$w_n = (4n-3)w_{n-1} - 2nw_{n-2}, \quad w_0 = 1, \quad w_1 = 1.$$

or from the equation

$$w_n = \left\{1 + n + 1\cdot 9C_{n,2} + 1\cdot 9\cdot 17C_{n,3} + \ldots\ldots\right\}\frac{1}{2^n},$$

but the assertion is incorrect,* the number of distinct terms being 2, 36 when n is 1, 2.

PAIGE, C. LE (1880).

[Sur les déterminants hémisymétriques d'ordre pair. *Sitzungsb. . . . böhm. Ges. d. Wiss.* (Prag), Jahrg. 1880, pp. 125–127.]

[Sur une propriété des déterminants hémisymétriques d'ordre pair.† *Nouv. Corresp. Math.*, vi. pp. 155–158.]

The property in question is Cayley's of 1847, but is stated in the form first reached by Bellavitis in 1857 (*Hist.*, ii. p. 278). The process of proof is at the outset that adopted by Trudi in 1862 (*Hist.*, ii. pp. 288–289).

*If the coefficient of w_{n-2} be changed into $2(n-1)$, the two modes of determining w_n will be brought into accord : but this is not all that is wanted.

† The two papers cover almost the same ground.

CHAPTER XI.

ORTHOGONANTS, FROM 1855 TO 1879.

From the study of the subject of linear transformation there arise two quite distinct matters for consideration in the theory of determinants; namely, first, the properties of a special form of determinant known as an Orthogonant, and, second, the character of the roots of Lagrange's determinantal equation. At first the two matters were interlocked: in the present volume this is no longer quite the case, the papers being without much difficulty separable into the two relevant groups. As, however, the subject of the second group is still on the whole viewed in connection with orthogonal transformation, the separation has not been made. The reader, therefore, who is interested in what came soon to be known as the subject of the *Latent Roots of a Determinant* must still turn to Orthogonants.

We have only further to remark that the first of the papers belongs to the twenty-year period immediately preceding, and that between the two periods there is little difference to be noted in the matter of fruitfulness.

CAUCHY, A. (1855).

[Sur le dénombrement des racines qui, dans une équation algébrique ou transcendante, satisfont à des conditions données. *Comptes Rendus . . . Acad. des Sci.* (Paris), xl. pp. 1329–1335; or *Œuvres complètes*, (1) xii. pp. 293–299.]

What concerns us here is the one or two corollaries to the second of three theorems on the subject matter of the title, the main corollary being that the equation

$$\begin{vmatrix} \sigma_{11}-x & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22}-x & \dots & \sigma_{2n} \\ \dots & \dots & \dots & \dots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn}-x \end{vmatrix} = 0$$

will have all its roots real if σ_{rs} be any integral function of ξ whose form depends on r and s, and which is not altered by the interchange of r and s. One of the cases in which the σ's are linear is taken for special comment; it should be distinguished, however, from Sylvester's extension of 1853 (*Hist.*, ii. p. 314).

HESSE, O. (1861, 1869).

[Vorlesungen über analytische Geometrie des Raumes, insbesondere über Oberflächen zweiter Ordnung. xv+368 pp. (2te Aufl., xvi+456 pp.) Leipzig.]

As has already been implied, there is more of Algebra in these lectures than is usual in a book on Solid Geometry, and naturally the algebra of orthogonal transformation in the case of three variables receives considerable attention. A series of excerpts from five or six of the chapters would provide an excellent exposition of the subject.

CLEBSCH, A. (1861).

[Ueber eine Classe von Gleichungen welche nur reellen Wurzeln besitzen. *Crelle's Journ.*, lxii. pp. 232–245.]

Taking the theorem (*Hist.*, ii. pp. 449–451) regarding the reality of the roots of the equation

$$\begin{vmatrix} a_{11}+\lambda & a_{12}+b_{12}\sqrt{-1} & a_{13}+b_{13}\sqrt{-1} & \dots \\ a_{12}-b_{12}\sqrt{-1} & a_{22}+\lambda & a_{23}+b_{23}\sqrt{-1} & \dots \\ a_{13}-b_{13}\sqrt{-1} & a_{23}-b_{23}\sqrt{-1} & a_{33}+\lambda & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n = 0,$$

where the a's and b's are real, Clebsch makes the a's vanish, and thence draws the conclusion that the roots of the equation

$$\begin{vmatrix} \mu & b_{12} & b_{13} & \dots \\ -b_{12} & \mu & b_{23} & \dots \\ -b_{13} & -b_{23} & \mu & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n = 0$$

are purely imaginary. This he also proves separately, by expanding the determinant in descending powers of μ as originally done by Cayley (*Hist.*, ii. p. 261).

He is then led to an equation of another type, the reality of whose roots depends on persistence of sign in a given quadric: and an extension is thus given to Hesse's process for expressing the discriminant as a sum of squares (*Hist.*, ii. p. 321).

TRUDI, N. (1862).

[Teoria dei Determinanti, . . . ix+268 pp. Napoli.]

Trudi's chapter on Linear Substitution (pp. 206–217), though for a certain length closely resembling Baltzer's, deals more clearly and fully with the complementary minors of an orthogonant.

In extension of Jacobi's theorem on the subject he affirms that *the product of an orthogonant by any one of its minors is equal to the 'algebraic complement' (i.e. cofactor) of the said minor.* The orthogonant being $|\omega_{1n}|$, M_k any k-line minor of it, and $\overline{M}_k$ the corresponding minor of the adjugate $|\Omega_{1n}|$, we have

$$\overline{M}_k = |\omega_{1n}|^{k-1} \cdot \text{cof. of } M_k \text{ in } |\omega_{1n}| \cdot$$

But, it having been already proved as a first case that any element Ω_{rs} of $\overline{M}_k$ is equal to $|\omega_{1n}| \cdot \omega_{rs}$, it follows that we also have

$$\overline{M}_k = |\omega_{1n}|^k \cdot M_k.$$

By equatement of these two expressions for $\overline{M}_k$ we at once obtain, as desired,

$$|\omega_{1n}| \cdot M_k = \text{cof. of } M_k.$$

Multiplying both sides of this by M_k and then performing the summation required by Laplace's expansion-theorem, Trudi in effect derives the result

$$|\omega_{1n}| \cdot \sum M_k^2 = |\omega_{1n}|,$$

and thus concludes that *the sum of the squares of the* k*-line minors formable from any array of* k *rows of an orthogonant is equal to* 1.

After explaining the construction of Cayley's orthogonant and showing rather unnecessarily that its value is $+1$, he adds, by way

of rounding off the subject, that an orthogonant equal to -1 is readily obtained by simply changing the sign of an odd number of rows or columns.

His exposition (pp. 237–241) of the mode of transforming the n-ary quadric $\Sigma a_{rs}x_r x_s$ into $\Sigma \mathrm{A}_r y_r^2$ by means of the orthogonal substitution

$$x_r = \omega_{r1}y_1 + \omega_{r2}y_2 + \ldots + \omega_{rn}y_n$$

resembles Lebesgue's in simplicity and clearness (*Hist.*, i. pp. 463–467). Thus, to obtain the equation for the determination of the A's he takes the reverse substitution

$$y_r = \omega_{1r}x_1 + \omega_{2r}x_2 + \ldots + \omega_{nr}x_n,$$

and from the identity of the two forms of the quadric deduces at once

$$a_{rs} = \mathrm{A}_1\omega_{r1}\omega_{s1} + \mathrm{A}_2\omega_{r2}\omega_{s2} + \ldots + \mathrm{A}_n\omega_{rn}\omega_{sn}.$$

Then taking the cases of this where $s = 1, 2, \ldots, n$, and using the multipliers $\omega_{1s}, \omega_{2s}, \ldots$, he obtains on addition

$$a_{r1}\omega_{1s} + a_{r2}\omega_{2s} + \ldots + a_{rn}\omega_{ns} = \mathrm{A}_s\omega_{rs},$$

whence, after putting $r = 1, 2, \ldots, n$, there is deduced

$$\begin{vmatrix} a_{11}-\mathrm{A}_s & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\mathrm{A}_s & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\mathrm{A}_s \end{vmatrix} = 0.$$

CAYLEY, A. (1862).

[Report on the progress of the solution of certain special problems of dynamics. *Report . . . British Assoc. . . .* xxxii. (Cambridge), pp. 184–252; or *Collected Math. Papers*, iv. pp. 513–593.]

Under the heading 'Transformation of Coordinates' (§§ 124–131), Cayley gives an account of some of the more important memoirs from 1758 to 1845 (*Hist.*, i. pp. 407–410), that is to say, practically up to the date of his own notable paper on the construction of an orthogonant.

HESSE, O. (1862): HENRICI, O. (1864).

[Zerlegung der Bedingung für die Gleichheit der Hauptaxen eines auf einer Oberfläche zweiter Ordnung liegenden Kegelschnittes in die Summe von Quadraten. *Crelle's Journ.*, lx. pp. 305–312; or *Werke*, pp. 497–506.]

[Bemerkung zu "Hesse, Zerlegung der Bedingung. . . . ." *Crelle's Journ.*, lxiv. pp. 187–192.]

The equations of the surface and the plane being respectively

$$ax^2+by^2+cz^2+2lyz+2mzx+2nxy = 0,$$
$$\text{and} \quad Ax+By+Cz+D = 0,$$

where $A^2+B^2+C^2 = 1$, and consequently

$$\begin{vmatrix} a-\lambda & n & m & A \\ n & b-\lambda & l & B \\ m & l & c-\lambda & C \\ A & B & C & . \end{vmatrix} = 0$$

being the equation for determining the axes of the section, Hesse expresses the discriminant, $(\lambda_1-\lambda_2)^2$, of this equation in four different ways as a sum of squares, namely, as a sum of 10, of 7, of 6, and of 5 squares. To this Henrici contributes an expression in the form of the sum of two squares.

The subject is continued by C. Souillart, G. Bauer, C. F. Geiser, E. Sourander, and C. Souillart.*

Geiser makes an extension to the case where the determinant is of the $(n+1)^{\text{th}}$ order and the equation is of the $(n-1)^{\text{th}}$ degree: he also recurs in another paper to the original allied problem of Kummer.†

SYLVESTER, J. J. (1867).

[Thoughts on inverse orthogonal matrices, simultaneous sign-successions, ... *Philos. Magazine*, xxxiv. pp. 461–475; or *Collected Math. Papers*, ii. pp. 615–628.]

Sylvester, having been led to study determinants whose elements are inversely proportional to the corresponding elements of the

* *Crelle's Journ.*, lxv. pp. 320–334: lxxi. pp. 46–52: *Annali di Mat.*, (2) viii. pp. 113–120; *Crelle's Journ.*, lxxxv. pp. 339–344: lxxxvii. pp. 220–221.

† *Crelle's Journ.*, lxxxii. pp. 47–53.

adjugate determinant, naturally digresses a little to speak of those in which the proportionality is *direct*. He even goes so far as to suggest that the latter property should be that used for the definition of an orthogonant; but he is careful enough to point out that this would leave the *value* of the determinant unfixed instead of having it equal to ± 1.

Later on, when considering the construction of inversely orthogonal determinants whose order is a power of 2, he makes another digression to draw attention to the so-called orthogonants (strictly *penorthogonants*) of the 2nd, 4th, 8th orders whose values are

$$-(a^2+b^2),\quad -(a^2+b^2+c^2+d^2)^2,\quad -(a^2+b^2+c^2+d^2+l^2+m^2+n^2+p^2),^4$$

respectively, writing them in the form

$$\begin{vmatrix} a & b \\ b & -a \end{vmatrix},$$

$$\begin{vmatrix} a & b & c & d \\ b & -a & d & -c \\ c & -d & -a & b \\ d & c & -b & -a \end{vmatrix},$$

$$\begin{vmatrix} a & b & c & d & l & m & n & p \\ b & -a & -d & c & m & -l & p & -n \\ c & d & -a & -b & n & -p & -l & m \\ d & -c & b & -a & p & n & -m & -l \\ l & -m & -n & -p & -a & b & c & d \\ m & l & p & -n & -b & -a & d & -c \\ n & -p & l & m & -c & -d & -a & b \\ p & n & -m & l & -d & c & -b & -a \end{vmatrix}.$$

The second of the three is essentially the same as Souillart's of 1860 (*Hist.*, ii. pp. 287–288), and becomes identical with it on altering the signs of the last three rows.*

We may note for ourselves that by making the elements in the first columns negative the determinants would be wholly skew, and

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* Souillart's result could readily have been arrived at as a case of Cayley's illustrative example of 1847, namely,

$$\begin{vmatrix} x & a & b & c \\ -a & x & d & e \\ -b & -d & x & f \\ -c & -e & -f & x \end{vmatrix} = x^4 + x^2(a^2+b^2+c^2+d^2+e^2+f^2) + (af-be+cd)^2;$$

for the right-hand member here is seen to be equal to

$$(x^2+af-be+cd)^2 + (a-f)^2x^2 + (b+e)^2x^2 + (c-d)^2x^2,$$

and therefore to become

$$(x^2+a^2+b^2+c^2)^2$$

when we put $f,\ e,\ d = a,\ -b,\ c$.

their values would be positive : also that one would possibly have expected the second determinant to be identical with the first four-line minor in the third.

BAUER, G. (1868).

[Von der Zerlegung der Discriminante des cubischen Gleichung, welche die Hauptaxen einer Fläche zweiter Ordnung bestimmen (*sic*), in eine Summe von Quadraten. *Crelle's Journ.*, lxxi. pp. 40–45.]

Bauer, with his eye on Kummer (*Hist.*, ii. p. 295), takes the discriminant of

$$\begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix} = 0, \quad \text{or, say,} \quad x^3-\mathrm{P}x^2+\mathrm{Q}x-\mathrm{R} = 0,$$

in the form of Bezout's axisymmetric eliminant

$$\begin{vmatrix} 6\mathrm{Q}-2\mathrm{P}^2 & -9\mathrm{R}+\mathrm{PQ} \\ -9\mathrm{R}+\mathrm{PQ} & 6\mathrm{RP}-2\mathrm{Q}^2 \end{vmatrix},$$

and, partitioning it into four determinants, succeeds in evolving Borchardt's, his own, and Kummer's expressions for it in terms of 13, 10 and 7 different squares respectively.

GERONO, E. (1870) : WISSELINK, D. B. (1877).

[Note sur une application de la théorie des déterminants. *Nouv. Annales de Math.*, (2) ix. pp. 392–398.]

[Merkwaardige eigenschappen van eenen determinant van den derden graad. *Nieuw Archief v. Wisk.*, iii. pp. 84–89.]

These are simple expositions of the properties of the three-line orthogonant, Wisselink following the lead of Gerono.

VELTMANN, W. (1871).

[Beiträge zur Theorie der Determinanten. *Zeitschrift f. Math. u. Phys.*, xvi. pp. 516–525.]

Veltmann's second contribution (pp. 523–525) concerns Cayley's construction of an orthogonal substitution. He dispenses with

the intermediary variables θ_1, θ_2, θ_3, ... , laying down the proposition that x_1, x_2, x_3, ... , and ξ_1, ξ_2, ξ_3, ... , are orthogonally related if

$$\left.\begin{aligned} l_{11}x_1+l_{12}x_2+l_{13}x_3+\ldots. &= l_{11}\xi_1+l_{21}\xi_2+l_{31}\xi_3+\ldots. \\ l_{21}x_1+l_{22}x_2+l_{23}x_3+\ldots. &= l_{12}\xi_1+l_{22}\xi_2+l_{32}\xi_3+\ldots. \\ l_{31}x_1+l_{32}x_2+l_{33}x_3+\ldots. &= l_{13}\xi_1+l_{23}\xi_2+l_{33}\xi_3+\ldots. \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\},$$

where the determinant of the l's on the left is meant to be unit-axial skew, and is manifestly the conjugate of that on the right. To establish this, he has only got to solve for each variable of the one set in terms of the variables of the other. Thus, from the equations

$$\left.\begin{aligned} x_1+\nu x_2-\mu x_3 &= \xi_1-\nu\xi_2+\mu\xi_3 \\ -\nu x_1+x_2+\lambda x_3 &= \nu\xi_1+\xi_2-\lambda\xi_3 \\ \mu x_1-\lambda x_2+x_3 &= -\mu\xi_1+\lambda\xi_2+\xi_3 \end{aligned}\right\},$$

there is obtained in the usual way

$$x_1 = \begin{vmatrix} \xi_1-\nu\xi_2+\mu\xi_3 & \nu & -\mu \\ \nu\xi_1+\xi_2-\lambda\xi_3 & 1 & \lambda \\ -\mu\xi_1+\lambda\xi_2+\xi_3 & -\lambda & 1 \end{vmatrix} \div \begin{vmatrix} 1 & \nu & -\mu \\ -\nu & 1 & \lambda \\ \mu & -\lambda & 1 \end{vmatrix}$$

$$= \frac{1+\lambda^2-\mu^2-\nu^2}{1+\lambda^2+\mu^2+\nu^2}\xi_1 + \frac{2(\lambda\mu-\nu)}{1+\lambda^2+\mu^2+\nu^2}\xi_2 + \frac{2(\nu\lambda+\mu)}{1+\lambda^2+\mu^2+\nu^2}\xi_3,$$

and similar expressions for x_2 and x_3.

We may also note that from the constitution of the unsolved set of equations the changing of the signs of λ, μ, ν in the coefficients here obtained must give us the coefficients in the expressions for ξ_1 in terms of x_1, x_2, x_3.

SIACCI, F. (1872).

[Questioni 1, 2, 3, 11. *Giornale di Mat.*, x. p. 188, p. 360. Solution of 3 by P. Cassani, x. pp. 239–240 ; Solution of 1, 2, 3 by M. Albeggiani, x. pp. 285–290.]

The first three ' questions,' sent to the *Giornale* on 17th May, are special cases of a general theorem which is the subject of a

communication read to the Turin Academy * on 9th June: and 'question' 11 is another special case sent similarly for proof while a paper containing a treatment of it was being prepared for insertion in the *Annali di Matematica*.† Unfortunately, no hint of this double publication of the theorems is given anywhere. The formal enunciations of them may be put thus:

(1) *It is immaterial whether the elements of the main diagonal of an* n*-line orthogonant be increased by* $x_1, x_2, \ldots, x_n$ *respectively or by* $x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}$ *so long as* $x_1 x_2 \ldots x_n = 1$.

(2) *If from each element of an odd-ordered orthogonant there be subtracted the corresponding element of another orthogonant of the same order, the determinant so formed vanishes.*

(3) *If two* n*-line orthogonants be taken, and the rows of one be multiplied by* $h_1, h_2, \ldots, h_n$ *respectively, and the rows of the other by* $k_1, k_2, \ldots, k_n$ *respectively, and a new determinant be formed, each of whose elements is the sum of the corresponding elements in the orthogonants thus modified, this determinant is such that it will remain unaltered on the interchange of the* h*'s and* k*'s.*

(4) *The determinant formed by subtracting* 1 *from each element of the main diagonal of an even-ordered orthogonant has the complementary minors of its diagonal elements all equal to the negative half of itself.*

Cassani's proof of the third begins by partitioning the determinant in question into a series of determinants with monomial elements: and Albeggiani follows the same tiresome method, with this difference, that he devotes the first part of his paper (6 pp.) to finding a general expression for the development of a determinant with binomial elements, and then in the following six pages uses the resulting theorem to establish (1), (2), (3). This process of specializing is of little interest; and, as for the general theorem, we give attention to it, like Siacci's, in its proper place.

For ourselves we may note that a very natural and simple way of obtaining all of them is by multiplying the determinant concerned in each case by the orthogonant used in its construction. Thus, taking the first, we have

* *Atti*, vii. pp. 772–783. † v. pp. 296–304.

$$\begin{vmatrix} \alpha_1+x_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2+x_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3+x_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} 1+\alpha_1x_1 & \beta_1x_1 & \gamma_1x_1 \\ \alpha_2x_2 & 1+\beta_2x_2 & \gamma_2x_2 \\ \alpha_3x_3 & \beta_3x_3 & 1+\gamma_3x_3 \end{vmatrix} = x_1x_2x_3 \begin{vmatrix} \alpha_1+\frac{1}{x_1} & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2+\frac{1}{x_2} & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3+\frac{1}{x_3} \end{vmatrix};$$

and taking the fourth, we have

$$\begin{vmatrix} 1 & . & . & . \\ \beta_1 & \beta_2-1 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3-1 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4-1 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_4 & \delta_4 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ -\alpha_2 & 1-\beta_2 & -\gamma_2 & -\delta_2 \\ -\alpha_3 & -\beta_3 & 1-\gamma_3 & -\delta_3 \\ -\alpha_4 & -\beta_4 & -\gamma_4 & 1-\delta_4 \end{vmatrix}$$

$$= -\begin{vmatrix} \alpha_1-1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2-1 & \gamma_2 & \delta_2 \\ \alpha_3 & \beta_3 & \gamma_3-1 & \delta_3 \\ \alpha_4 & \beta_4 & \gamma_4 & \delta_4-1 \end{vmatrix} - \begin{vmatrix} \beta_2-1 & \beta_3 & \beta_4 \\ \gamma_2 & \gamma_3-1 & \gamma_4 \\ \delta_2 & \delta_3 & \delta_4-1 \end{vmatrix}.$$

The second manifestly follows at once from the third by putting all the h's equal to 1 and all the k's equal to -1. As for the third itself, we first multiply by one of the orthogonants involved, the result being

$$\begin{vmatrix} h_1\alpha_1+k_1\lambda_1 & h_1\alpha_2+k_1\lambda_2 & h_1\alpha_3+k_1\lambda_3 \\ h_2\beta_1+k_2\mu_1 & h_2\beta_2+k_2\mu_2 & h_2\beta_3+k_2\mu_3 \\ h_3\gamma_1+k_3\nu_1 & h_3\gamma_2+k_3\nu_2 & h_3\gamma_3+k_3\nu_3 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} h_1+k_1\Sigma\lambda\alpha & k_1\Sigma\lambda\beta & k_1\Sigma\lambda\gamma \\ k_2\Sigma\mu\alpha & h_2+k_2\Sigma\mu\beta & k_2\Sigma\mu\gamma \\ k_3\Sigma\nu\alpha & k_3\Sigma\nu\beta & h_3+k_3\Sigma\nu\gamma \end{vmatrix},$$

and then we note the difference in the product which multiplication by the other orthogonant $|\lambda_1\mu_2\nu_3|$ would have brought about.

Finally, we draw attention to the fact that the orthogonants involved are all here taken as positive, and that if they be not so taken Siacci's enunciations will require modification.

BELTRAMI, E. (1873).

[Sulle funzioni bilineari. *Giornale di Mat.*, xi. pp. 98–106.]

In the treatment of the problem * of the transformation of $\sum_{rs} c_{rs} x_r y_s$ into the canonic form $\sum_{m} \gamma_m \xi_m \eta_m$, it is inferred that *if*

$$\begin{vmatrix} a & f & e & \dots \\ f & b & d & \dots \\ e & d & c & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}, \quad \begin{vmatrix} a' & f' & e' & \dots \\ f' & b' & d' & \dots \\ e' & d' & c' & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

be two forms of the square of any determinant, the one obtained by row-multiplication and the other by column-multiplication, then

$$\begin{vmatrix} a-x & f & e & \dots \\ f & b-x & d & \dots \\ e & d & c-x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}, \quad \begin{vmatrix} a'-x & f' & e' & \dots \\ f' & b'-x & d' & \dots \\ e' & d' & c'-x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

viewed as functions of x, *are identical.* The ground for the inference is that while the functions are of the n^{th} degree in x, they are found to be equal for $n+1$ values of x.

A closely allied theorem by Hamburger will be found dealt with under the same year in chap. i.

GRAVELAAR, N. L. W. A. (1875).

[Neuer Beweis für die Realität der Wurzeln einer wichtigen Gleichung. *Archiv d. Math. u. Phys.*, lviii. pp. 301–318.]

Gravelaar's proof is not really new : what his paper provides is a criticism of a previous proof and a mode of removing the defects.

In discussing the character of the roots of the equation

$$\begin{vmatrix} a_{11}-x & a_{12} & a_{13} & \dots \\ a_{12} & a_{22}-x & a_{23} & \dots \\ a_{13} & a_{23} & a_{33}-x & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n = 0 = \mathrm{L}_n(x), \text{ say,}$$

* Jacobi's of 1831 (see *Hist.*, i. p. 436) : but Beltrami begins by showing generally how the determinant of a bilinear form is altered by the performance of the two linear substitutions (see Cayley's memoir referred to in *Hist.*, ii. p. 313).

Salmon, seeking to shorten Cauchy's procedure of 1829, had used as a substitute for Sturm's series of derived functions the series

$$L_n(x),\quad L_{n-1}(x),\quad \ldots\ldots,\quad L_1(x),\quad 1.$$

To this Gravelaar objects that, though the substituted series has one fundamental property in common with Sturm's series, it does not possess a further essential property, namely, the impossibility of two consecutive members of it simultaneously vanishing. In effect the objection means that Salmon did not, like his predecessor Cauchy, consider the case where the equation under discussion has equal roots.

The proof offered (§§ 10, 11, pp. 306–310) is of less interest than the preparatory theorems on which it depends, namely, (1) that *the* r^{th} *differential coefficient of* $L_n(x)$ *is, save for the arithmetical factor* $(-1)^r r!$, *the sum of the coaxial minors of the* $(n-r)^{th}$ *order*: (2) that *the necessary and sufficient condition for the equation* $L_n(x) = 0$ *having a real root* a *repeated* m *times is that all the minors of the* $(n-m+1)^{th}$ *order vanish when* x *is put equal to* a; (3) that *if* $L_n(x)$ *contain the factor* $(x-a)^m$, *then all the primary minors contain the factor* $(x-a)^{m-1}$.

No proof of the first theorem is given, but it is readily derivable from Trudi's form of Jacobi's theorem regarding the complete differential of a determinant. In support of the second theorem, but really proving little more than the converse, it is pointed out that if $L_n(x)$ contain the factor $(x-a)^m$, the putting of x equal to a must cause all the differential coefficients of $L_n(x)$ up to and including the $(m-1)^{th}$ to vanish; that therefore, from the first theorem, with the help of a property of axisymmetric determinants, all the coaxial minors of the $(n-m+1)^{th}$ and higher orders must vanish; and that thence, with the help of another property of axisymmetric determinants, the other minors of the $(n-m+1)^{th}$ and higher orders must vanish also. As for the third theorem, it is first shown to be true of the primary minors that are coaxial, and thereafter of the others by considering a two-line minor of the adjugate of $L_n(x)$ and by assuming that a secondary coaxial minor is divisible by $(x-a)^{m-2}$.

WEIHRAUCH, K. (1876): FROBENIUS, G. (1878).

[Zur Construction einer unimodularen Determinante. *Zeitschrift f. Math. u. Phys.*, xxi. pp. 134–137.]

[Theorie der linearen Formen mit ganzen Coefficienten. *Crelle's Journ.*, lxxxvi. pp. 146–208.]

What is here given by Weihrauch is an alternative solution of the problem which Hermite set himself in 1849 (*Hist.*, ii. p. 306).

Frobenius devotes a section (§ 3, pp. 150–151) of his extensive memoir to the same subject, referring his solution back to Gauss (*Disq. Arith.*, § 279), and drawing attention to four solutions given in a paper of Jacobi's published in 1868, but written probably early in 1850 (see *Crelle's Journ.*, lxix. pp. 1–28).

BARDELLI, G. (1876).

[Alcune proprietà dei coefficienti di uno sostituzione ortogonale. *Rendic. . . . Istituto Lombardo . . .* (2) ix. pp. 167–174.]

The basis of the work here is any orthogonant A which has all its elements functions of one and the same variable; and the investigation is mainly concerned with the determinant A_i, each element of which a'_{rs} is the derivate of the corresponding element a_{rs} of A.

From the fact that

$$\left.\begin{aligned} a_{r1}^2 + a_{r2}^2 + \ldots\ldots + a_{rn}^2 &= 1 \\ a_{r1}a_{s1} + a_{r2}a_{s2} + \ldots\ldots + a_{rn}a_{sn} &= 0 \end{aligned}\right\}$$

there is obtained by differentiation

$$\left.\begin{aligned} a_{r1}a'_{r1} + \ldots\ldots + a_{rn}a'_{rn} &= 0 \\ (a_{r1}a'_{s1} + \ldots\ldots + a_{rn}a'_{sn}) + (a_{s1}a'_{r1} + \ldots\ldots + a_{sn}a'_{rn}) &= 0 \end{aligned}\right\},$$

which equalities, in view of the fact that the expressions on their left are exactly such as occur when A_i is multiplied rowwise by A, may be conveniently shortened into

$$p_{rr} = 0, \quad p_{rs} + p_{sr} = 0.$$

We thus have A_iA, *i.e.* A_i, equal to the determinant of the p's, P say: and, this latter being zero-axial and skew, it follows that

A_i *is equal to* 0 *or the square of a Pfaffian according as* n *is odd or even.* It is equally readily seen that any determinant containing some rows taken from A_i and the others from A can in the same way, namely, by multiplication by A, be shown to be expressible as a determinant of lower order whose elements are p's. Thus, taking

$$A \equiv |\, \alpha_1\beta_2\gamma_3\delta_4 \,|,$$

and the determinant of mixed origin to be

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha'_1 & \alpha'_2 & \alpha'_3 & \alpha'_4 \\ \gamma'_1 & \gamma'_2 & \gamma'_3 & \gamma'_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{vmatrix}, \text{ or M, say,}$$

we obtain

$$MA = \begin{vmatrix} 1 & p_{11} & p_{13} & . \\ . & p_{21} & p_{23} & . \\ . & p_{31} & p_{33} & . \\ . & p_{41} & p_{43} & 1 \end{vmatrix} = \begin{vmatrix} p_{21} & p_{23} \\ p_{31} & . \end{vmatrix} = p_{13}p_{23}.$$

When the rows taken from A_i correspond with those not taken from A, the resulting determinant is zero-axial and skew : thus

$$\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta'_1 & \beta'_2 & \beta'_3 & \beta'_4 \\ \gamma'_1 & \gamma'_2 & \gamma'_3 & \gamma'_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{vmatrix} \cdot A = \begin{vmatrix} 1 & p_{12} & p_{13} & . \\ . & p_{22} & p_{23} & . \\ . & p_{32} & p_{33} & . \\ . & p_{42} & p_{43} & 1 \end{vmatrix} = \begin{vmatrix} . & p_{23} \\ p_{32} & . \end{vmatrix} = (p_{23})^2.$$

Next taking the equations which define a column of p's, namely,

$$\left.\begin{aligned} \alpha_{11}\alpha'_{s1} + \alpha_{12}\alpha'_{s2} + \ldots + \alpha_{1n}\alpha'_{sn} &= p_{1s} \\ \alpha_{21}\alpha'_{s1} + \alpha_{22}\alpha'_{s2} + \ldots + \alpha_{2n}\alpha'_{sn} &= p_{2s} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \alpha_{n1}\alpha'_{s1} + \alpha_{n2}\alpha'_{s2} + \ldots + \alpha_{nn}\alpha'_{sn} &= p_{ns} \end{aligned}\right\},$$

Bardelli, apparently without knowing of Jacobi's general theorem of 1831 (*Hist.*, i. pp. 438–439), deduces

$$\begin{aligned} \alpha_{1r}p_{1s} + \alpha_{2r}p_{2s} + \ldots + \alpha_{nr}p_{ns} &= \alpha'_{sr} \\ p_{1s}^2 + p_{2s}^2 + \ldots + p_{ns}^2 &= \alpha'^2_{s1} + \alpha'^2_{s2} + \ldots + \alpha'^2_{sn}. \end{aligned}$$

Going a step further, however, he arrives at the result

$$p_{r1}p_{s1} + p_{r2}p_{s2} + \ldots + p_{rn}p_{sn} = \alpha'_{r1}\alpha'_{s1} + \alpha'_{r2}\alpha'_{s2} + \ldots + \alpha'_{rn}\alpha'_{sn},$$

and thus is able to assert that P^2 and A_i^2 are element-by-element identical.*

Proceeding to a second differentiation, he readily obtains

$$a_{r1}a''_{s1}+a_{r2}a''_{s2}+\ldots+a_{rn}a''_{sn}=t_{rs},$$

where t_{rs} stands for the excess of p'_{rs} over the $(rs)^{\text{th}}$ element of P^2, and thence deduces (§ 3) a quite analogous series of results.

The paper concludes with a statement of the form which p_{rs} takes when the orthogonant employed is Cayley's of 1846.

MANSION, P. (1877).

[On a pair of algebraical equations. *Messenger of Math.*, (2) vii. pp. 57–58.]

The equations are

$$\begin{vmatrix} a_1-x & a_2 & a_3 \\ b_1 & b_2-x & b_3 \\ c_1 & c_2 & c_3-x \end{vmatrix} = 0, \qquad \begin{vmatrix} A_1-\xi & A_2 & A_3 \\ B_1 & B_2-\xi & B_3 \\ C_1 & C_2 & C_3-\xi \end{vmatrix} = 0:$$

* Here again, however, there is a more general theorem, quite unconnected with the differentiation of the elements of an orthogonant. For we know (*Hist.*, i. p. 450, footnote) that *if, when* $|\alpha_1\beta_2\gamma_3|$ *is an orthogonant, there be given*

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}(x_1, y_1, z_1) = X_1, Y_1, Z_1,$$

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}(x_2, y_2, z_2) = X_2, Y_2, Z_2,$$

then

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (X_1, Y_1, Z_1)(X_2, Y_2, Z_2).$$

Consequently, if we were given another triad of equations, namely,

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}(x_3, y_3, z_3) = X_3, Y_3, Z_3,$$

it would follow that

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}^2 = \begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix};$$

and that, if the squaring in both cases were done by row-by-row multiplication, the resulting determinants would be identical, element-by-element.

and Mansion's observation is that they are connected by the relation

$$x\xi = |\, a_1 b_2 c_3 \,|.$$

This we have already seen suggested in connection with a result of Spottiswoode's (*Hist.*, ii. p. 308).

VOSS, A. (1877).

[Zur Theorie der orthogonalen Substitutionen. *Math. Annalen*, xiii. pp. 320–374.]

It is only the first two sections (pp. 322–333) of this long paper which strictly concern orthogonants.

A partial revise is therein given of their properties, the definition being widened by starting with the condition that the substitution

$$y_r = a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n \Big\}_{r=1}^{r=n}$$

is to be such that

$$y_1^2 + y_2^2 + \ldots + y_n^2 = s(x_1^2 + x_2^2 + \ldots + x_n^2).$$

In this way the square of each of the rows of $|\, a_{1n} \,|$ is equal to s instead of 1 : the square of each of the columns also is s : the square of the orthogonant as thus defined is s^n : and the words 'proper' and 'improper' as applied to a substitution correspond to the values $+\sqrt{s^n}$, $-\sqrt{s^n}$ of the orthogonant.

This widening, it may be remarked, is not essentially different from that proposed by Sylvester (see above, p. 288). For it can readily be shown that any element of the adjugate of Voss' $|\, a_{1n} \,|$ is equal to the corresponding element of $|\, a_{1n} \,|$ itself multiplied by $s^{\frac{1}{2}n-1}$, and that therefore the ratio of the two elements is constant,—which is Sylvester's criterion.

Lagrange's determinantal equation is reached through the somewhat loose statement that "the elements x_r, which by the substitution pass over into themselves, are determined by the equations

$$\rho x_r = a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n \Big\}_{r=1}^{r=n},$$

that is to say, by the roots of the equation

$$\begin{vmatrix} a_{11}-\rho & a_{12} & \ldots \\ a_{21} & a_{22}-\rho & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}_n = 0."$$

The reciprocity of these roots is then established in Bruno's manner (*Hist.*, ii. p. 318) by multiplying by $|\ a_{1n}\ |$, the equation in ρ,

$$\mathrm{L}(\rho) = 0, \text{ say,}$$

being thus thrown into the form

$$(-1)^n \rho^n \mathrm{L}\left(\frac{s}{\rho}\right) = 0,$$

and the word 'reciprocal' being temporarily widened so as to apply to two roots whose product is s. This enables us to note the so-called 'singular' roots, namely,

$+\sqrt{s}$, $-\sqrt{s}$ when n is even and $|\ a_{1n}\ | = -\sqrt{s^n}$;
$+\sqrt{s}$ when n is odd and $|\ a_{1n}\ | = \sqrt{s^n}$;
$-\sqrt{s}$ when n is odd and $|\ a_{1n}\ | = -\sqrt{s^n}$.

Similarly, from the fact that (*Hist.*, ii. p. 320) the multiplication by $\mathrm{L}(-\rho)$ gives

$$0 = \mathrm{L}(\rho).\mathrm{L}(-\rho) = \rho^n \begin{vmatrix} \frac{s}{\rho}-\rho & a_{21}-a_{12} & a_{31}-a_{13} & \dots \\ a_{12}-a_{21} & \frac{s}{\rho}-\rho & a_{32}-a_{23} & \dots \\ a_{13}-a_{31} & a_{23}-a_{32} & \frac{s}{\rho}-\rho & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n$$

he deduced that in the first of the three cases just specified, namely, *when* n *is even and* $|\ a_{1n}\ | = -\sqrt{s^n}$, *the determinant whose* $(rs)^{th}$ *element is* $a_{rs}-a_{sr}$ *vanishes, and so do all its primary minors.* In the third place he multiplies by $\mathrm{L}(\rho)$ itself, obtaining

$$0 = \{\mathrm{L}(\rho)\}^2 = \begin{vmatrix} a_{11}-\frac{1}{2}\left(\rho+\frac{s}{\rho}\right) & \frac{1}{2}(a_{12}+a_{21}) & \dots & \frac{1}{2}(a_{1n}+a_{n1}) \\ \frac{1}{2}(a_{21}+a_{12}) & a_{22}-\frac{1}{2}\left(\rho+\frac{s}{\rho}\right) & \dots & \frac{1}{2}(a_{2n}+a_{n2}) \\ \dots & \dots & \dots & \dots \\ \frac{1}{2}(a_{n1}+a_{1n}) & \frac{1}{2}(a_{n2}+a_{2n}) & \dots & a_{nn}-\frac{1}{2}\left(\rho+\frac{s}{\rho}\right) \end{vmatrix} 2^n(-1)^n$$

and thence concludes that *when* n *is even and* $|\ a_{1n}\ | = +\sqrt{s^n}$ *the determinant on the right is expressible as the square of a polynomial in* $\rho+s\rho^{-1}$. Also, he affirms that *for any one of the repeated roots the said determinant, together with all its primary minors, vanishes.*

The existence of equal roots in the equation $L(\rho) = 0$ and the evanescence of systems of minors of $L(\rho)$ are next considered, the k-line minors being disentangled, so to say, and drawn out in series by bordering $L(\rho)$ by k rows and k columns with a common minor of k^2 zeros; for example, if n be 5 and k be 2, the determinant formed is

$$\begin{vmatrix} a_{11}-\rho & a_{12} & a_{13} & a_{14} & a_{15} & b_{11} & b_{12} \\ a_{21} & a_{22}-\rho & a_{23} & a_{24} & a_{25} & b_{21} & b_{22} \\ a_{31} & a_{32} & a_{33}-\rho & a_{34} & a_{35} & b_{31} & b_{32} \\ a_{41} & a_{42} & a_{43} & a_{44}-\rho & a_{45} & b_{41} & b_{42} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}-\rho & b_{51} & b_{52} \\ c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & . & . \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & . & . \end{vmatrix},$$

this being expressible as an aggregate of 10^2 terms each composed of three factors,—a two-line minor of the b's, a two-line minor of the c's, and a three-line minor of the original.* Multiplication by $|a_{1n}|$ is then performed, with a result analogous to that obtained when the unbordered $L(\rho)$ was multiplied by $|a_{1n}|$, the product being of the same form as the multiplicand and s/ρ taking the place of ρ. A theorem is thence readily deduced, and several minor results follow.

IGEL, B. (1877).

[Ueber die orthogonalen und einige ihren verwandte Substitutionen. *Denksch. d. k. Acad. d. Wiss.* (Wien), xxxix. pp. 29–40.]

Instead of beginning as usual with

$$\left.\begin{aligned} x &= \lambda_1\xi+\lambda_2\eta+\lambda_3\zeta \\ y &= \mu_1\xi+\mu_2\eta+\mu_3\zeta \\ z &= \nu_1\xi+\nu_2\eta+\nu_3\zeta \end{aligned}\right\}, \quad x^2+y^2+z^2 = \xi^2+\eta^2+\zeta^2,$$

* Such an aggregate can be most appropriately viewed as a *bilinear form*; thus, denoting the 3rd compound of the original by $|A_{11}A_{22}\ldots A_{10,10}|$, the determinants of the b-array by $B_1, B_2, \ldots, B_{10}$, and the determinants of the c-array by $C_1, C_2, \ldots, C_{10}$, we have the interesting alternative expression

$$\begin{array}{cccc|c} B_{10} & B_9 & \ldots & B_1 & \\ \hline A_{11} & A_{12} & \ldots & A_{1,10} & C_{10} \\ A_{21} & A_{22} & \ldots & A_{2,10} & C_9 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{10,1} & A_{10,2} & \ldots & A_{10,10} & C_1. \end{array}$$

and then making the four familiar deductions (*Hist.*, i. pp. 437–438), Igel takes as his second datum the reverse substitution

$$\left.\begin{aligned} \xi &= \lambda_1 x+\mu_1 y+\nu_1 z \\ \eta &= \lambda_2 x+\mu_2 y+\nu_2 z \\ \zeta &= \lambda_3 x+\mu_3 y+\nu_3 z \end{aligned}\right\}$$

and deduces the rest. (But see Hesse's *Vorlesungen* of 1861, p. 192.)

Similarly, he starts with the two substitutions,

$$\left.\begin{aligned} x &= \lambda_1\xi+\lambda_2\eta+\lambda_3\zeta \\ y &= \mu_1\xi+\mu_2\eta+\mu_3\zeta \\ z &= \nu_1\xi+\nu_2\eta+\nu_3\zeta \end{aligned}\right\}, \quad \left.\begin{aligned} \mathrm{S}\xi &= \lambda_1 x+\lambda_2 y+\lambda_3 z \\ \mathrm{S}\eta &= \mu_1 x+\mu_2 y+\mu_3 z \\ \mathrm{S}\zeta &= \nu_1 x+\nu_2 y+\nu_3 z \end{aligned}\right\},$$

and deduces as far as he can analogous results, thereafter making an application to Aronhold's theory of ternary quantics. He uses throughout n variables.

GRAVELAAR, N. L. W. A. (1878).

[Eene bijzondere vergelijking. *Nieuw Archief v. Wisk.*, iv. pp. 113–124.]

The equation in question is

$$\begin{vmatrix} a_{11}-x & a_{12} & \dots & a_{1n} & b_{11} & b_{12} & \dots & b_{1m} \\ a_{12} & a_{22}-x & \dots & a_{2n} & b_{21} & b_{22} & \dots & b_{2m} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn}-x & b_{n1} & b_{n2} & \dots & b_{nm} \\ b_{11} & b_{21} & \dots & b_{n1} & . & . & \dots & . \\ b_{12} & b_{22} & \dots & b_{n2} & . & . & \dots & . \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{1m} & b_{2m} & \dots & b_{nm} & . & . & \dots & . \end{vmatrix} = 0,$$

or, say, $\mathrm{L}_{n,m}(x) = 0$,

being thus a generalization of Hesse's of 1862 and Geiser's of 1876. The object of course is to prove that all the n–m roots are real: and the procedure followed (pp. 118–124) is similar to that of 1875 in the case of $m = 0$.

SYLVESTER, J. J. (1878).

[Preuve instantanée d'après la méthode de Fourier de la réalité des racines de l'équation séculaire. *Crelle's Journ.*, lxxxviii. pp. 4–5 ; or *Collected Math. Papers*, iii. pp. 451–452.]

The proof in question is only a slight variant of Salmon's of 1859, which again, as we have seen, is in essence not really different from Cauchy's of 1829.

Following upon it is a short paper (pp. 6–9) intended to show that the property possessed by the determinant in question is shared by one that, though still axisymmetric, is much more general. Sylvester's first step in this direction had been taken twenty-five years before (*Hist.*, ii. p. 314), and Cauchy's two years later.

SOURANDER, E. (1879).

[Sur l'équation dont dépendent les inégalités séculaires des planètes. *Journ.* (*de Liouville*) *de Math.*, (3) v. pp. 195–208 ; or *Acta Soc. Sci. Fenn.* (Helsingfors), xi. pp. 257–271.]

Sourander, like Bauer, takes the product of the squared differences of the roots of the equation

$$\begin{vmatrix} a-x & h & g \\ h & b-x & f \\ g & f & c-x \end{vmatrix} = 0, \quad \text{or, say,} \quad x^3-\mathrm{P}x^2+\mathrm{Q}x-\mathrm{R} = 0,$$

in the form

$$\tfrac{1}{3} \begin{vmatrix} 2(\mathrm{Q}^2-3\mathrm{RP}) & -\mathrm{PQ}+9\mathrm{R} \\ -\mathrm{PQ}+9\mathrm{R} & 2(\mathrm{P}^2-3\mathrm{Q}) \end{vmatrix};$$

but, unlike Bauer and improving on him, he effects the transformation of it into 12 times the square of a 2-by-6 array, and therefore into 12 times the sum of the squares of fifteen determinants of the second order. The essential part of his procedure consists in introducing

$$\left.\begin{matrix} u,\ v,\ w \\ \mathrm{U},\ \mathrm{V},\ \mathrm{W} \end{matrix}\right\} \text{ to stand for } \begin{cases} \dfrac{b-c}{\sqrt{6}}, & \dfrac{c-a}{\sqrt{6}}, & \dfrac{a-b}{\sqrt{6}} \\ \dfrac{\mathrm{B}-\mathrm{C}}{\sqrt{6}}, & \dfrac{\mathrm{C}-\mathrm{A}}{\sqrt{6}}, & \dfrac{\mathrm{A}-\mathrm{B}}{\sqrt{6}} \end{cases}$$

respectively, and showing that

$$\begin{aligned} 2(Q^2-3RP) &= 6(F^2+G^2+H^2+U^2+V^2+W^2), \\ -PQ+9R &= 6(fF+gG+hH+uU+vV+wW), \\ 2(P^2-3Q) &= 6(f^2+g^2+h^2+u^2+v^2+w^2); \end{aligned}$$

for, clearly, this at once gives

$$12 \left\| \begin{matrix} f & g & h & u & v & w \\ F & G & H & U & V & W \end{matrix} \right\|^2,$$

i.e.

$$\begin{aligned} 12\{ &|fG|^2+|gH|^2+|hF|^2 \\ &+|fU|^2+|gV|^2+|hW|^2 \\ &+|fV|^2+|gW|^2+|hU|^2 \\ &+|fW|^2+|gU|^2+|hV|^2 \\ &+|uV|^2+|vW|^2+|wU|^2\} \end{aligned} \quad \text{or} \quad \begin{aligned} 12\{ &\mathring{\sum}|fG|^2 \\ &+\mathring{\sum}|fU|^2 \\ &+\mathring{\sum}|fV|^2 \\ &+\mathring{\sum}|fW|^2 \\ &+\mathring{\sum}|uV|^2\}. \end{aligned}$$

From this Borchardt's form (*Hist.*, ii. p. 302) is got by showing that

$$\mathring{\sum}|uV|^2 = 3|uV|^2 = \tfrac{1}{12}\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ A & B & C \end{vmatrix}^2,$$

—a consequence of the identities

$$u+v+w = 0 = U+V+W.$$

Another slight condensation, leading to Bauer's form, is possible, because

$$|fU| = -\frac{1}{\sqrt{6}}|gH| \quad \text{and} \quad \therefore \quad 12\mathring{\sum}|fU|^2 = 2\mathring{\sum}|fG|^2;$$

and a still further, because the derived identity

$$|fU|+|fV|+|fW| = 0$$

enables us to change the identity

$$|fV|^2+|fW|^2 = \tfrac{1}{2}\{|fV|+|fW|\}^2+\tfrac{1}{2}\{|fV|-|fW|\}^2$$

into

$$|fV|^2+|fW|^2 = \tfrac{1}{2}|fU|^2+\tfrac{1}{2}\{|fV|^2-|fW|^2\},$$

and therefore gives

$$\begin{aligned} 12\sum|fV|^2+12\sum|fW|^2 &= 6\sum|fU|^2+6\sum(|fV|-|fW|)^2 \\ &= \sum|fG|^2+\sum\begin{vmatrix} f & b+c-2a \\ F & B+C-2A \end{vmatrix}^2. \end{aligned}$$

Using all, we are led to Kummer's form,

$$15\sum|fG|^2+\sum\begin{vmatrix} f & b+c-2a \\ F & B+C-2A \end{vmatrix}^2+\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ A & B & C \end{vmatrix}^2.$$

BÖRSCH, A. (1879).

[Ueber ein den Gleichungen der orthogonalen Substitution verwandtes Gleichungssystem. *Zeitschrift f. Math. u. Phys.*, xxiv. pp. 391–399.]

The interesting theorem here established is that *if the elements of the determinant*

$$\begin{vmatrix} 1 & x_{01} & x_{02} & \dots & x_{0n} \\ 1 & x_{11} & x_{12} & \dots & x_{1n} \\ 1 & x_{21} & x_{22} & \dots & x_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & x_{n1} & x_{n2} & \dots & x_{nn} \end{vmatrix}$$

be such that the square of each row is equal to 2, *the maximum value of the determinant is*

$$\sqrt{\frac{(n+1)^{n+1}}{n^n}}.$$

This is established by showing that as a consequence of the data and of the requirements for an extreme value the product of any row by any other row must be $1-1/n$. It follows then that the square of an extreme value of the determinant is

$$\begin{vmatrix} 2 & 1-\frac{1}{n} & 1-\frac{1}{n} & \dots & 1-\frac{1}{n} \\ 1-\frac{1}{n} & 2 & 1-\frac{1}{n} & \dots & 1-\frac{1}{n} \\ 1-\frac{1}{n} & 1-\frac{1}{n} & 2 & \dots & 1-\frac{1}{n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1-\frac{1}{n} & 1-\frac{1}{n} & 1-\frac{1}{n} & \dots & 2 \end{vmatrix}_{n+1};$$

and this being known to be equal to

$$(n+1)^{n+1} \div n^n$$

the desired result is reached.

In the case where $n = 2, 3, 4$, Börsch is able to give to the elements values which satisfy the conditioning equations. These values are as seen in the determinants

$$\begin{vmatrix} 1 & . & 1 \\ 1 & -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ 1 & \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{vmatrix}, \quad \begin{vmatrix} 1 & . & . & 1 \\ 1 & -\frac{2}{3}\sqrt{2} & . & -\frac{1}{3} \\ 1 & -\frac{1}{3}\sqrt{2} & \frac{1}{3}\sqrt{6} & -\frac{1}{3} \\ 1 & -\frac{1}{3}\sqrt{2} & -\frac{1}{3}\sqrt{6} & -\frac{1}{3} \end{vmatrix}, \quad \begin{vmatrix} 1 & . & . & . & 1 \\ 1 & \frac{1}{4}\sqrt{5} & \frac{1}{4}\sqrt{5} & \frac{1}{4}\sqrt{5} & -\frac{1}{4} \\ 1 & -\frac{1}{4}\sqrt{5} & \frac{1}{4}\sqrt{5} & -\frac{1}{4}\sqrt{5} & -\frac{1}{4} \\ 1 & -\frac{1}{4}\sqrt{5} & -\frac{1}{4}\sqrt{5} & \frac{1}{4}\sqrt{5} & -\frac{1}{4} \\ 1 & \frac{1}{4}\sqrt{5} & -\frac{1}{4}\sqrt{5} & -\frac{1}{4}\sqrt{5} & -\frac{1}{4} \end{vmatrix},$$

the squares of which are

$$\frac{27}{4}, \quad \frac{256}{27}, \quad \frac{3125}{256}.$$

He might also have pointed out that in his procedure is implied the theorem that *if the elements of an* n*-line determinant be such that the square of every row is equal to* a, *and the product of any two different rows is equal to* b, *the value of the determinant is*

$$\sqrt{(a-b)^n(a+nb)}.$$

A result known otherwise (see under Roberts 1864, Sardi 1868, etc.) would thus have been brought into touch with orthogonants.

Bibliographical Note.

It is probable that the writings dealt with in the foregoing pages are practically all belonging to the period 1860–1880 that are strictly relevant to the theory of orthogonants. The inclusion of papers on the wider subject of orthogonal transformation, though justified at an earlier date when orthogonants had scarcely a separate existence, would evidently be improper now unless the papers involved some new property of the determinant in question. For the convenience, however, of students of this branch of linear substitution, the following titles are added :

1853. Hermite, Ch. Sur la théorie des formes quadratiques. *Crelle's Journ.*, xlvii. pp. 313–368.

1856. Brioschi, F. Sur les séries qui donnent le nombre de racines reelles . . . *Nouv. Annales de Math.*, xv. pp. 264–286.

1857. Cayley, A. A memoir on the theory of matrices. *Philos. Transac. R. Soc.* (London), cxlviii. pp. 17–37 ; or *Collected Math. Papers*, ii. pp. 475–496.

1858. Weierstrass, C. Ueber ein die homogenen Functionen zweiten Grades betreffendes Theorem. *Monatsb. . . . Akad. d. Wiss.* (Berlin), pp. 207–220 ; or *Werke*, i. pp. 233–246. An account of it is given by Brioschi in *Annali di Mat.*, i. pp. 250–255.

1864. Christoffel, E. B. Verallgemeinerung einiger Theoreme des Herrn Weierstrass. *Crelle's Journ.*, lxiii. pp. 255–272.

1865. Henrici, O. Transformation von Differentialausdrücken erster Ordnung zweiten Grades . . . *Crelle's Journ.*, lxv. pp. 1–25.

1868. Weierstrass, C. Zur Theorie der bilinearen und quadratischen Formen. *Monatsb. . . . Akad. d. Wiss.* (Berlin), pp. 310–338 ; or *Werke*, ii. pp. 25– .

1869. Cantor, G. De transformatione formarum ternarium quadraticarum. Dissert. Halle.

1873. Bachmann, P. Untersuchungen über quadratische Formen. *Crelle's Journ.*, lxxvi. pp. 331–341.

1874. Darboux, G. Mémoire sur la théorie algébrique des formes quadratiques. *Journ.* (*de Liouville*) *de Math.*, (2) xix. pp. 347–396.

1874. Hermite, Ch. Sur la transformation des formes quadratiques ternaires en elles-mêmes. *Crelle's Journ.*, lxxviii. pp. 323–328.

1874. Rosanes, J. Ueber die Transformation einer quadratischen Form in sich selbst. *Crelle's Journ.*, lxxx. pp. 52–72.

1874. Lemonnier, H. Mémoire sur la transformation des formes quadratiques. *Bull. Soc. Math. de France*, iii. pp. 48–76.

1876. Tannery, J. Sur les substitutions linéaires par lesquelles une forme quadratique ternaire se reproduit elle-même. *Bull. des Sci. Math.*, xi. pp. 221–233.

1877. Frobenius, G. Ueber lineare Substitutionen und bilineare Formen. *Crelle's Journ.*, lxxxiv. pp. 1–63.

1878. Stickelberger, L. Ueber Schaaren von bilinearen und quadratischen Formen. *Crelle's Journ.*, lxxxvii. pp. 20–43.

These, of course, would be taken in conjunction with the footnotes on pp. 313, 314 of *Hist.* ii. and the matter to which the footnotes refer.

CHAPTER XII.

PERSYMMETRIC DETERMINANTS, FROM 1836 TO 1879.

It is necessary to note that two of the papers dealt with here do not belong to the period of the present volume, and that, indeed, one of the two bears a date prior to the period of the second volume.

SCHELLBACH, K. H. (1836).

[Ueber die Gaussischen Formeln zur näherungsweise Berechnung eines bestimmten Integrals. *Crelle's Journ.*, xvi. pp. 192–195.]

Schellbach reaches the set of $2n$ equations

$$\left.\begin{array}{rrrrr}
\mathrm{A}+ & \mathrm{B}+ & \mathrm{C}+\ldots+ & \mathrm{N} & = 1 \\
\alpha\mathrm{A}+ & \beta\mathrm{B}+ & \gamma\mathrm{C}+\ldots+ & \nu\mathrm{N} & = 0 \\
\alpha^2\mathrm{A}+ & \beta^2\mathrm{B}+ & \gamma^2\mathrm{C}+\ldots+ & \nu^2\mathrm{N} & = \tfrac{1}{3} \\
\alpha^3\mathrm{A}+ & \beta^3\mathrm{B}+ & \gamma^3\mathrm{C}+\ldots+ & \nu^3\mathrm{N} & = 0 \\
\alpha^4\mathrm{A}+ & \beta^4\mathrm{B}+ & \gamma^4\mathrm{C}+\ldots+ & \nu^4\mathrm{N} & = \tfrac{1}{5} \\
\ldots & \ldots & \ldots & \ldots & \\
\alpha^{2n-1}\mathrm{A}+ & \beta^{2n-1}\mathrm{B}+ & \gamma^{2n-1}\mathrm{C}+\ldots+ & \nu^{2n-1}\mathrm{N} & = 0
\end{array}\right\}$$

in seeking to make

$$\mathrm{A}\phi(a+\alpha b) + \mathrm{B}\phi(a+\beta b) + \mathrm{C}\phi(a+\gamma b) + \ldots + \mathrm{N}\phi(a+\nu b)$$

a satisfactory approximation to

$$\int_{a-b}^{a+b} \phi(x)\,dx.$$

The determination of the $n+n$ quantities

$$\mathrm{A},\ \mathrm{B},\ \mathrm{C},\ \ldots,\ \mathrm{N},$$
$$\alpha,\ \beta,\ \gamma,\ \ldots,\ \nu,$$

so as to satisfy the set of equations, ensures that the integral when expressed by Maclaurin's theorem as a series in ascending powers of b will not differ from the approximation until the term containing b^{2n+1} is reached. The essential part of Schellbach's solution consists in showing* that $\alpha, \beta, \gamma, \ldots, \nu$ are the roots of the equation

$$\frac{\partial^2}{\partial x^2}(x^2-1)^n = 0,$$

i.e. $$x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2(2n-1)(2n-3)2^2}x^{n-4} - \ldots = 0,$$

or, as we may write it,

$$x^n - \frac{C_{n,1}C_{n-1,1}}{2^2 C_{n-\frac{1}{2},1}}x^{n-2} + \frac{C_{n,2}C_{n-2,2}}{2^4 C_{n-\frac{1}{2},2}}x^{n-4} - \frac{C_{n,3}C_{n-3,3}}{2^6 C_{n-\frac{1}{2},3}}x^{n-6} + \ldots = 0.$$

JOACHIMSTHAL, F. (1854).

[Bemerkungen über den *Sturm*'schen Satz. *Crelle's Journ.*, xlviii. pp. 386–416.]

Joachimsthal brings forward the set of equations connected with Gauss' problem merely as an illustration (§ 16, pp. 411–413) of his general subject. The set, however, differs from Schellbach's in having for the right-hand members

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots.$$

instead of $$1, 0, \tfrac{1}{3}, 0, \tfrac{1}{5}, \ldots.$$

His method of solution is the same as Sylvester's. (*Hist.*, ii. pp. 333–335.)

The recurrence-formula which had been given in 1835 by Jacobi for the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_n & 1 \\ a_2 & a_3 & a_4 & \ldots & a_{n+1} & x \\ a_3 & a_4 & a_5 & \ldots & a_{n+2} & x^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n+1} & a_{n+2} & a_{n+3} & \ldots & a_{2n} & x^n \end{vmatrix}$$

he is led to establish anew (§ 9, pp. 397–398), the method, however, being still not purely determinantal.

* But see Jacobi, C. G. J. Ueber Gauss' neue Methode, die Werthe der Integrale näherungsweise zu finden. *Crelle's Journ.*, i. (1826), pp. 301–308.

LIGOWSKI, W. (1861).

[Nachtrag zu der Abhandlung "Ueber die Inhaltsberechnung der Körper." *Archiv d. Math. u. Phys.*, xxxvi. pp. 181–185.]

Ligowski, while on the same track as Schellbach (1836), reaches the same set of equations as Joachimsthal, and solves them in Sylvester's way. What is new and interesting is his evaluation of the special determinants to which Sylvester's method leads. The first of these is the determinant got by leaving out the last column of the array

$$\begin{array}{cccccc}
1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} & \frac{1}{n+2} \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+2} & \frac{1}{n+3} \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n-1} & \frac{1}{2n}
\end{array}.$$

This he has the insight to recognize as a case of Cauchy's double alternant

$$\begin{vmatrix}
\frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_1-\beta_2} & \cdots & \frac{1}{\alpha_1-\beta_n} \\
\frac{1}{\alpha_2-\beta_1} & \frac{1}{\alpha_2-\beta_2} & \cdots & \frac{1}{\alpha_2-\beta_n} \\
\cdot & \cdot & \cdots & \cdot \\
\frac{1}{\alpha_n-\beta_1} & \frac{1}{\alpha_n-\beta_2} & \cdots & \frac{1}{\alpha_n-\beta_n}
\end{vmatrix},$$

namely, the case where the α's are

$$n+1,\quad n+2,\quad n+3,\ \ldots\ldots,\ 2n,$$

and the β's

$$n,\quad n-1,\quad n-2,\ \ldots\ldots,\ 1,$$

and where therefore the general value

$$(-1)^{\frac{1}{2}n(n-1)}\zeta^{\frac{1}{2}}(\alpha_1, \ldots, \alpha_n)\cdot\zeta^{\frac{1}{2}}(\beta_1, \ldots, \beta_n) \div \prod_{r=1}^{r=n}(\alpha_r-\beta_1)(\alpha_r-\beta_2)\ldots(\alpha_r-\beta_n)$$

becomes

$$(-1)^{\frac{1}{2}n(n-1)}\frac{[1!\,2!\,3!\,\ldots\,(n-1)!]\,[(-1)^{\frac{1}{2}n(n-1)}1!\,2!\,3!\,\ldots\,(n-1)!]}{\frac{n!}{1}\cdot\frac{(n+1)!}{1!}\cdot\frac{(n+2)!}{2!}\cdot\ldots\cdot\frac{(2n-1)!}{(n-1)!}},$$

i.e.

$$\frac{[1!\,2!\,3!\,\ldots\,(n-1)!]^3}{n!\,(n+1)!\,(n+2)!\,\ldots\,(2n-1)!}.$$

The other determinant* is that got from the same array by leaving out any other column, say the $(k+1)^{\text{th}}$, and is made dependent on the double alternant also, the α's this time being

$$n+2,\quad n+3,\quad n+4,\quad \ldots,\quad 2n+1,$$

the β's

$$n+1,\quad n,\quad n-1,\quad \ldots,\quad n-k+2,\quad n-k,\quad n-k-1,\quad \ldots,\quad 1,$$

and the value of the determinant

$$\frac{[1!\,2!\,3!\,\ldots\,(n-1)!]^3\,n!\,(n+k)!}{(k!)^2\,(n-k)!\,(n+1)!\,(n+2)!\,\ldots\,(2n)!}.$$

The quotient of the latter determinant by the former, namely,

$$\frac{(n!)^2\,(n+k)!}{(k!)^2\,(n-k)!\,(2n)!}$$

is, save for a sign-factor, the value of one of the unknowns corresponding to those which, in Sylvester's paper, were denoted by $\Sigma\lambda_1$, $\Sigma\lambda_1\lambda_2$, ... The value had already been obtained (1832) by Murphy without the use of determinants.

HANKEL, H. (1861).

[Ueber eine besondere Classe der symmetrischen Determinanten. Dissert. 29 pp. Göttingen.]

The special determinant with which Hankel is concerned is the persymmetric, but he does not speak of it as having been previously considered, and names it *orthosymmetric*. It and the special form

$$\begin{vmatrix} 1 & a_1 & a_1a_2 & \ldots & a_1a_2\ldots a_{n-1} \\ 1 & a_2 & a_2a^3 & \ldots & a_2a_3\ldots a_n \\ 1 & a_3 & a_3a_4 & \ldots & a_3a_4\ldots a_{n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a_n & a_na_{n+1} & \ldots & a_na_{n+1}\ldots a_{2n-2} \end{vmatrix}, \qquad (P')$$

* Apparently not seen to include the former.

he points out, are fundamentally related. By dividing each row of a persymmetric determinant by the first element of the row a determinant of the form (P′) is evolved, and conversely, by multiplying each row of (P′) by the first element of the corresponding column, persymmetry makes its appearance, the connecting equation being

$$a_1^{n-1}a_2^{n-2}\dots a_{n-1}^1 \cdot \mathrm{P}'(a_1, a_2, \dots, a_{2n-2}) = \mathrm{P}(1, a_1, a_1a_2, a_1a_2a_3, \dots, a_1a_2\dots a_{2n-2}).$$

By performing on the persymmetric determinant of

$$a_0,\ a_1,\ a_2,\ \dots,\ a_{2n-2}$$

the operations

$$\mathrm{col}_n - \mathrm{col}_{n-1},\quad \mathrm{col}_{n-1} - \mathrm{col}_{n-2},\quad \dots,\quad \mathrm{col}_2 - \mathrm{col}_1,$$

then on the resulting determinant the operations

$$\mathrm{col}_n - \mathrm{col}_{n-1},\quad \mathrm{col}_{n-1} - \mathrm{col}_{n-2},\quad \dots,\quad \mathrm{col}_3 - \mathrm{col}_2,$$

and so on, there being one operation fewer each time, there is obtained the determinant

$$\begin{vmatrix} a_0 & \Delta_0^{(1)} & \Delta_0^{(2)} & \dots & \Delta_0^{(n-1)} \\ a_1 & \Delta_1^{(1)} & \Delta_1^{(2)} & \dots & \Delta_1^{(n-1)} \\ a_2 & \Delta_2^{(1)} & \Delta_2^{(2)} & \dots & \Delta_2^{(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & \Delta_{n-1}^{(1)} & \Delta_{n-1}^{(2)} & \dots & \Delta_{n-1}^{(n-1)} \end{vmatrix}.$$

Next this is treated in an exactly similar way, save that it is now the rows that are operated on, the outcome being

$$\begin{vmatrix} \Delta^{(0)} & \Delta^{(1)} & \Delta^{(2)} & \dots & \Delta^{(n-1)} \\ \Delta^{(1)} & \Delta^{(2)} & \Delta^{(3)} & \dots & \Delta^{(n)} \\ \Delta^{(2)} & \Delta^{(3)} & \Delta^{(4)} & \dots & \Delta^{(n+1)} \\ \dots & \dots & \dots & \dots & \dots \\ \Delta^{(n-1)} & \Delta^{(n)} & \Delta^{(n+1)} & \dots & \Delta^{(2n-2)} \end{vmatrix}.$$

Hankel has thus his fundamental theorem, namely, that *the persymmetric determinant of* $a_0, a_1, a_2, \dots, a_{2n-2}$ *is equal to the persymmetric determinant of the first members of the* $2n-2$ *difference-series of the* a*'s*. As illustrations he uses the well-known case

where $a_k = s_k = a_1^k + a_2^k + \ldots + a_n^k$ and the case where $a_k = a_k +$ a constant.

The next section (§ 3, pp. 7–11) is occupied with the discussion of the case where

$$a_k = (x+k-a_1)(x+k-a_2) \ldots (x+k-a_\mu),$$

and where therefore the constituents of the μ^{th} difference-series are each equal to $\mu!$ The value of the determinant is thus seen to be

$$(-1)^{\frac{1}{2}n(n-1)}\{(n-1)!\}^n \quad \text{or} \quad 0,$$

according as $\mu =$ or $< n-1$, that is to say, is in both cases independent of the a's. By giving the a's the values

$$0, \quad -1, \quad -2, \ldots, -\mu+1$$

in the first of these two results, and performing on the determinant the operations

$$\text{col}_1 - \text{col}_2, \quad \text{col}_2 - \text{col}_3, \ldots,$$

it is possible to remove the factor $(n-1)^{n-1}$ from both sides, then by similar means the factor $(n-2)^{n-2}$, and so on, the final outcome being

$$\text{P}'(x+n-1, \; x+n, \ldots, x+3n-4) = (n-1)^1(n-2)^2 \ldots (2)^{n-2}1^{n-1},$$

a result easily proved otherwise.

The fourth section is devoted to the case where

$$a_k = \frac{(1-q^\alpha)(1-q^{\alpha+1}) \ldots (1-q^{\alpha+k-1})}{(1-q^\gamma)(1-q^{\gamma+1}) \ldots (1-q^{\gamma+k-1})}.$$

Here, on the removal of the requisite factors, the P′ form is readily seen to be

$$\text{P}'\left(\frac{1-q^\alpha}{1-q^\gamma}, \; \frac{1-q^{\alpha+1}}{1-q^{\gamma+1}}, \ldots, \frac{1-q^{\alpha+2n-3}}{1-q^{\gamma+2n-3}}\right), \quad \text{or, say,} \quad \text{F}_n(\alpha, \gamma),$$

and on utilizing the column of 1's to reduce the order it is found that $\text{F}_n(\alpha, \gamma)$ and $\text{F}_{n-1}(\alpha+1, \gamma+2)$ are connected by a factor. Repeated use of this result is then all that is wanted for the purpose in view; but the connecting factor being

$$(-1)^{n-1} \cdot q^{\frac{1}{2}(n-1)(n-2)} \cdot (q^\gamma - q^\alpha)^{n-1} \cdot \frac{\text{A}}{\text{B}},$$

where $A = (1-q)(1-q^2) \ldots (1-q^{n-1}),$

and $B = (1-q^\gamma)(1-q^{\gamma+1})^2(1-q^{\gamma+2})^2 \ldots$
$\times (1-q^{\gamma+n-3})^2(1-q^{\gamma+n-2})^2(1-q^{\gamma+n-1}),$

the expression obtained for the determinant is neither short nor simple. Among the interesting specializations made there may be noted

(1) $q = 1$, and $\therefore \dfrac{1-q^x}{1-q^y} = \dfrac{x}{y}$;

(2) $q \gtrless 1$, $\gamma = \mp\infty$, and $\therefore q^\gamma = 0$;

(3) $q \gtrless 1$, $\alpha = \mp\infty$, and $\therefore q^\alpha = 0.$

Section fifth concerns sets of linear equations of the type

$$a_r x_1 + a_{r+1}x_2 + \ldots + a_{r+n-1}x_n + a_{r+n} = 0 \Big\}_{r=1}^{r=n},$$

whose determinants are the persymmetric forms dealt with in the preceding sections. Section sixth is a continuation, the special set of equations considered being

$$x_1 + \frac{a_r}{\gamma_r}x_2 + \frac{a_r a_{r+1}}{\gamma_r\gamma_{r+1}}x_3 + \ldots + \frac{a_r \ldots a_{r+n-2}}{\gamma_r \ldots \gamma_{r+n-2}}x_n + \frac{a_r \ldots a_{r+n-1}}{\gamma_r \ldots \gamma_{r+n-1}} = 0 \Big\}_{r=1}^{r=n}.$$

The last section (pp. 25–29) is occupied with the transformation of

$$1 + a_1 x + a_2 x^2 + \ldots$$

into a continued fraction of the form

$$\frac{1}{1} - \frac{\rho_1 x}{1} - \frac{\rho_2 x}{1} - \ldots.$$

The resolution obtained is accurate so far as it goes: it leaves, however, a misleading impression in that the expression reached for ρ_{2n} is different from and more complicated than that for ρ_{2n+1}. The details need not be given, as, on putting

$$b_0 = 1, \quad b_2 = b_3 = \ldots = 0$$

in Heilermann's result of the year 1845, we obtain

$$a_0 + a_1 x + a_2 x^2 + \ldots$$

$$= \frac{\Delta_0}{1} - \frac{\Delta_1 x}{\Delta_0} - \frac{\Delta_2 x}{\Delta_1} - \frac{\Delta_0\Delta_3 x}{\Delta_2} - \frac{\Delta_1\Delta_4 x}{\Delta_3} - \frac{\Delta_2\Delta_5 x}{\Delta_4} - \ldots$$

$$\text{or} = \frac{\Delta_0}{1} - \frac{x\Delta_1/\Delta_0}{1} - \frac{x\Delta_2/\Delta_0\Delta_1}{1} - \frac{x\Delta_0\Delta_3/\Delta_1\Delta_2}{1} - \frac{x\Delta_1\Delta_4/\Delta_2\Delta_3}{1} - \ldots,$$

where $\Delta_0 = a_0$, $\Delta_1 = a_1$,

$$\Delta_2 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}, \quad \Delta_5 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \quad \ldots,$$

which is what Hankel would have been led to had he gone a little farther.

SYLVESTER, J. J. (1862).

[Sur une classe nouvelle d'équations différentielles et d'équations aux différences finies d'une forme intégrable. *Comptes Rendus . . . . Acad. des Sci.* (Paris), liv. pp. 129–132 ; or *Collected Math. Papers*, ii. pp. 308–312.]

The existence of the equation

$$u_x - p_1 u_{x+1} + p_2 u_{x+2} - p_3 u_{x+3} + u_{x+4} = 0,$$

where p_1, p_2, p_3 are constants, implies that

$$\begin{vmatrix} u_x & u_{x+1} & u_{x+2} & u_{x+3} \\ u_{x+1} & u_{x+2} & u_{x+3} & u_{x+4} \\ u_{x+2} & u_{x+3} & u_{x+4} & u_{x+5} \\ u_{x+3} & u_{x+4} & u_{x+5} & u_{x+6} \end{vmatrix}$$

is independent of x ; for if we perform the operation

$$\text{row}_1 - p_1 \text{row}_2 + p_2 \text{row}_3 - p_3 \text{row}_4$$

and make use of the said equation, the determinant is simply changed into

$$- \begin{vmatrix} u_{x+4} & u_{x+5} & u_{x+6} & u_{x+7} \\ u_{x+1} & u_{x+2} & u_{x+3} & u_{x+4} \\ u_{x+2} & u_{x+3} & u_{x+4} & u_{x+5} \\ u_{x+3} & u_{x+4} & u_{x+5} & u_{x+6} \end{vmatrix},$$

—in other words, we learn that the given determinant is not altered by substituting $x+1$ for x. From this Sylvester concludes that the difference-equation

$$\begin{vmatrix} u_x & u_{x+1} & u_{x+2} & \cdots & u_{x+n-1} \\ u_{x+1} & u_{x+2} & u_{x+3} & \cdots & u_{x+n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{x+n-1} & u_{x+n} & u_{x+n+1} & \cdots & u_{x+2n-2} \end{vmatrix} = \text{constant}$$

is satisfied by the same integrals as the equation

$$u_x - p_1 u_{x+1} + p_2 u_{x+2} - \cdots + (-1)^{n-1} p_{n-1} u_{x+n-1} + (-1)^n u_{x+n} = 0.$$

An analogous result is obtained in regard to the differential equation

$$\begin{vmatrix} y & \frac{dy}{dx} & \frac{d^2y}{dx^2} & \cdots & \frac{d^{n-1}y}{dx^{n-1}} \\ \frac{dy}{dx} & \frac{d^2y}{dx^2} & \frac{d^3y}{dx^3} & \cdots & \frac{d^ny}{dx^n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{d^{n-1}y}{dx^{n-1}} & \frac{d^ny}{dx^n} & \frac{d^{n-1}y}{dx^{n+1}} & \cdots & \frac{d^{2n-2}y}{dx^{2n-2}} \end{vmatrix} = \text{constant},$$

where the determinant is seen to be a special form of Wronskian.

HANKEL, H. (1862).

[Ueber die Transformation von Reihen in Kettenbrüche. *Zeitschrift f. Math. u. Phys.*, vii. pp. 338–343.]

This is merely a fresh presentation of results already known: for example, from Brioschi (1854) and Sylvester (1850). (*Hist.*, ii. pp. 343–346, 333–335.)

SYLVESTER, J. J. (1863).

[Sequel to the theorems relating to 'canonic roots,' . . . . *Philos. Magazine*, (4) xxv. pp. 453–460; or *Collected Math. Papers*, ii. pp. 331–337.]

A simple problem here solved is the elimination of x from two such equations as

$$\begin{vmatrix} 1 & x & x^2 & x^3 \\ a & b & c & d \\ b & c & d & e \\ c & d & e & f \end{vmatrix} = 0, \qquad \begin{vmatrix} 1 & x & x^2 \\ a & b & c \\ b & c & d \end{vmatrix} = 0,$$

the eliminant found being a power of

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

Possible extensions of the problem are indicated.

WHITWORTH, W. A. (1865): ZEIPEL, V. v. (1865).

[Of recurring series. *Oxford, Cambr. and Dubl. Messenger of Math.*, iii. pp. 117–121; or *Math. from Educ. Times*, iii. p. 100.]

[Determinanter, hvars elementer äro binomialkoefficienter. *Lunds Univ. Årsskrift*, ii. pp. 1–68.]

Whitworth's paper is summed up in one proposition, namely, *If* $c_0+c_1x+c_2x^2+\ldots$ *be any convergent recurring series of the* r^{th} *order of recurrence whose first* 2r *terms are given, its scale of relation will be*

$$\begin{vmatrix} x^r & x^{r-1} & x^{r-2} & \ldots & 1 \\ c_0 & c_1 & c_2 & \ldots & c_r \\ c_1 & c_2 & c_3 & \ldots & c_{r+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{r-1} & c_r & c_{r+1} & \ldots & c_{2r-1} \end{vmatrix} \div \begin{vmatrix} c_0 & c_1 & c_2 & \ldots & c_{r-1} \\ c_1 & c_2 & c_3 & \ldots & c_r \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{r-1} & c_r & c_{r+1} & \ldots & c_{2r-2} \end{vmatrix},$$

and its sum to infinity will be

$$\begin{vmatrix} 0 & s_1x^{r-1} & s_2x^{r-2} & \ldots & s_r \\ c_0 & c_1 & c_2 & \ldots & c_r \\ c_1 & c_2 & c_3 & \ldots & c_{r+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{r-1} & c_r & c_{r+1} & \ldots & c_{2r-1} \end{vmatrix} \div \begin{vmatrix} x^r & x^{r-1} & x^{r-2} & \ldots & 1 \\ c_0 & c_1 & c_2 & \ldots & c_r \\ c_1 & c_2 & c_3 & \ldots & c_{r+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{r-1} & c_r & c_{r+1} & \ldots & c_{2r-1} \end{vmatrix},$$

where s_m *denotes the sum of the first* m *of the given terms.*

The 'scale of relation' being of the form

$$1+\beta_1x+\beta_2x^2+\ldots+\beta_rx^r,$$

there are r equations

$$\left.\begin{aligned} c_0\beta_r+c_1\beta_{r-1}+\ldots+c_r &= 0 \\ c_1\beta_r+c_2\beta_{r-1}+\ldots+c_{r+1} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ c_{r-1}\beta_r+c_r\beta_{r-1}+\ldots+c_{2r-1} &= 0 \end{aligned}\right\},$$

which lead to the determination of the β's and the finding of the actual scale as enunciated. Now, by multiplying the series by the scale of relation, it naturally is found that the product, unlike the multiplicand, is limited in the number of its terms, and we have then little more to do than to take the quotient of this product by the scale in order to reach the desired expression for the sum.

If the idea of a 'formula of recurrence' be used instead of a 'scale of relation,' the first of these results takes the form: *If* $c_0, c_1, c_2, \ldots$ *be a recurring series of the* r^{th} *order, the recurrence-formula is*

$$\begin{vmatrix} c_0 & c_1 & c_2 & \ldots & c_r \\ c_1 & c_2 & c_3 & \ldots & c_{r+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_{r-1} & c_r & c_{r+1} & \ldots & c_{2r-1} \\ c_{n-r} & c_{n-r+1} & c_{n-r+2} & \ldots & c_n \end{vmatrix} = 0.$$

Zeipel's memoir we deal with under its own heading (Chap. XX.); its single persymmetric determinant

$$\begin{vmatrix} (m)_{p+r} & (m)_{p+r-1} & (m)_{p+r-2} & \ldots & (m)_p \\ (m)_{p+r-1} & (m)_{p+r-2} & (m)_{p+r-3} & \ldots & (m)_{p-1} \\ (m)_{p+r-2} & (m)_{p+r-3} & (m)_{p+r-4} & \ldots & (m)_{p-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (m)_p & (m)_{p-1} & (m)_{p-2} & \ldots & (m)_{p-r} \end{vmatrix}$$

cannot well be considered apart from other forms.

FONTEBASSO, D. (1872).

[I PRIMI ELEMENTI DELLA TEORIA DEI DETERMINANTI, ... viii+134 pp Treviso.]

Fontebasso proves that the n-line determinant whose elements in the secondary diagonal of the complementary minor of the $(n, n)^{th}$ element are zeros and all the others units is ± 1. He does not observe that the set of operations

$$\text{row}_n - \text{row}_{n-1}, \quad \text{row}_{n-1} - \text{row}_{n-2}, \quad \ldots .$$

gives at once the result

$$(-1)^{\frac{1}{2}n(n-1)}.$$

McKENZIE, J. L. (1874).

[Question 4526. *Educ. Times*, xxvii. p. 166; solution by R. F. Scott in *Math. from Educ. Times*, xxv. pp. 106–107.]

The theorem brought forward by McKenzie is Joachimsthal's of 1854 (*Hist.*, ii. pp. 169–170). In Scott's proof the determinant concerned appears as the result of eliminating the coefficients of the related equation.

GÜNTHER, S. (1875): MUIR, T. (1876).

[Lehrbuch der Determinanten-Theorie, . . . viii+236 pp. Erlangen.]

[New general formulae for the transformation of infinite series into continued fractions. *Transac. R. Soc. Edinburgh*, xxvii. pp. 467–471.]

Günther's contribution (p. 88) is that when the series of distinct elements in a persymmetric determinant are in equirational progression the determinant vanishes; and Muir's result, involving persymmetric determinants, is a rediscovery.

McKENZIE, J. L. (1878).

[Question 5622. *Educ. Times*, xxxi. p. 114; solution by R. F. Scott and others in *Math. from Educ. Times*, xxxviii. pp. 86–87.]

The 'question' here concerns the primary minors of

$$\begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_n \\ s_1 & s_2 & s_3 & \cdots & s_{n+1} \\ s_2 & s_3 & s_4 & \cdots & s_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_n & s_{n+1} & s_{n+2} & \cdots & s_{2n} \end{vmatrix},$$

in which s_r stands for the sum of the r^{th} powers of the roots of an equation of the n^{th} degree, and the value of which is identically 0 (*Hist.*, ii. p. 353).

Taking the case where n is 4, we clearly have

$$\begin{vmatrix} 1 & . & . & . & . & . \\ . & 1 & 1 & 1 & 1 & 1 \\ . & \alpha & \beta & \gamma & \delta & x \\ . & \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & x^2 \\ . & \alpha^3 & \beta^3 & \gamma^3 & \delta^3 & x^3 \\ . & \alpha^4 & \beta^4 & \gamma^4 & \delta^4 & x^4 \end{vmatrix} = |\alpha^0\beta^1\gamma^2\delta^3x^4| = |\alpha^0\beta^1\gamma^2\delta^3| \cdot (x^4 - \Sigma a \cdot x^3 + \Sigma a\beta \cdot x^2 - \Sigma a\beta\gamma \cdot x + a\beta\gamma\delta);$$

and putting y for x in this and exchanging the first and last columns, we obtain

$$-\begin{vmatrix} . & . & . & . & . & 1 \\ 1 & 1 & 1 & 1 & 1 & . \\ y & \alpha & \beta & \gamma & \delta & . \\ y^2 & \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & . \\ y^3 & \alpha^3 & \beta^3 & \gamma^3 & \delta^3 & . \\ y^4 & \alpha^4 & \beta^4 & \gamma^4 & \delta^4 & . \end{vmatrix} = |\alpha^0\beta^1\gamma^2\delta^3| \cdot (y^4 - \Sigma a \cdot y^3 + \Sigma a\beta \cdot y^2 - \Sigma a\beta\gamma \cdot y + a\beta\gamma\delta).$$

From these by multiplication there results

$$\begin{vmatrix} . & 1 & y & y^2 & y^3 & y^4 \\ 1 & s_0 & s_1 & s_2 & s_3 & s_4 \\ x & s_1 & s_2 & s_3 & s_4 & s_5 \\ x^2 & s_2 & s_3 & s_4 & s_5 & s_6 \\ x^3 & s_3 & s_4 & s_5 & s_6 & s_7 \\ x^4 & s_4 & s_5 & s_6 & s_7 & s_8 \end{vmatrix}$$

$$= |\alpha^0\beta^1\gamma^2\delta^3|^2 \cdot (x^4 - \Sigma a \cdot x^3 + \Sigma a\beta \cdot x^2 - \ . \ . \ .) \ (y^4 - \Sigma a \cdot y^3 + \Sigma a\beta \cdot y^2 - \ . \ . \ .).$$

But if we denote the adjugate of the cofactor of the zero element in the six-line determinant here by $|\,S_{00}S_{11}S_{22}S_{33}S_{44}\,|$, the left-hand member is equal to

$$\begin{array}{ccccc|c} 1 & y & y^2 & y^3 & y^4 & \\ \hline S_{00} & S_{01} & S_{02} & S_{03} & S_{04} & 1 \\ S_{10} & S_{11} & S_{12} & S_{13} & S_{14} & x \\ S_{20} & S_{21} & S_{22} & S_{23} & S_{24} & x^2 \\ S_{30} & S_{31} & S_{32} & S_{33} & S_{34} & x^3 \\ S_{40} & S_{41} & S_{42} & S_{43} & S_{44} & x^4; \end{array}$$

and if we denote

$$1,\ \Sigma a,\ \Sigma a\beta,\ \Sigma a\beta\gamma,\ \Sigma a\beta\gamma\delta,\quad \text{by}\quad c_0,\ c_1,\ c_2,\ c_3,\ c_4,$$

the right-hand member is equal to

$$|\,a^0\beta^1\gamma^2\delta^3\,|^2\cdot\begin{array}{|ccccc|c}
y^4 & y^3 & y^2 & y^1 & 1 & \\
\hline
c_0^2 & -c_0c_1 & c_0c_2 & -c_0c_3 & c_0c_4 & x^4 \\
-c_1c_0 & c_1^2 & -c_1c_2 & c_1c_3 & -c_1c_4 & x^3 \\
c_2c_0 & -c_2c_1 & c_2^2 & -c_2c_3 & c_2c_4 & x^2 \\
-c_3c_0 & c_3c_1 & -c_3c_2 & c_3^2 & -c_3c_4 & x \\
c_4c_0 & -c_4c_1 & c_4c_2 & -c_4c_3 & c_4^2 & 1.
\end{array}$$

Consequently we have, by equating cofactors of $x^r y^s$,

$$S_{rs} = |\,a^0\beta^1\gamma^2\delta^3\,|^2\cdot(-1)^{r+s}c_{n-r}c_{n-s}.$$

This may be viewed as a generalization of a previous result, for on putting $s = n$ it becomes

$$S_{rn} = |\,a^0\beta^1\gamma^2\delta^3\,|^2\cdot(-1)^{r+n}c_{n-r},$$

which is not essentially different from Bellavitis' statement regarding the c's (*Hist.*, ii. p. 182).

WOLSTENHOLME, J. (1878).

[MATHEMATICAL PROBLEMS, ... 2nd edition. x+480 pp. London.]

The result intended to be stated (p. 276) is that

$$P(1,\ \cos\theta,\ \cos 2\theta,\ .\ .\ .\ .\ .,\ \cos\overline{2n-2}\cdot\theta) = 0$$

for all values of n greater than 2. Scott (1879) affirms the same for the similar determinant whose elements are

$$\cos a,\quad \cos(a+\theta),\ .\ .\ .\ .\ .,\ \cos(a+\overline{2n-2}\cdot\theta),$$

and adds that when n is 2 the value is $-\sin^2\theta$.

SYLVESTER, J. J. (1878).

[On the theorem connected with Newton's rule for the discovery of imaginary roots of equations. *Messenger of Math.*, ix. pp. 71–84; or *Collected Math. Papers*, iii. pp. 414–425.]

In a postscript to a postscript Sylvester enunciates the proposi-

tion that *if* $f_i = (a_0, a_1, a_2, \ldots, a_i)(x, y)^i$, *then the persymmetric determinant*

$$\begin{vmatrix} f_{i-e} & f_{i-e+1} & \ldots & f_{i+e} \\ f_{i-e+1} & f_{i-e+2} & \ldots & f_{i+e+1} \\ \ldots & \ldots & \ldots & \ldots \\ f_{i+e} & f_{i+e+1} & \ldots & f_{i+3e} \end{vmatrix} = y^{e(e+1)} \cdot H_e(f_{i+e}),$$

where the cofactor of $y^{e(e+1)}$ *is that covariant of* f_{i+e} *whose highest power of* x *has for its coefficient the persymmetric determinant*

$$\begin{vmatrix} a_0 & a_1 & a_2 & \ldots & a_e \\ a_1 & a_2 & a_3 & \ldots & a_{e+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_e & a_{e+1} & a_{e+2} & \ldots & a_{2e} \end{vmatrix}.$$

There can be little doubt that this is incorrect, even when made consistent by altering f_{i+2e} into f_{i+3e}, and that the determinant of the f's was meant to be not of the $(2e+1)^{th}$ order but of the $(e+1)^{th}$ order, so that the last row and last column should be changed into

$$f_i, \quad f_{i+1}, \quad f_{i+2}, \ldots, f_{i+e}.$$

The process of proof followed shows signs of equally hasty construction; and is quite unnecessarily complicated, since all that is wanted is a simple diminution of the columns in backward order, and thereafter of the rows in like manner. Thus, taking the case where $i = 3$, $e = 1$, we have

$$\begin{vmatrix} a_0x+a_1y & a_0x^2+2a_1xy+a_2y^2 & a_0x^3+\ldots+a_3y^3 \\ a_0x^2+2a_1xy+a_2y^2 & a_0x^3+\ldots+a_3y^3 & a_0x^4+\ldots+a_4y^4 \\ a_0x^3+\ldots+a_3y^3 & a_0x^4+\ldots+a_4y^4 & a_0x^5+\ldots+a_5y^5 \end{vmatrix}$$

$$= \begin{vmatrix} a_0x+a_1y & a_1x+a_2y & a_2x+a_3y \\ a_0x^2+\ldots+a_2y^2 & a_1x^2+\ldots+a_3y^2 & a_2x^2+\ldots+a_4y^2 \\ a_0x^3+\ldots+a_3y^3 & a_1x^3+\ldots+a_4y^3 & a_3x^3+\ldots+a_5y^3 \end{vmatrix} y^{2+1}$$

$$= \begin{vmatrix} a_0x+a_1y & a_1x+a_2y & a_2x+a_3y \\ a_1x+a_2y & a_2x+a_3y & a_3x+a_4y \\ a_2x+a_3y & a_3x+a_4y & a_4x+a_5y \end{vmatrix} y^{2(2+1)}$$

$$= \left\{ \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} x^3 + \begin{vmatrix} a_0 & a_1 & a_3 \\ a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \end{vmatrix} x^2y + \begin{vmatrix} a_0 & a_2 & a_3 \\ a_1 & a_3 & a_4 \\ a_2 & a_4 & a_5 \end{vmatrix} xy^2 + \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} y^3 \right\} y^{2(2+1)}.$$

Indeed, if we put

$$(a_r, a_{r+1}, \ldots . \}(x, y)^s \equiv (r, s)$$

and take note of the relation repeatedly used in the transformation just made, namely,

$$(r, s) - x(r, s-1) = y(r+1, s-1),$$

we obtain the identity

$$\begin{vmatrix} (0, i-e) & (0, i-e+1) & (0, i-e+2) & \ldots . & (0, i) \\ (0, i-e+1) & (0, i-e+2) & (0, i-e+3) & \ldots . & (0, i+1) \\ (0, i-e+2) & (0, i-e+3) & (0, i-e+4) & \ldots . & (0, i+2) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (0, i) & (0, i+1) & (0, i+2) & \ldots . & (0, i+e) \end{vmatrix}$$

$$= \begin{vmatrix} (0, i-e) & (1, i-e) & (2, i-e) & \ldots . & (e, i-e) \\ (1, i-e) & (2, i-e) & (3, i-e) & \ldots . & (e+1, i-e) \\ (2, i-e) & (3, i-e) & (4, i-e) & \ldots . & (e+2, i-e) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (e, i-e) & (e+1, i-e) & (e+2, i-e) & \ldots . & (2e, i-e) \end{vmatrix} y^{e(e+1)},$$

which may be viewed as a preferable form of Sylvester's theorem. In it, it is interesting to note, the persymmetry of the given determinant is due to the set of second index-numbers of the elements, the first index-number being 0 throughout, whereas in the determinant which results after the removal of the factor $y^{e(e+1)}$ it is the second index-number that is invariable, the persymmetry being due to the set of first numbers. This means that in every element of the latter determinant the highest power of x is the $(i-e)^{\text{th}}$, and that therefore the first term of the resulting binary quantic is

$$\begin{vmatrix} a_0 & a_1 & a_2 & \ldots . & a_e \\ a_1 & a_2 & a_3 & \ldots . & a_{e+1} \\ a_2 & a_3 & a_4 & \ldots . & a_{e+2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_e & a_{e+1} & a_{e+2} & \ldots . & a_{2e} \end{vmatrix} x^{(i-e)(e+1)}.$$

SCOTT, R. F. (1879).

[On some symmetrical forms of determinants. *Messenger of Math.*, viii. pp. 131–138, 145–150.]

Beginning with the evaluation of the persymmetric continuant $K(b_c^a b_c^a b \ldots)$ (see chap. xvii. at end), Scott passes on by two easy stages to the result

$$\begin{vmatrix} c & a & a & \ldots & a \\ b & c & a & \ldots & a \\ b & b & c & \ldots & a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b & b & b & \ldots & c \end{vmatrix}_n = \frac{a(c-b)^n - b(c-a)^n}{a-b},$$

adding that the determinant got from this by bordering it axisymmetrically with 0, 1, 1, 1, . . . , 1 is equal to

$$\frac{(c-a)^{n-1} - (c-b)^{n-1}}{a-b},$$

and that the determinant, no longer persymmetric, got by changing the c's into $c_1, c_2, \ldots, c_n$, is equal to

$$\frac{a\phi(b) - b\phi(a)}{a-b}, \quad \text{where} \quad \phi(x) = (c_1-x)(c_2-x) \ldots (c_n-x).$$

He next devotes considerable space to the determinant which is identical with the above as regards its main diagonal and the two contiguous minor diagonals, but differs in having a c in all its other places. This is followed by one still more fanciful, and then he returns to forms like Sardi's of 1868.

On account of the connection of persymmetric determinants with Sturm's Functions, the following writings dealing with the latter and reproducing previous results in more or less fresh forms are worth noting :

1862. HATTENDORF, K. Ueber die Sturm'schen Functionen. Dissert. 54 pp. Göttingen. 2te Aufl., Hannover, 1874.

1866–7. Gilbert, Ph. Sur les fonctions de Sturm. *Journ.* (*de Liouville*) *de Math.*, (2) xii. pp. 87–97.

1874–5 Darboux, G. Mémoire sur le théorème de Sturm. *Bull. des Sci. Math.*, viii. pp. 56–63, 92–112.

1878. Wendlandt, H. Die Sturm'schen Functionen zweiter Gattung. *Archiv d. Math. u. Phys.*, lxii. pp. 1–77.

It is also worth noting that Jacobi's paper of 1845 (*Hist.*, ii. pp. 326–330), dealing with a special problem connected with the approximate representation of functions, is followed up by Frobenius in his paper 'Ueber Relationen zwischen den Näherungsbrüchen von Potenzreihen' in *Crelle's Journ.*, xc. (1879), pp. 1–17.

CHAPTER XIII.

BIGRADIENTS, FROM 1859 TO 1880.

HITHERTO the only bigradient considered had been that which originates in the process of dialytic elimination: and although in the period now reached bigradients of other origins make their appearance, it is still Sylvester's that claims the greatest share of attention. Note needs consequently to be taken that on account of the existence of Bezout's eliminant, which of course is equivalent in substance but which in form is axisymmetric, a difficulty of classification arises, and that we adopt what seems the most generally convenient course of not attempting to separate the two. Propositions, therefore, which concern the axisymmetric eliminant of two binary quantics will be found as a rule under our present heading.

BRUNO, F. FAÀ DI (1859).

[THÉORIE GÉNÉRALE DE L'ÉLIMINATION, . . . x+224 pp. Paris.]

In his section on the highest-common-divisor (pp. 47–52), Bruno, denoting the r^{th} Sturmian remainder of

$$a_0x^m+a_1x^{m-1}+\ldots, \qquad b_0x^{m-1}+b_1x^{m-2}+\ldots$$

by R_r, finds for it the expression

$$\lambda_r \cdot \sum_{l=0}^{l=m-r+1} \left\{ x^{m+r-l-1} \left| \begin{array}{cccccc} a_0 & a_1 & a_2 & \ldots & a_{2r-1} & a_{2r+l} \\ . & a_0 & a_1 & \ldots & a_{2r-2} & a_{2r+l-1} \\ . & . & a_0 & \ldots & a_{2r-3} & a_{2r+l-2} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & a_r & a_{r+l-1} \\ b_0 & b_1 & b_2 & \ldots & b_{2r-1} & b_{2r+l} \\ . & b_0 & b_1 & \ldots & b_{2r-2} & b_{2r+l-1} \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & b_{r-1} & b_{r+l} \end{array} \right| \right\},$$

where λ_r and R_r/λ_r are the "allotrious factor" and "simplified residue" of Sylvester (1853). He must, however, have overlooked the question of sign, for the example which he gives, namely,

$$\frac{1}{b_0^2}\left\{\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ . & b_0 & b_1 \end{vmatrix} x^2 + \begin{vmatrix} a_0 & a_1 & a_3 \\ b_0 & b_1 & b_3 \\ . & b_0 & b_2 \end{vmatrix} x + \begin{vmatrix} a_0 & a_1 & a_4 \\ b_0 & b_1 & . \\ . & b_0 & b_3 \end{vmatrix}\right\}$$

as the first Sturmian remainder of

$$a_0x^4+a_1x^3+\ldots+a_4, \qquad b_0x^3+b_1x^2+b_2x+b_3$$

is incorrect in this particular. His method is more direct than Sylvester's: indeed, it would seem that the expressions which the latter obtained in 1840 (*Hist.*, i. pp. 236–238) as the result of dialytic elimination were only thought of at a later date as being connected with the Sturmian remainders. He, however, claims the earlier date,* and ought on the like ground to have viewed Jacobi as having forestalled him in the matter of his other set of expressions for the same remainders, namely, those in which the cofficients are minors of the condensed eliminant.†

In the section on the properties of the resultant (pp. 68–81) he recalls Richelot's theorem of 1840, that *if* w *be a common root of the equations*

$$a_0x^m+a_1x^{m-1}+\ldots=0, \qquad b_0x^n+b_1x^{n-1}+\ldots=0,$$

whose resultant is R, *then we have*

$$\frac{\partial R}{\partial a_0}:\frac{\partial R}{\partial a_1}:\ldots:\frac{\partial R}{\partial a_m} :: w^m : w^{m-1} :\ldots:1,$$

$$\frac{\partial R}{\partial b_0}:\frac{\partial R}{\partial b_1}:\ldots:\frac{\partial R}{\partial b_n} :: w^n : w^{n-1} :\ldots:1.$$

This is not brought forward as a theorem in determinants, but for comparison, when $n = m$, with Jacobi's theorem of 1835 to the effect that *the signed primary minors associated with the elements of any row of Bezout's condensed eliminant*

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*Sylvester, J. J. On the theory of the syzygetic relations . . . *Philos. Transac. R. Soc.* (London) (1853), art. 9 pp. 426–427.

†Jacobi, C. G. J. De eliminatione variabilis . . . *Crelle's Journ.*, xv. (1835), §15, pp. 119–121.

$$\begin{vmatrix} |a_0b_1| & |a_0b_2| & \dots \\ |a_0b_2| & |a_0b_3| + |a_1b_2| & \dots \\ \dots & \dots & \dots \end{vmatrix}_m$$

are proportional to

$$w^{m-1}, \quad w^{m-2}, \dots, w, \quad 1.$$

In the section on common roots (pp. 81–84) he obtains such a root when it is solitary by taking any one of these three series of proportionals and dividing one member of the series by the member immediately following. When the number of such roots is k he has recourse to the Sturmian remainders previously found, stating for comparison Lagrange's set of conditions : *

$$R = 0, \quad \frac{\partial R}{\partial a_m} = 0, \quad \frac{\partial^2 R}{\partial a_m^2} = 0, \dots, \quad \frac{\partial^{k-1} R}{\partial a_m^{k-1}} = 0.$$

TRUDI, N. (1862).

[Teoria de' Determinanti, ... xii+268 pp. Napoli.]

To Trudi is due the first methodical exposition of bigradients, a nineteen-page chapter of the first part (*Teoria*) of his text-book being specially devoted to them, and several chapters of his second part (*Applicazioni*) making constant use of them.

The nineteen-page chapter or section (§ xi. pp. 94–112) bears the heading "Matrici e determinanti a due scale." It contains, first of all, careful explanations of the various expressions which he finds necessary to use in a special sense while dealing with such determinants ; for example, *scala, scala di grado* r, *scala diretta, scala inversa, scala completa*, etc. He next gives an account of the simple properties of bigradient arrays, or, as he calls them, *two-scale matrices*, and introduces a notation for them, writing

$$\begin{vmatrix} (a_0)_r \\ (b_0)_s \end{vmatrix}$$

to denote the bigradient array in which the elements $a_0, a_1, \dots, a_m$ are repeated r times, and the elements $b_0, b_1, \dots, b_n$ are

* Lagrange. Réflexions sur la résolution algébrique des équations. *Nouv. Mémoires . . . Acad. . . . Berlin*, 1770, 1771 ; or *Œuvres complètes*, iii. pp. 205–421 (227–229).

repeated s times, a_0 and b_0 when furthest to the left being in the first column, and a_m and b_n when furthest to the right being in the last column. Clearly, the notation would have been less imperfect if written

$$\left\| \begin{matrix} (a_0, \ldots, a_m)_r \\ (b_0, \ldots, b_n)_s \end{matrix} \right\|.$$

For example, the bigradient array

$$\left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & . & . \\ . & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & . \\ . & . & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ . & . & . & . & . & b_0 & b_1 & b_2 & b_3 & b_4 \\ . & . & . & . & b_0 & b_1 & b_2 & b_3 & b_4 & . \\ . & . & . & b_0 & b_1 & b_2 & b_3 & b_4 & . & . \\ . & . & b_0 & b_1 & b_2 & b_3 & b_4 & . & . & . \\ . & b_0 & b_1 & b_2 & b_3 & b_4 & . & . & . & . \\ b_0 & b_1 & b_2 & b_3 & b_4 & . & . & . & . & . \end{matrix} \right\|$$

might with fair appropriateness be denoted by

$$\left\| \begin{matrix} (a_0, \ldots, a_7)_3 \\ (b_0, \ldots, b_4)_6 \end{matrix} \right\|,$$

the only weak point then being that the introduction of the 6 is uncalled for, on account of the necessary equality of $m+r$ and $n+s$, either of which specifies the number of columns in the array. It is a convenience, however, to have both the outside suffixes 3, 6 in front of us, because their sum gives the number of the rows, a sum we should otherwise have to know from $m+2r-n$. Instead of all the determinants of such an array being viewed, as hitherto, of equal prominence, Trudi only concerns himself with the first two of the ten, namely, those which have in common the first eight columns of the array. These $n-r+1$ determinants he designates not very happily "the successive determinants of the array." The name "principal" which he gives to the first determinant of all may be advantageously translated "leading."

The number of bigradient arrays associated with the two sets of elements

$$a_0, a_1, \ldots, a_m, \qquad b_0, b_1, \ldots, b_n$$

is evidently n: thus, in the case where $m,n = 7,4$ the arrays are

$$\left\| \begin{matrix} (a_0, \ldots, a_7)_1 \\ (b_0, \ldots, b_4)_4 \end{matrix} \right\|, \quad \left\| \begin{matrix} (a_0, \ldots, a_7)_2 \\ (b_0, \ldots, b_4)_5 \end{matrix} \right\|, \quad \left\| \begin{matrix} (a_0, \ldots, a_7)_3 \\ (b_0, \ldots, b_4)_6 \end{matrix} \right\|, \quad \left\| \begin{matrix} (a_0, \ldots, a_7)_4 \\ (b_0, \ldots, b_4)_7 \end{matrix} \right\|,$$

the last, where $r = n$, being square, and therefore preferably written in the form

$$\left| \begin{matrix} (a_0, \ldots, a_7)_4 \\ (b_0, \ldots, b_4)_7 \end{matrix} \right|.$$

These and other preliminaries being settled, he is in a position to deal with an important theorem on the subject of what we may call *the condensation of a bigradient array*. The proof given is, unfortunately, not at all so simple as it might have been. We shall therefore substitute for it one of our own, which Trudi himself would probably have devised had he been aware of Cayley's work of 1845. Taking, first, a case in which $m = n$, say the case

$$\left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & . & . \\ . & a_0 & a_1 & a_2 & a_3 & a_4 & . \\ . & . & a_0 & a_1 & a_2 & a_3 & a_4 \\ . & . & b_0 & b_1 & b_2 & b_3 & b_4 \\ . & b_0 & b_1 & b_2 & b_3 & b_4 & . \\ b_0 & b_1 & b_2 & b_3 & b_4 & . & . \end{matrix} \right\| \quad \text{or} \quad \left\| \begin{matrix} (a_0, \ldots, a_4)_3 \\ (b_0, \ldots, b_4)_3 \end{matrix} \right\|,$$

we multiply the determinant

$$\left| \begin{matrix} 1 & . & . & . & . & -b_0 \\ . & 1 & . & . & -b_0 & -b_1 \\ . & . & 1 & -b_0 & -b_1 & -b_2 \\ . & . & . & a_0 & a_1 & a_2 \\ . & . & . & . & a_0 & a_1 \\ . & . & . & . & . & a_0 \end{matrix} \right|$$

by the given array in column-by-column fashion, obtaining (*Hist.*, ii. p. 34)

$$\left\| \begin{matrix} (a_0, \ldots, a_4)_3 \\ (b_0, \ldots, b_4)_3 \end{matrix} \right\| = \left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & . & . \\ . & a_0 & a_1 & a_2 & a_3 & a_4 & . \\ . & . & a_0 & a_1 & a_2 & a_3 & a_4 \\ . & . & . & |a_0b_1| & |a_0b_2| & |a_0b_3| & |a_0b_4| \\ . & . & . & |a_0b_2| & |a_0b_3| + |a_1b_2| & |a_0b_4| + |a_1b_3| & |a_1b_4| \\ . & . & . & |a_0b_3| & |a_0b_4| + |a_1b_3| & |a_1b_4| + |a_2b_3| & |a_2b_4| \end{matrix} \right\|.$$

Of the seven identities included in this the first two are Trudi's, and these he writes in combination, thus :

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & . & . \\ . & a_0 & a_1 & a_2 & a_3 & a_4 & . \\ . & . & a_0 & a_1 & a_2 & a_3 & a_4 \\ . & . & b_0 & b_1 & b_2 & b_3 & b_4 \\ . & b_0 & b_1 & b_2 & b_3 & b_4 & . \\ b_0 & b_1 & b_2 & b_3 & b_4 & . & . \end{vmatrix} = \begin{Vmatrix} |a_0b_1| & |a_0b_2| & |a_0b_3| & |a_0b_4 \\ |a_0b_2| & |a_0b_3| + |a_1b_2| & |a_0b_4| + |a_1b_3| & |a_1b_4 \\ |a_0b_3| & |a_0b_4| + |a_1b_3| & |a_1b_4| + |a_2b_3| & |a_2b_4 \end{Vmatrix}$$

meaning thereby that the determinants got by leaving out the 7th column on the left and the 4th on the right are equal to one another, and also those determinants got by leaving out the 6th column on the left and the 3rd on the right.*

He then draws attention to the fact that the two-line determinants involved in the array on the right are principal minors of the array

$$\begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4, \end{matrix}$$

and he formulates a mnemonic rule like Sylvester's (*Hist.*, ii. p. 340) for the formation of the condensed array. His own illustrative examples are

$$\begin{vmatrix} a & b & c & d & . & . \\ . & a & b & c & d & . \\ . & . & a & b & c & d \\ . & . & t & u & v & x \\ . & t & u & v & x & . \\ t & u & v & x & . & . \end{vmatrix} = \begin{vmatrix} au-bt & av-ct & ax-dt \\ av-ct & \begin{matrix} ax-dt \\ +bv-cu \end{matrix} & bx-du \\ ax-dt & bx-du & cx-dv \end{vmatrix},$$

* With Cayley the assertion

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix} \right\| = \left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix} \right\|$$

included 6 equations, whereas with Trudi it only includes 3, namely, the first 3 of Cayley's 6 : and with Cayley the assertion

$$\left\| \begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix} \right\| = \left\| \begin{matrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{matrix} \right\|$$

was meaningless, whereas with Trudi it includes 2 equations. Since in the former case Trudi's 3 equations are known to necessitate the other 3, there is clearly no good reason for refusing to profit by the new usage. What is common to any two arrays which Trudi may equate is the excess of the number of columns over the number of rows : and evidently if his excess be δ, the number of included equations is $\delta + 1$.

$$\left\|\begin{matrix} a & b & c & d & . \\ . & a & b & c & d \\ . & t & u & v & x \\ t & u & v & x & . \end{matrix}\right\| = \left\|\begin{matrix} au-bt & av-ct & ax-dt \\ av-ct & ax-dt & bx-du \\ & +bv-cu & \end{matrix}\right\|,$$

$$\left\|\begin{matrix} a & b & c & d \\ t & u & v & x \end{matrix}\right\| = \| au-bt \quad av-ct \quad ax-dt \|,$$

where each array on the left is got from the one that precedes it by deleting the first row, the first column, and the last row ; and each array on the right by merely deleting the last row. It is noted that the leading determinant of the condensed array is axisymmetric.

Lastly, it is pointed out that cases where $m > n$ present almost no additional difficulty, as they are readily brought under the foregoing. Thus, if the case be

$$\left|\begin{matrix} (a,\ b,\ c,\ d,\ e,\ f)_2 \\ (t,\ u,\ v)_5 \end{matrix}\right|,$$

we have only to take

a	b	c	d	e	f
0	0	0	t	u	v

for our generating array and proceed exactly as before, the results being

$$\left\|\begin{matrix} a & b & c & d & e & f & . \\ . & . & . & t & u & v & \\ . & . & t & u & v & . & \\ . & t & u & v & . & . & \\ t & u & v & . & . & . & \\ u & v & . & . & . & . & \end{matrix}\right\| = \left|\begin{matrix} . & . & at & au & av \\ . & at & au+bt & av+bu & bv \\ at & au+bt & av+bu+ct & bv+cu & cv \\ au & av+bu & bv+cu & cv+du-et & dv-ft \\ av & bv & cv & dv-ft & ev-fu \end{matrix}\right| \div a^3$$

$$= \left|\begin{matrix} . & . & t & u & v \\ . & t & u & v & . \\ t & u & v & . & . \\ au & av+bu & bv+cu & cv+du-et & dv-ft \\ av & bv & cv & dv-ft & eu-fu \end{matrix}\right|,$$

$$\left\|\begin{matrix} a & b & c & d & e & f \\ . & . & t & u & v & \\ . & t & u & v & . & \\ t & u & v & . & . & \\ u & v & . & . & . & \end{matrix}\right\| = \left\|\begin{matrix} . & . & t & u & v \\ . & t & u & v & . \\ t & u & v & & . \\ au & av+bu & bv+cu & cu+du-et & dv-ft \end{matrix}\right\|.$$

The requisite division by a^3 (in general a^{m-n}) may be performed by removing the a's one at a time, or by using the divisor in the form

$$\begin{vmatrix} a & b & c & . & . \\ . & a & b & . & . \\ . & . & a & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{vmatrix}.$$

Another theorem of a similar kind, but introduced for a different purpose, namely, for dilatation rather than condensation (pp. 129–131), is

$$\left\| \begin{matrix} (b_0, \ldots, b_n)_r \\ (c_0, \ldots, c_{n-1}) \end{matrix} \right\| = \frac{(-1)^\sigma}{b_0^{m-n+1}} \left\| \begin{matrix} (a_0, \ldots, a_m)_{r+1} \\ (b_0, \ldots, b_n) \end{matrix} \right\|,$$

where the c's are determinants defined by the postulated identity

$$\frac{a_0x^m + \ldots}{b_0x^n + \ldots} = q_0x^{m-n} + \ldots + \frac{c_0x^{n-1} + \ldots}{b_0x^n + \ldots}$$

(see under *Recurrents*) and where $\sigma = r+1+\frac{1}{2}(m-n)(m-n-1)$. For example, when $m = 5$, $n = 4$, $r = 2$, the identity is

$$\left\| \begin{matrix} b_0 & b_1 & b_2 & b_3 & b_4 & . \\ . & b_0 & b_1 & b_2 & b_3 & b_4 \\ . & . & c_0 & c_1 & c_2 & c_3 \\ . & c_0 & c_1 & c_2 & c_3 & . \\ c_0 & c_1 & c_2 & c_3 & . & . \end{matrix} \right\| = \frac{(-1)^3}{b_0^2} \left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & . & . \\ . & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & . \\ . & . & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ . & . & . & b_0 & b_1 & b_2 & b_3 & b_4 \\ . & . & b_0 & b_1 & b_2 & b_3 & b_4 & . \\ . & b_0 & b_1 & b_2 & b_3 & b_4 & . & . \\ b_0 & b_1 & b_2 & b_3 & b_4 & . & . & . \end{matrix} \right\|.$$

Trudi's proof consists in evolving the second member from the first, but here again it is simpler to use Cayley's multiplication-theorem of 1845. Thus, taking the second array as multiplier and the determinant

$$\begin{vmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & -q_1 & 1 & . & . & . \\ . & -q_1 & -q_0 & . & 1 & . & . \\ -q_1 & -q_0 & . & . & . & 1 & . \\ -q_0 & . & . & . & . & . & 1 \end{vmatrix}$$

as multiplicand, we at once find the product to be

$$\left\|\begin{matrix} . & . & c_0 & c_1 & c_2 & c_3 & . & . \\ . & . & . & c_0 & c_1 & c_2 & c_3 & . \\ . & . & . & . & c_0 & c_1 & c_2 & c_3 \\ . & . & . & b_0 & b_1 & b_2 & b_3 & b_4 \\ . & . & b_0 & b_1 & b_2 & b_3 & b_4 & . \\ . & b_0 & b_1 & b_2 & b_3 & b_4 & . & . \\ b_0 & b_1 & b_2 & b_3 & b_4 & . & . & . \end{matrix}\right\|,$$

which is equal to

$$-b_0{}^2\left\|\begin{matrix} c_0 & c_1 & c_2 & c_3 & . & . \\ . & c_0 & c_1 & c_2 & c_3 & . \\ . & . & c_0 & c_1 & c_2 & c_3 \\ . & b_0 & b_1 & b_2 & b_3 & b_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 & . \end{matrix}\right\| = -b_0{}^2\left\|\begin{matrix} (b_0, \ldots, b_4)_2 \\ (c_0, \ldots, c_3)_3 \end{matrix}\right\|,$$

as was to be proved.

The use to which this second theorem is put (pp. 132–137) is in connection with the division-process for finding the highest-common-divisor of two integral functions, and, in particular, with the modification of the said process employed by Sturm in obtaining his so-called "remainders." From the general theorem* connecting dividend, divisor, quotient, and remainder we know that the coefficients of the first remainder in such a process are proportional to the successive determinants of a bigradient array composed of the coefficients of the dividend and divisor. We thus also know that, this remainder having been made the divisor and the previous divisor the dividend, the new remainder must be expressible in like fashion. In the second bigradient array thus arising, however, one of the two sets of elements is complicated, being in fact the successive determinants of the previous array: and what Trudi's "dilatation"-theorem enables us to do is to supplant it by another array whose elements are simply the coefficients of the original functions. In this way the theorem finally reached is: *The coefficients of the* r^{th} *remainder* R_r *arising in the course of the performance of Sturm's division-process on*

$$a_0x^m + \ldots + a_m, \qquad b_0x^n + \ldots + b_n$$

* See above under *Recurrents*, Trudi (1862).

are equal to the successive determinants of the array

$$\left\| \begin{matrix} (a_0, \ldots, a_m)_r \\ (b_0, \ldots, b_n) \end{matrix} \right\|$$

divided by the product of the squares of the first coefficients of all the preceding remainders and by b_0^{m-n+1} *and by the sign-factor* $(-1)^{\frac{1}{2}(m-n)(m-n-1)}$. The r^{th} remainder, when divested of the three-fold divisor here specified, a_r say, Trudi follows Sylvester in calling the r^{th} *residuo semplificato*,* and denotes by ρ_r, so that $R_r = \rho_r/a_r$. For example, when the originating functions A and B are

$$ax^4+bx^3+cx^2+dx+e,$$
$$px^3+qx^2+rx+s,$$

the three Sturmian remainders in their 'simplified' or disencumbered form are

$$\begin{vmatrix} a & b & c \\ . & p & q \\ p & q & r \end{vmatrix} x^2 + \begin{vmatrix} a & b & d \\ . & p & r \\ p & q & s \end{vmatrix} x + \begin{vmatrix} a & b & e \\ . & p & s \\ p & q & . \end{vmatrix},$$

$$\begin{vmatrix} a & b & c & d & e \\ . & a & b & c & d \\ . & . & p & q & r \\ . & p & q & r & s \\ p & q & r & s & . \end{vmatrix} x + \begin{vmatrix} a & b & c & d & . \\ . & a & b & c & e \\ . & . & p & q & s \\ . & p & q & r & . \\ p & q & r & s & . \end{vmatrix},$$

$$\begin{vmatrix} a & b & c & d & e & . & . \\ . & a & b & c & d & e & . \\ . & . & a & b & c & d & e \\ . & . & . & p & q & r & s \\ . & . & p & q & r & s & . \\ . & p & q & r & s & . & . \\ p & q & r & s & . & . & . \end{vmatrix},$$

* A most natural and helpful notation for such a remainder would be

$$\left\| \begin{matrix} (a_0, \ldots, a_m)_r \\ (b_0, \ldots, b_n) \end{matrix} \right\| (x^{n-r}, \ldots, x, 1).$$

Thus, in the case here used for purposes of illustration, the remainders would be written

$$\left\| \begin{matrix} a & b & c & d & e \\ . & p & q & r & s \\ p & q & r & s & . \end{matrix} \right\| (x^2, x, 1), \qquad \left\| \begin{matrix} (a\ b\ c\ d\ e)_2 \\ (p\ q\ r\ s)_3 \end{matrix} \right\| (x, 1), \qquad \ldots.$$

the removed encumbrances being the factors

$$\frac{1}{p^2}, \quad \frac{p^2}{\begin{vmatrix} a & b & c \\ . & p & q \\ p & q & r \end{vmatrix}^2}, \quad \frac{\begin{vmatrix} a & b & c \\ . & p & q \\ p & q & r \end{vmatrix}^2}{p^2 \begin{vmatrix} a & b & c & d & e \\ . & a & b & c & d \\ . & . & p & q & r \\ . & p & q & r & s \\ p & q & r & s & . \end{vmatrix}^2},$$

respectively. The general expression for the factor, a_r, connecting the simplified and unsimplified forms of a remainder, is readily got (pp. 138–139) in Sylvester's way by using the fact that *the product* $a_{r-1}a_r$ *is equal to the square of the first coefficient of* ρ_r. For, this is the same as saying that, if we denote the first determinant of

$$\left\| \begin{matrix} (a_0, \ldots, a_m)_r \\ (b_0, \ldots, b_n) \end{matrix} \right\|$$

by D_r, we have

$$a_2a_2 = D_1^2, \quad a_2a_3 = D_2^2, \ldots\ldots$$

and these lead to

$$a_{2\mu} = \frac{1}{a_1} \cdot \frac{D_1^2 D_3^2 D_5^2 \ldots D_{2\mu-1}^2}{D_2^2 D_4^2 \ldots D_{2\mu-2}^2},$$

and

$$a_{2\mu+1} = a_1 \cdot \frac{D_2^2 D_4^2 \ldots D_{2\mu}^2}{D_1^2 D_3^2 \ldots D_{2\mu-1}^2},$$

where

$$a_1 = (-1)^{\frac{1}{2}(m-n)(m-n-1)} \cdot b_0^{\,m-n+1}.$$

An important observation made in passing is that any simplified remainder can be condensed into a single determinant: for example, the three just given above are equal to

$$\begin{vmatrix} a & b & cx^2+dx+e \\ . & p & qx^2+rx+s \\ p & q & rx^2+sx \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a & b & \mathrm{A} \\ . & p & \mathrm{B} \\ p & q & \mathrm{B}x \end{vmatrix},$$

$$\begin{vmatrix} a & b & c & d & ex \\ . & a & b & c & dx+e \\ . & . & p & q & rx+s \\ . & p & q & r & sx \\ p & q & r & s & . \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a & b & c & d & \mathrm{A}x \\ . & a & b & c & \mathrm{A} \\ . & . & p & q & \mathrm{B} \\ . & p & q & r & \mathrm{B}x \\ p & q & r & s & \mathrm{B}x^2 \end{vmatrix},$$

$$\begin{vmatrix} a & b & c & d & e & . & \mathrm{A}x^2 \\ . & a & b & c & d & e & \mathrm{A}x \\ . & . & a & b & c & d & \mathrm{A} \\ . & . & . & p & q & r & \mathrm{B} \\ . & . & p & q & r & s & \mathrm{B}x \\ . & p & q & r & s & . & \mathrm{B}x^2 \\ p & q & r & s & . & . & \mathrm{B}x^3 \end{vmatrix},$$

where, be it remarked, the ultimate forms, namely, those explicitly involving A and B, are Cayley's of 1848.

Cayley's relation between any three consecutive 'simplified remainders' is next given (pp. 140–142), the proof arising quite naturally and being mainly dependent on the equality $a_{r-1}a_r = \mathrm{D}^2_{r-1}$. Thus, taking the equations that indicate the nature of the division-process, namely,

$$\begin{aligned} \mathrm{A} &= \mathrm{Q}_1\mathrm{B} - \mathrm{R}_1 \\ \mathrm{B} &= \mathrm{Q}_2\mathrm{R}_1 - \mathrm{R}_2 \\ \mathrm{R}_1 &= \mathrm{Q}_3\mathrm{R}_2 - \mathrm{R}_3 \\ \mathrm{R}_2 &= \mathrm{Q}_4\mathrm{R}_3 - \mathrm{R}_4 \\ &\cdots\cdots, \end{aligned}$$

and substituting ρ_r/a_r for R_r, we obtain

$$\begin{aligned} a_1 \cdot \mathrm{A} &= a_1\mathrm{Q}_1 \cdot \mathrm{B} - \rho_1 \\ a_1a_2 \cdot \mathrm{B} &= a_2\mathrm{Q}_2 \cdot \rho_1 - a_1 \cdot \rho_2 \\ a_2a_3 \cdot \rho_1 &= a_1a_3\mathrm{Q}_3 \cdot \rho_2 - a_1a_2 \cdot \rho_3 \\ a_3a_4 \cdot \rho_2 &= a_2a_4\mathrm{Q}_4 \cdot \rho_3 - a_2a_3 \cdot \rho_4 \\ &\cdots\cdots\cdots\cdots. \end{aligned}$$

In this way there results the general equation

$$a_{r-1}a_r \cdot \rho_{r-2} = a_{r-2}a_r\mathrm{Q}_r \cdot \rho_{r-1} - a_{r-2}a_{r-1} \cdot \rho_r,$$

and thence

$$\mathrm{D}^2_{r-1} \cdot \rho_{r-2} = a_{r-2}a_r\mathrm{Q}_r \cdot \rho_{r-1} - \mathrm{D}^2_{r-2} \cdot \rho_r,$$

showing that ρ_{r-2} and ρ_r have different signs for any value of x that makes ρ_{r-1} vanish.

Proceeding from the above-noted Cayleyan mode of expressing the 'simplified remainders,' Trudi puts forward (pp. 145–152) another mode, each remainder now appearing as *a sum of a multiple of* A *and a multiple of* B; or, in Sylvester's words,* *as a syzygetic function of* A *and* B. For example, the three remainders above given he considers in the form

$$\begin{vmatrix} a & b & 1 \\ . & p & . \\ p & q & . \end{vmatrix} A + \begin{vmatrix} a & b & . \\ . & p & 1 \\ p & q & x \end{vmatrix} B,$$

$$\begin{vmatrix} a & b & c & d & x \\ . & a & b & c & 1 \\ . & . & p & q & . \\ . & p & q & r & . \\ p & q & r & s & . \end{vmatrix} A + \begin{vmatrix} a & b & c & d & . \\ . & a & b & c & . \\ . & . & p & q & 1 \\ . & p & q & r & x \\ p & q & r & s & x^2 \end{vmatrix} B, \quad \ldots ;$$

and generally he writes

$$\rho_r = U_r \cdot A + V_r \cdot B,$$

where it is readily seen that, as regards x,

$$\left.\begin{array}{l} U_r \text{ is of the degree } r-1, \\ V_r \quad \ldots \quad \ldots \quad m-n+r-1, \end{array}\right\}$$

$$\left.\begin{array}{l} U_r A \quad \ldots \quad \ldots \quad m+r-1, \\ V_r B \quad \ldots \quad \ldots \quad m+r-1, \end{array}\right\}$$

and where, as we know,

$$\rho_r \quad \ldots \quad \ldots \quad n-r.$$

Observation also shows that the coefficients of the highest powers of x in U_r, V_r are $-b_0 D_{r-1}$, $a_0 D_{r-1}$ respectively. By substituting the new forms for ρ_{r-2}, ρ_{r-1}, ρ_r in Cayley's relation

$$D_{r-1}^2 \cdot \rho_{r-2} - \alpha_{r-2}\alpha_r Q_r \cdot \rho_{r-1} + D_{r-1}^2 \cdot \rho_r = 0,$$

there is obtained

$$(D_{r-1}^2 \cdot U_{r-2} - \alpha_{r-2}\alpha_r Q_r \cdot U_{r-1} + D_{r-2}^2 \cdot U_r) A$$
$$+ (D_{r-1}^2 \cdot V_{r-2} - \alpha_{r-2}\alpha_r Q_r \cdot V_{r-1} + D_{r-2}^2 \cdot V_r) B = 0;$$

* It is worth noting that it was in this connection that the word 'syzygetic' was first used, the full title of the memoir of 1853 (which clearly had considerable influence on Trudi) being "On a theory of the *syzygetic* relations of two rational integral functions, comprising an application to the theory of Sturm's functions, and that of the greatest algebraical common measure."

and, as Trudi proves that this can only hold when the coefficients of A and B vanish, it follows that each of the two series of functions

$$U_0,\quad U_1,\ \ldots\ ,\ U_{n+1},\qquad V_0,\quad V_1,\ \ldots\ ,\ V_{n+1}$$

has one of the Sturmian properties which the ρ's have been shown to possess.

As regards the highest-common-divisor (pp. 142–144) his result is : *In order that two functions may have a common divisor of the* k^{th} *degree, it is necessary and sufficient that the first determinant of each of the last* k *of their bigradient arrays shall vanish : and, when this holds, the coefficients of the divisor in question are the successive determinants of the* $(n-k)^{th}$ *array.* For example, the functions being

$$a_0x^8+a_1x^7+\ \ldots\ +a_8,\qquad b_0x^5+b_1x^4+\ \ldots\ +b_5,$$

their bigradient arrays are

$$\left\|\begin{matrix}(a_0,\ \ldots\ ,\ a_8)_1\\(b_0,\ \ldots\ ,\ b_5)\end{matrix}\right\|,\quad \left\|\begin{matrix}(a_0,\ \ldots\ ,\ a_8)_2\\(b_0,\ \ldots\ ,\ b_5)\end{matrix}\right\|,\quad \left\|\begin{matrix}(a_0,\ \ldots\ ,\ a_8)_3\\(b_0,\ \ldots\ ,\ b_5)\end{matrix}\right\|,$$

$$\left\|\begin{matrix}(a_0,\ \ldots\ ,\ a_8)_4\\(b_0,\ \ldots\ ,\ b_5)\end{matrix}\right\|,\quad \left\|\begin{matrix}(a_0,\ \ldots\ ,\ a_8)_5\\(b_0,\ \ldots\ ,\ b_5)\end{matrix}\right\|,$$

and the proposition states that if the first determinant of each of the last three arrays vanishes, the functions have the common cubic factor

$$\left\|\begin{matrix}(a_0,\ \ldots\ ,\ a_8)_2\\(b_0,\ \ldots\ ,\ b_5)\end{matrix}\right\|(x^3,\ x^2,\ x,\ 1).$$

At a later stage (p. 151) there is given the supplementary proposition that *the quotients resulting from dividing* A *and* B *by the said highest-common-divisor are, save for an unimportant factor in each case, the coefficients of* B *and* A *in Trudi's form of the* $(n-k+1)^{th}$ '*simplified remainder*'—that is to say, are V_{n-k+1} and U_{n-k+1} as before defined.

The closely related question concerning the common roots of two equations he deals with at length in a section devoted to elimination (pp. 161–178). Starting with the proposition that, u and v being integral functions of x, $uA+vB$ must vanish for any common root of the equations $A = 0$, $B = 0$, he next points out that u and v may be so chosen as to make $uA+vB$ of a low degree in x, even of the degree zero. In the latter extreme case $uA+vB$ must contain the

eliminant as a factor, and if in addition it be of the proper degree in the coefficients of A and B, it is the eliminant pure and simple. Attention is then called to the fact that the division-process for finding the highest-common-divisor of A and B, or the Sturmian modification of this process, supplies a series of pairs of functions like u and v, and in particular that the last 'simplified remainder' D_n, as satisfying all the requirements mentioned, is the eliminant. The condition for the existence of more than one common root is investigated in like manner. If the number of the roots in question be k, the degree-number of $u\mathrm{A}+v\mathrm{B}$ cannot be less than k. Founding on this, it is asserted that functions of the form $u\mathrm{A}+v\mathrm{B}$, whose degree-number is less than k, must vanish identically, and that therefore in particular the last k 'simplified remainders' of A and B must so vanish. In the next place, proof is adduced that the vanishing of these remainders is equivalent to the vanishing of their first coefficients: and finally, there is reached the following variant to the above proposition regarding the highest-common-divisor: *In order that the equations* $\mathrm{A}=0$, $\mathrm{B}=0$ *may have* k *common roots, it is necessary and sufficient that* $\mathrm{D_n}, \mathrm{D_{n-1}}, \ldots, \mathrm{D_{n-k+1}}$ *vanish: and, this being the case, the equation of the said common roots is* $\rho_{n-k}=0$. The fact that the vanishing of the first coefficients of the 'simplified remainders' implies in each case the vanishing of all the coefficients following the first is merely commented on in passing. Attention, however, is more fully drawn to the important fact that the existence of the condensation-theorem makes it possible to put every proposition which, like the foregoing, involves bigradients into an alternative form. Thus, the condition that the equations

$$\left.\begin{aligned} x^3+a_1x^2+a_2x+a_3 &= 0 \\ x^2+b_1x+b_2 &= 0 \end{aligned}\right\}$$

may have two roots in common is, according to the said proposition, the vanishing of

$$\begin{vmatrix} 1 & a_1 & a_2 & a_3 & . \\ . & 1 & a_1 & a_2 & a_3 \\ . & . & 1 & b_1 & b_2 \\ . & 1 & b_1 & b_2 & . \\ 1 & b_1 & b_2 & . & . \end{vmatrix}, \qquad \begin{vmatrix} 1 & a_1 & a_2 \\ . & 1 & b_1 \\ 1 & b_1 & b_2 \end{vmatrix};$$

and this, by the condensation-theorem, is the same as the vanishing of

$$\begin{vmatrix} 1 & b_1 & b_2 \\ b_1 & b_2+a_1b_1-a_2 & a_1b_2-a_3 \\ b_2 & a_1b_2-a_3 & a_2b_2-a_3b_1 \end{vmatrix}, \quad \begin{vmatrix} 1 & b_1 \\ b_1 & b_2+a_1b_1-a_2 \end{vmatrix}.$$

Bezout's 'abridged method' and Sylvester's 'dialytic' method, which resemble each other in involving elimination of successive powers of a common root, are only introduced by Trudi for purposes of corroboration. In connection with the former method there is noted Sylvester's theorem * that the derived equations provide also an alternative way of obtaining the Sturmian 'simplified remainders,' the first remainder being the non-zero member of the first equation, the second remainder being the result of eliminating the highest power of x from the first two equations, the third remainder the result of eliminating the two highest powers of x from the first three equations, and so on. In other words, if the set of equations derived from A = 0, B = 0 by Bezout's method be

$$\left.\begin{aligned} c_{11}x^{m-1}+c_{12}x^{m-2}+\ldots+c_{1m} &= 0 \\ c_{21}x^{m-1}+c_{22}x^{m-2}+\ldots+c_{2m} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\},$$

then the second, third, . . . 'simplified remainders' of A and B are

$$\begin{vmatrix} c_{11} & c_{12}x^{m-2}+c_{13}x^{m-3}+\ldots+c_{1m} \\ c_{21} & c_{22}x^{m-2}+c_{23}x^{m-3}+\ldots+c_{2m} \end{vmatrix},$$

$$\begin{vmatrix} c_{11} & c_{12} & c_{13}x^{m-3}+\ldots+c_{1m} \\ c_{21} & c_{22} & c_{23}x^{m-3}+\ldots+c_{2m} \\ c_{31} & c_{32} & c_{33}x^{m-3}+\ldots+c_{3m} \end{vmatrix}, \ldots\ldots$$

Proceeding from this, Trudi then says that if the non-zero members of the said derived equations be denoted by $Y_1, Y_2 \ldots$, the 'simplified remainders' can clearly be put in the form

$$Y_1, \quad \begin{vmatrix} c_{11} & Y_1 \\ c_{21} & Y_2 \end{vmatrix}, \quad \begin{vmatrix} c_{11} & c_{12} & Y_1 \\ c_{21} & c_{22} & Y_2 \\ c_{31} & c_{32} & Y_3 \end{vmatrix}, \ldots\ldots;$$

* See Art. 5 of "On a theory of the syzygetic relations . . ."

and, as by definition,

$$\begin{aligned} Y_1 &= a_0B - b_0A, \\ Y_2 &= (a_0x + a_1)B - (b_0x + b_1)A, \\ Y_3 &= (a_0x^2 + a_1x + a_2)B - (b_0x^2 + b_1x + b_2)A, \\ &\cdots\cdots\cdots\cdots\cdots , \end{aligned}$$

it follows that the said remainders have still another form, namely,

$$a_0B - b_0A,$$

$$\begin{vmatrix} c_{11} & a_0 \\ c_{21} & a_0x + a_1 \end{vmatrix} B - \begin{vmatrix} c_{11} & b_0 \\ c_{21} & b_0x + b_1 \end{vmatrix} A,$$

$$\begin{vmatrix} c_{11} & c_{12} & a_0 \\ c_{21} & c_{22} & a_0x + a_1 \\ c_{31} & c_{32} & a_0x^2 + a_1x + a_2 \end{vmatrix} B - \begin{vmatrix} c_{11} & c_{12} & b_0 \\ c_{21} & c_{22} & b_0x + b_1 \\ c_{31} & c_{32} & b_0x^2 + b_1x + b_2 \end{vmatrix} A,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

—a result easily shown, by the use of the condensation-theorem, to be in agreement with a previous one in which the determinant coefficients of A and B are bigradients. He is also careful to note that although here, as usual, n is taken equal to m, no real restriction is thereby made, the case where $m > n$ being viewable as a case in which the coefficients of $x^{n+1}, x^{n+2}, \ldots, x^m$ in B are equal to 0. For example, if the given equations be

$$\left.\begin{aligned} ax^4 + bx^3 + cx^2 + dx + e &= 0 \\ qx^2 + rx + s &= 0 \end{aligned}\right\},$$

Bezout's derived equations (although not in Bezout's nor Trudi's notation) are

$$\left.\begin{aligned} \begin{vmatrix} a & bx^3 + cx^2 + dx + e \\ \cdot & qx^2 + rx + s \end{vmatrix} &= 0, \\ \begin{vmatrix} ax + b & cx^2 + dx + e \\ \cdot & qx^2 + rx + s \end{vmatrix} &= 0, \\ \begin{vmatrix} ax^2 + bx + c & dx + e \\ q & rx + s \end{vmatrix} &= 0, \\ \begin{vmatrix} ax^3 + bx^2 + cx + d & e \\ qx + r & s \end{vmatrix} &= 0, \end{aligned}\right\}$$

or, in their usual form,

$$\left.\begin{array}{rrrrl} & aq\cdot x^2 + & ar\cdot x + & as & = 0 \\ aqx^3 + (ar+bq)x^2 + & (as+br)x + & bs & = 0 \\ arx^3 + (as+br)x^2 + & (bs+cr-dq)x + & (cs-eq) & = 0 \\ asx^3 + & bsx^2 + & (cs-eq)x + & (ds-er) & = 0 \end{array}\right\}.$$

We then have for the simplified remainders of the 1st and 0th degrees the determinants

$$\begin{vmatrix} . & aq & ar\cdot x + as\cdot \\ aq & ar+bq & (as+br)\cdot x + bs \\ ar & as+br & (bs+cr-dq)\cdot x + (cs-eq) \end{vmatrix},$$

$$\begin{vmatrix} . & aq & ar & as \\ aq & ar+bq & as+br & bs \\ ar & as+br & bs+cr-dq & cs-eq \\ as & bs & cs-eq & ds-er \end{vmatrix},$$

being only careful to note that both of these contain the irrelevant factor a^2. Trudi, however, does not point out that this factor would not have troubled us if we had noted at the outset that for the first two derived equations we might have substituted

$$\begin{aligned} qx^2 + rx + s &= 0, \\ qx^3 + rx^2 + sx &= 0, \end{aligned}$$

thus using Sylvester's method of derivation for the first $m-n$ equations and Bezout's for the remaining n, as Cauchy had shown in 1840.

The case where B is the derivate of A receives special attention (pp. 152–160), the object of course being to show that the quantities D_r, U_r, V_r are then expressible in terms of sums of like powers of the roots of the equation $A = 0$. The reason for the possibility of this transformation lies in the fact that the coefficients

$$b_0,\quad b_1,\quad b_2,\ \ldots\ldots,\ b_{n-1}$$

are then equal to

$$a_0s_0,\quad a_0s_1+a_1s_0,\quad a_0s_2+a_1s_1+a_2s_0,\quad \ldots,\quad a_0s_{n-1}+\ldots+a_{n-1}s_0,$$

and that in addition we have

$$(a_0, a_1, \ldots, a_n \}\, s_n, \quad s_{n-1}, \ldots, s_0) = 0,$$
$$(a_0, a_1, \ldots, a_n \}\, s_{n+1}, s_n\ ., \ldots, s_1) = 0,$$
$$(a_0, a_1, \ldots, a_n \}\, s_{n+2}, s_{n+1}, \ldots, s_2) = 0.$$

The results arrived at are

$$D_r = a_0^{2r+1} \begin{vmatrix} s_0 & s_1 & s_2 & \ldots & s_{r-1} & s_r \\ s_1 & s_2 & s_3 & \ldots & s_r & s_{r+1} \\ s_2 & s_3 & s_4 & \ldots & s_{r+1} & s_{r+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_{r+1}$$

$$= 0 \text{ when } r > n-1,$$

$$V_r = a_0^{2r} \begin{vmatrix} s_0 & s_1 & s_2 & \ldots & s_{r-1} & 1 \\ s_1 & s_2 & s_3 & \ldots & s_r & x \\ s_2 & s_3 & s_4 & \ldots & s_{r+1} & x^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_{r+1},$$

$$U_r = -a_0^{2r} \begin{vmatrix} s_0 & s_1 & s_2 & \ldots & s_{r-1} & . \\ s_1 & s_2 & s_3 & \ldots & s_r & s_0 \\ s_2 & s_3 & s_4 & \ldots & s_{r+1} & s_0x + s_1 \\ s_3 & s_4 & s_5 & \ldots & s_{r+2} & s_0x^2 + s_1x + s_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_{r+1}.$$

Here again, however, Trudi loses his opportunity from not being acquainted with Cayley's multiplication-theorem of 1845, the use of which enables us to transform not only D_r, but the whole bigradient array of which D_r is the first determinant. In fact, it gives us for the case under consideration another condensation-theorem. For example, when

$$A = a_0x^5 + a_1x^4 + \ldots + a_5,$$

and we consequently have to consider the four 'simplified remainders'

$$\left\| \begin{matrix} (a_0, \ldots, a_5)_1 \\ (b_0, \ldots, b_4)_2 \end{matrix} \right\| (x^3, x^2, x, 1), \qquad \left\| \begin{matrix} (a_0, \ldots, a_5)_2 \\ (b_0, \ldots, b_4)_3 \end{matrix} \right\| (x^2, x, 1),$$

$$\left\| \begin{matrix} (a_0, \ldots, a_5)_3 \\ (b_0, \ldots, b_4)_4 \end{matrix} \right\| (x, 1), \qquad \left| \begin{matrix} (a_0, \ldots, a_5)_4 \\ (b_0, \ldots, b_4)_5 \end{matrix} \right|,$$

we find the condensation-results

$$\left\| \begin{matrix} (a_0, \ldots, a_5)_1 \\ (b_0, \ldots, b_4)_2 \end{matrix} \right\| = a_0{}^2 \left\| \begin{matrix} s_0 & a_0s_1 & a_0s_2+a_1s_1 & a_0s_3+\ldots & a_0s_4+\ldots \\ s_1 & a_0s_2 & a_0s_3+a_1s_2 & a_0s_4+\ldots & a_0s_5+\ldots \end{matrix} \right\|,$$

$$\left\| \begin{matrix} (a_0, \ldots, a_5)_2 \\ (b_0, \ldots, b_4)_3 \end{matrix} \right\| = a_0{}^4 \left\| \begin{matrix} s_0 & s_1 & a_0s_2 & a_0s_3+a_1s_2 & a_0s_4+a_1s_3+a_2s_2 \\ s_1 & s_2 & a_0s_3 & a_0s_4+a_1s_3 & a_0s_5+a_1s_4+a_2s_3 \\ s_2 & s_3 & a_0s_4 & a_0s_5+a_1s_4 & a_0s_6+a_1s_5+a_2s_4 \end{matrix} \right\|,$$

$$\left\| \begin{matrix} (a_0, \ldots, a_5)_3 \\ (b_0, \ldots, b_4)_4 \end{matrix} \right\| = a_0{}^6 \left\| \begin{matrix} s_0 & s_1 & s_2 & a_0s_3 & a_0s_4+a_1s_3 \\ s_1 & s_2 & s_3 & a_0s_4 & a_0s_5+a_1s_4 \\ s_2 & s_3 & s_4 & a_0s_5 & a_0s_6+a_1s_5 \\ s_3 & s_4 & s_5 & a_0s_6 & a_0s_7+a_1s_6 \end{matrix} \right\|,$$

$$\left| \begin{matrix} (a_0, \ldots, a_5)_4 \\ (b_0, \ldots, b_4)_5 \end{matrix} \right| = a_0{}^9 \begin{vmatrix} s_0 & s_1 & s_2 & s_3 & s_4 \\ s_1 & s_2 & s_3 & s_4 & s_5 \\ s_2 & s_3 & s_4 & s_5 & s_6 \\ s_3 & s_4 & s_5 & s_6 & s_7 \\ s_4 & s_5 & s_6 & s_7 & s_8 \end{vmatrix},$$

all in agreement with Cayley's original result of 1846 (*Hist.*, ii. pp. 162–164). Taking the second of these for proof, we multiply unity columnwise by the given bigradient array, obtaining

$$\left\| \begin{matrix} 1 & . & . & . & -s_0 \\ . & 1 & . & -s_0 & -s_1 \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & . \\ . & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ . & . & b_0 & b_1 & b_2 & b_3 & b_4 \\ . & b_0 & b_1 & b_2 & b_3 & b_4 & . \\ b_0 & b_1 & b_2 & b_3 & b_4 & . & . \end{matrix} \right\|,$$

$$= \left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & . \\ . & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ . & . & b_0 & b_1 & b_2 & b_3 & a_4 \\ . & . & a_0s_1 & a_0s_2+a_1s_1 & a_0s_3+a_1s_2+a_2s_1 & a_0s_4+\ldots & a_0s_5+\ldots \\ . & . & a_0s_2 & a_0s_3+a_1s_2 & a_0s_4+a_1s_3+a_2s_2 & a_0s_5+\ldots & a_0s_6+\ldots \end{matrix} \right\|$$

$$= a_0{}^2 \left\| \begin{matrix} a_0s_0 & a_0s_1+a_1s_0 & a_0s_2+a_1s_1+a_2s_0 & a_0s_3+\ldots & a_0s_4+\ldots \\ a_0s_1 & a_0s_2+a_1s_1 & a_0s_3+a_1s_2+a_2s_1 & a_0s_4+\ldots & a_0s_5+\ldots \\ a_0s_2 & a_0s_3+a_1s_2 & a_0s_4+a_1s_3+a_2s_2 & a_0s_5+\ldots & a_0s_6+\ldots \end{matrix} \right\|$$

and thence the final form desired.

The question of the existence of multiple or repeated roots in an equation is next taken up (pp. 178–190), the main result being: *The equation* A = 0 *will admit of only* k *distinct roots if the first determinant of each of the last* n−k *bigradient arrays arising from* A *and its derivative vanishes: and this condition being fulfilled, the equation of the said roots will be got by equating to* 0 *the determinant formed by replacing the last column of* D_k *by* k *zeros and the* 0^{th}, 1^{st}, 2^{nd}, . . . , k^{th} *powers of* x. For example, A and its derivate being

$$a^5+x^4-5x^3-x^2+8x+4,$$
$$5x^4+4x^3-15x^2-2x+8,$$

and it having been found that the first determinants of the arrays

$$\left\| \begin{matrix} (1,\ldots,4)_1 \\ (5,\ldots,8)_2 \end{matrix} \right\|, \quad \left\| \begin{matrix} (1,\ldots,4)_2 \\ (5,\ldots,8)_3 \end{matrix} \right\|, \quad \left\| \begin{matrix} (1,\ldots,4)_3 \\ (5,\ldots,8)_4 \end{matrix} \right\|, \quad \left| \begin{matrix} (1,\ldots,4)_4 \\ (5,\ldots,8)_5 \end{matrix} \right|,$$

have the values

$$54, \quad 0, \quad 0, \quad 0,$$

the proposition states that the number of distinct roots of the equation A = 0 is 2, *i.e.* 5−3, and that the equation of these two roots is

$$\begin{vmatrix} 1 & 1 & -5 & -1 & . \\ . & 1 & 1 & -5 & . \\ . & . & 5 & 4 & 1 \\ . & 5 & 4 & -15 & x \\ 5 & 4 & -15 & -2 & x^2 \end{vmatrix} = 0.$$

As a matter of fact, this equation is $54(x+2)(x-1) = 0$, and $A \equiv (x+2)^2(x-1)^3$.

In the next place (pp. 191–196) Trudi takes up Sturm's theorem for determining the number of roots of an equation which lie between two given values. In dealing with it he brings forward a new series of functions as a substitute for Sturm's series, namely,

$$1, \quad \begin{vmatrix} a_0 & a_1 & . \\ . & b_0 & 1 \\ b_0 & b_1 & x \end{vmatrix}, \quad \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & . \\ . & a_0 & a_1 & a_2 & . \\ . & . & b_0 & b_1 & 1 \\ . & b_0 & b_1 & b_2 & x \\ b_0 & b_1 & b_2 & b_3 & x^2 \end{vmatrix}, \quad \ldots\ldots,$$

where $b_0 = na_0$, $b_1 = na_1^{n-1}$, etc.

Further, he points out that the individual members of this series can be lowered in grade by the use of his condensation-theorem, thus providing a variant of the series. He also notes that by means of the theorem which we have extended above into another condensation-theorem they can be transformed into

$$1, \quad a_0^2 \begin{vmatrix} s_0 & 1 \\ s_1 & x \end{vmatrix}, \quad a_0^4 \begin{vmatrix} s_0 & s_1 & 1 \\ s_1 & s_2 & x \\ s_2 & s_3 & x^2 \end{vmatrix}, \quad \ldots\ldots,$$

and so he arrives by a different route at Joachimsthal's series of 1854 (*Hist.*, ii. p. 171).

Lastly, it may be noted that in introducing (p. 234) the discriminant of a quantic as the resultant of the first differential-coefficients of the quantic,* and in showing like Baltzer that for a binary n-thic, $\phi(x, y)$, the discriminant must be the resultant (bigradient or other) of

$$\frac{\partial}{\partial x}\phi(x, y) = 0, \quad n\phi(x, y) - x\frac{\partial}{\partial x}\phi(x, y) = 0,$$

he puts 1 for y and asserts that the discriminant of $\phi(x, y)$ must be identical with the resultant got by eliminating x from

$$\frac{\partial}{\partial x}\phi(x, 1) = 0, \quad n\phi(x, 1) - x\frac{\partial}{\partial x}\phi(x, 1) = 0,$$

and therefore must equal the resultant of

$$\frac{\partial}{\partial x}\phi(x, 1) = 0, \quad \phi(x, 1) = 0;$$

and consequently must be equivalent to the expression which with unfortunate ambiguity had for long† been known as the 'deter-

* So used from 1851 by Sylvester (*Hist.*, ii. pp. 63, 388) and so defined in the glossary at the end of his memoir on Syzygetic Relations (1853).

† In 1801 Gauss spoke of the determinant of a function of two or more variables, $b^2 - 4ac$ being the determinant of $ax^2 + 2bxy + cy^2$ (*Hist.*, i. pp. 64, 65); in 1815 he spoke of the determinant of a function of one variable, $-l'^2 + 4l''$ being the determinant of $x^2 + 2l'x + l''$ (*Werke*, iii. pp. 33–56, § 6). By 1852 we find in use 'the determinant of the equation $ax^2 + 2bxy + cy^2 = 0$' (Salmon's *Higher Plane Curves*, pp. 296–7); and by 1857 the determinant of the equation $f(x) = 0$ (*Hist.*, ii. p. 184).

All through this there was nothing implied in regard to the precise form of the expression so named, which might be and indeed was, even with Gauss, a product of squares of differences. When, however, 'determinant' as thus understood came to be expressed in a particular form which also bore the name 'determinant,' it was clearly imperative to make a change.

minant' of $\phi(x, 1)$. Nevertheless, he does not note that in this there is ample justification for extending the use of the term 'discriminant' to the case of the rational integral function $\phi(x, 1)$, and even in connection with the equation $\phi(x, 1) = 0$.

Trudi's work on bigradients, extending to 94 pages if both *Teoria* and *Applicazioni* be included, has suffered undeserved neglect. Why this should have been the case it is a little difficult to understand, its only demerits being an occasional wordiness, a not very acceptable notation, and a paucity of concrete examples. In his preface (p. vii) he tells us that it was first communicated in a number of papers to the Naples Academy of Sciences in the year 1857. This being so, it was two years in advance of Zeipel's memoir on the same subject (*Hist.*, ii. pp. 370–372) and Bruno's text-book, a fact which it is important for the reader to recall if any small point of similarity between two modes of treatment should attract attention.

SALMON, G. (1866).

[Lessons Introductory to the Modern Higher Algebra. 2nd ed. viii+296. Dublin.]

In a table of resultants (pp. 283–285) the final expansion of R_{25} is given, and the discriminant of

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4.$$

SARDI, C. (1866): RAJOLA, L. (1866): TORELLI, G. (1866).

[Questione 47. *Giornale di Mat.*, iv. pp. 239–240: solution by L. Rajola, iv. p. 297.]

[Teorema sui determinanti a due scale, e soluzione della questione 47. *Giornale di Mat.*, iv. pp. 294–296.]

We have already seen how, from equating two forms of the resultant of a pair of rational integral equations, interesting identities may be obtained (*Hist.*, i. p. 487 at bottom: ii. pp. 369–370, 374–375). Another instance is here reached, the forms of eliminant used being Sylvester's bigradient and the eliminant which arises from successively substituting the roots of one of the equations in

the non-zero member of the other equation and taking the product of the resulting expressions. If in connection with the latter we make use of Spottiswoode's determinant expression (*Hist.*, ii. p. 111) for such a non-zero member, the identity evolved will be purely and almost alarmingly determinantal.

BALTZER, R. (1864, 1870, 1875).

[THEORIE UND ANWENDUNG DER DETERMINANTEN, . . . 2te Aufl. 3te Aufl. 4te Aufl. Leipzig.]

Putting (§ 11. 4)

$$\left.\begin{aligned} \mathrm{A}(x) &\equiv a_0x^m+a_1x^{m-1}+\ldots+a_m \equiv a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_m)\\ \mathrm{B}(x) &\equiv b_0x^n+b_1x^{n-1}+\ldots+b_n \equiv b_0(x-\beta_1)(x-\beta_2)\ldots(x-\beta_n)\end{aligned}\right\} m \leqq n$$

and supposing x to be one of the roots of the equation $\mathrm{B}(x) = 0$, Baltzer predicates the n equations

$$\begin{aligned} 0 &= \{a_m-\mathrm{A}(x)\} + a_{m-1}x + a_{m-2}x^2 + \ldots.\\ 0 &= \{a_m-\mathrm{A}(x)\}x + a_{m-1}x^2 + \ldots.\\ 0 &= \{a_m-\mathrm{A}(x)\}x^2 + \ldots.\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

and the m equations

$$\begin{aligned} 0 &= b_n + b_{n-1}x + b_{n-2}x^2 + \ldots.\\ 0 &= b_nx + b_{n-1}x^2 + \ldots.\\ 0 &= b_nx^2 + \ldots.,\end{aligned}$$

and so deduces

$$\begin{vmatrix} a_m-\mathrm{A}(x) & a_{m-1} & a_{m-2} & \ldots.\\ . & a_m-\mathrm{A}(x) & a_{m-2} & \ldots.\\ . & . & a_m-\mathrm{A}(x) & \ldots.\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\ b_n & b_{n-1} & b_{n-1} & \ldots.\\ . & b_n & b_{n-1} & \ldots.\\ . & . & b_n & \ldots.\\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \end{vmatrix}_{n+m} = 0,$$

which must thus be the equation in $\mathrm{A}(x)$ whose roots are $\mathrm{A}(\beta_1)$, $\mathrm{A}(\beta_2)$, . . . , $\mathrm{A}(\beta_n)$. Since the coefficient of the highest power of $\mathrm{A}(x)$ in it is $(-1)^n{b_0}^m$, it follows that

$$(-1)^n b_0^m \cdot \mathrm{A}(\beta_1)\mathrm{A}(\beta_2) \ldots \mathrm{A}(\beta_n) = \begin{vmatrix} a_m & a_{m-1} & a_{m-2} & \ldots \\ . & a_m & a_{m-1} & \ldots \\ . & . & a_m & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ b_n & b_{n-1} & b_{n-2} & \ldots \\ . & b_n & b_{n-1} & \ldots \\ . & . & b_n & \ldots \end{vmatrix}_{n+m,}$$

as Hesse in 1858 had shown by direct transformation.

The bigradient form of resultant is also used (§ 11. 7) to show that when A and B are of the same degree

$$\text{resultant (A, B}+\lambda\text{A)} = \text{resultant (A, B)}.$$

A fresh proof is given of Jacobi's theorem* that *if* ϕ *be a given function of the* $(\mathrm{m}+\mathrm{n}-1)^{th}$ *degree in* x, *it is possible to determine two functions* u, v *of the* $(\mathrm{n}-1)^{th}$, $(\mathrm{m}-1)^{th}$ *degrees so as to have*

$$\mathrm{uA}+\mathrm{vB} = \mathrm{S}\phi,$$

where S *is Sylvester's bigradient.* This consists simply in taking the $1+n+m$ equations

$$\left.\begin{array}{rcccccccc} \phi &=& c_{m+n-1} &+& c_{m+n-2}x &+& c_{m+n-3}x^2 &+& \ldots \\ \mathrm{A} &=& a_m &+& a_{m-1}x &+& a_{m-2}x^2 &+& \ldots \\ x\mathrm{A} &=& & & a_m x &+& a_{m-1}x^2 &+& \ldots \\ x^2\mathrm{A} &=& & & & & a_m x^2 &+& \ldots \\ \ldots & & \ldots & & \ldots & & \ldots & & \ldots \\ \mathrm{B} &=& b_n &+& b_{n-1}x &+& b_{n-2}x^2 &+& \ldots \\ x\mathrm{B} &=& & & b_n x &+& b_{n-1}x^2 &+& \ldots \\ \ldots & & \ldots & & \ldots & & \ldots & & \ldots \end{array}\right\}$$

and deducing

$$\begin{vmatrix} \phi & c_{m+n-1} & c_{m+n-2} & c_{m+n-3} & \ldots \\ \mathrm{A} & a_m & a_{m-1} & a_{m-2} & \ldots \\ x\mathrm{A} & . & a_m & a_{m-1} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathrm{B} & b_n & b_{n-1}x & b_{n-2} & \ldots \\ x\mathrm{B} & . & b_n & b_{n-1} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_{1+n+m} = 0.$$

* *Crelle's Journ.*, xv. (1835), p. 108, where however $m = n$.

Jacobi's theorem of 1835 regarding Bezout's condensed eliminant suggests the similar theorem regarding the bigradient eliminant,* namely, *if* w *be a common root of the equations*

$$a_0x^m + a_1x^{m-1} + \ldots = 0, \qquad b_0x^n + b_1x^{n-1} + \ldots = 0,$$

then the signed primary minors associated with any row of

$$\begin{vmatrix} (a_0, \ldots, a_m)_n \\ (b_0, \ldots, b_n)_m \end{vmatrix}$$

are proportional to

$$w^{m+n-1},\ w^{m+n-2},\ \ldots,\ w,\ 1.$$

In dealing with the highest-common-factor of A and B and with the subject of elimination, Baltzer profits far less than he ought to have done from the work of Trudi, whom indeed he does not mention.

In conclusion, it is just worth noting that he now unreservedly adopts the name 'discriminant,' but, strangely enough, introduces it at first (§ 11. 19) not in its quite general sense, but in connection with the bigradient and other forms of the resultant of $f(x)$ and $f'(x)$.

ISÈ, E. (1873): JANNI, V. (1874).

[Sul grado della risultante. *Giornale di Mat.*, xi. p. 253.]

[Sul grado dell' eliminante del sistema di due equazioni. *Giornale di Mat.*, xii. p. 27.]

The bigradient form of eliminant is here used in the establishing of the proposition that *if the coefficients* a_r, b_r, *be functions of the* r^{th} *degree in one and the same variable* y, *the eliminant is of the* $(mn)^{th}$ *degree in the same variable.* Janni's proof, though not quite so good as it might have been, is the more interesting. The eliminant being

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & . \\ . & a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & . & . \\ . & b_0 & b_1 & b_2 & . \\ . & . & b_0 & b_1 & b_2 \end{vmatrix},$$

* Gordan (1870), in quoting the two from Baltzer, says that *mn* of the primary minors of the former eliminant are secondary minors of the latter. (*Math. Annalen*, iii. p. 356.)

he, in effect, multiplies the columns in reverse order by y^0, y^1, y^2, y^3, y^4 respectively, and then divides the rows in order by y^4, y^3, y^2, y^1, y^0 respectively, thus obtaining

$$\begin{vmatrix} a_0 & a_1y^{-1} & a_2y^{-2} & a_3y^{-3} & . \\ . & a_0 & a_1y^{-1} & a_2y^{-2} & a_3y^{-3} \\ b_0y^2 & b_1y & b_2 & . & . \\ . & b_0y^2 & b_1y & b_2 & . \\ . & . & b_0y^2 & b_1y & b_2 \end{vmatrix}.$$

In this equivalent form the elements of the first two rows are all now of the degree 0 in y, and those of the last three rows are all of the degree 2, whence comes at once the desired result.

It should be noted that the procedure shows *each term* to be of the $(mn)^{\text{th}}$ degree in y; in other words, that the eliminant is homogeneous. Also, dispensing in the end with y, we may deduce the isobarism of the eliminant, its weight being mn.

BJÖRLING, C. F. E. (1873): ZEUTHEN, H. G. (1874): GARBIERI, G. (1874): MADSEN, V. H. O. (1875).

[Sur les relations qui doivent exister entre les coëfficients d'un polynôme $F(x)$ pour qu'il contienne un facteur de la forme $x^n - a^n$. *Archiv d. Math. u. Phys.*, lv. pp. 429–440.]

[En Bemærking om Beviserne for Hovedsœtningen om Elimination mellem to algebraiske Ligninger. *Tidsskrift for Math.* (3), iv. pp. 165–171.]

[Determinanti, con . . . . xiii+267 pp. Bologna.]

[En Bemærking om Sylvesters dialytiske Eliminationsmethode. *Tidsskrift for Math.* (3), v. pp. 144–145.]

All these deal with the subject of common roots, Björling incidentally (§§ 3–6, pp. 431–437) and the others of set purpose.

Zeuthen repeats Salmon's mode of 1859 (*Hist.*, ii. pp. 373–374) of using Euler's treatment of two integral equations in x which have more than one common root: he is, however, more detailed, and takes the number of roots to be p.

Garbieri also repeats Salmon, taking indeed the same example, but he is careful to add a proof (pp. 120–121), that of the six conditions there arrived at only two are independent.

LEMONNIER, H. (1875, 1878).

[Théorèmes concernant les équations qui ont des racines communes. *Comptes Rendus . . . . Acad. des Sci.* (Paris), lxxx. pp. 111–112, 252–255.]

[Mémoire sur l'élimination. *Annales de l'École Norm. Sup.* (2), vii. pp. 77–96, 151–214.]

Lemonnier's condition for the equations

$$a_0x^m + \ldots + a_m = 0, \qquad b_0x^n + \ldots + b_n = 0$$

having k common roots is different from Trudi's, but fortunately for comparison is very easily expressed in Trudi's notation. It is* that *the first* k *determinants of*

$$\left\| \begin{matrix} (a_0, \ldots, a_m)_{n-k+1} \\ (b_0, \ldots, b_n)_{m-k+1} \end{matrix} \right\|$$

shall vanish, and the first determinant of

$$\left\| \begin{matrix} (a_0, \ldots, a_m)_{n-k} \\ (b_0, \ldots, b_n)_{m-k} \end{matrix} \right\|$$

shall not vanish. The former part of the condition recalls Zeipel's of 1859: the latter is an important necessary adjunct. When, however, the equation of the common roots

$$\left\| \begin{matrix} (a_0, \ldots, a_m)_{n-k} \\ (b_0, \ldots, b_n)_{m-k} \end{matrix} \right\| (x^k, x^{k-1}, \ldots, x^0) = 0$$

happens to be given along with the condition, it is less necessary to mention the latter part, as the determinant involved is the coefficient of x^k in the said equation.

The whole memoir is valuable, viewed either as an exposition or as a repository.

DARBOUX, G. (1876, 1877).

[Sur la théorie l'élimination entre deux équations à une inconnue. *Bull. des Sci. Math.* x. (1), pp. 56–64.]

[Sur l'élimination entre deux équations algébriques à une inconnue. *Bull. des Sci. Math.* (2), i. (1), pp. 54–64.]

Darboux uses Bezout's eliminant in the extended form given it by Cauchy for the case of unequal degrees: and he establishes the

* This is in accordance with the statement in § 13 of the complete memoir, and is somewhat different from that first published.

theorem that *the necessary and sufficient condition for the existence of* k *common roots is the vanishing of all the minors of the said eliminant down to but excluding those of the* $(n-k)^{th}$ *order,* n *being the higher of the two degree-numbers.* The proof given in the second paper is especially interesting.

It is also worth noting that Darboux gives at the end a mode of directly transforming the eliminant into Euler's product of differences.

MUIR, T. (1876).

[New general formulæ for the transformation of infinite series into continued fractions. *Transac. R. Soc. Edinburgh*, xxvii. pp. 467–471.]

[On the transformation of Gauss' hypergeometric series into a continued fraction. *Proc. London Math. Soc.*, vii. 112–118.]

The fundamental theorem, which is established in two different ways, is not essentially different from Heilermann's of 1845 (*Hist.*, ii. p. 361). The second of the two ways is the more interesting. Beginning with the series

$$a_0+a_1x+a_2x^2+a_3x^3+\ldots, \quad \text{or} \quad f_0,$$
$$b_0+b_1x+b_2x^2+b_3x^3+\ldots, \quad \text{or} \quad f_1,$$

and subtracting b_0 times the first from a_0 times the second, and dividing the result by x, we obtain

$$\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} + \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} x + \begin{vmatrix} a_0 & a_3 \\ b_0 & b_3 \end{vmatrix} x^2 + \ldots, \quad \text{or} \quad f_2 \text{ say};$$

and by subtracting $|a_0b_1|$ times the second from b_0 times this third series and dividing by x, there results

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ . & b_0 & b_1 \\ b_0 & b_1 & b_2 \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & a_3 \\ . & b_0 & b_2 \\ b_0 & b_1 & b_3 \end{vmatrix} x + \begin{vmatrix} a_0 & a_1 & a_4 \\ . & b_0 & b_3 \\ b_0 & b_1 & b_4 \end{vmatrix} x^2 + \ldots, \quad \text{or} \quad f_3 \text{ say};$$

and so on. The outcome is

$$\frac{a_0+a_1x+a_2x^2+\ldots}{b_0+b_1x+b_2x^2+\ldots} = \frac{\theta_1}{\theta_0} - \frac{\theta_2 x}{\theta_1} - \frac{\theta_3 x}{\theta_2} - \frac{\theta_1\theta_4 x}{\theta_3} - \ldots$$

where $\theta_0, \theta_1, \theta_2, \ldots$ are the first terms of $f_0, f_1, f_2 \ldots$ respectively.

VENTÉJOLS, . (1877).

[Sur un problème comprenant la théorie de l'élimination. *Comptes Rendus . . . Acad. des Sci.* (Paris), lxxxiv. pp. 546–549.]

Ventéjols' subject would have been much better described by Lemonnier's title of 1875. In substance nothing fresh is brought forward.

DICKSON, J. D. H. (1877).

[A class of determinants. *Transac. R. Soc. Edinburgh*, xxviii. pp. 625–631.]

[The numerical calculation of a class of determinants, and a continued fraction. *Proc. London Math. Soc.*, x. pp. 226–228.]

The determinants here considered are the bigradients dealt with by Heilermann (1845) and Muir (1876). They also arise in the same connection.

MANSION, P. (1878).

[Sur l'élimination. *Bulletin . . . Acad. . . . de Belgique*, xlvi. pp. 899–903.]

What is interesting here is Mansion's mode of obtaining the evanescent bigradient array that results from the existence of common roots. The equations being

$$\mathrm{A}(x) \equiv a_0x^5 + \ldots + a_5 = 0, \qquad \mathrm{B}(x) \equiv b_0x^4 + \ldots + b_4 = 0$$

and the common roots α, β, γ, it follows that

$$\begin{vmatrix} \alpha^m & \alpha^n & \mathrm{A}(\alpha) \\ \beta^m & \beta^n & \mathrm{A}(\beta) \\ \gamma^m & \gamma^n & \mathrm{A}(\gamma) \end{vmatrix}, \quad \begin{vmatrix} \alpha^m & \alpha^n & \alpha\mathrm{A}(\alpha) \\ \beta^m & \beta^n & \beta\mathrm{A}(\beta) \\ \gamma^m & \gamma^n & \gamma\mathrm{A}(\gamma) \end{vmatrix}, \quad \begin{vmatrix} \alpha^m & \alpha^n & \mathrm{B}(\alpha) \\ \beta^m & \beta^n & \mathrm{B}(\beta) \\ \gamma^m & \gamma^n & \mathrm{B}(\gamma) \end{vmatrix},$$

$$\begin{vmatrix} \alpha^m & \alpha^n & \alpha\mathrm{B}(\alpha) \\ \beta^m & \beta^n & \beta\mathrm{B}(\beta) \\ \gamma^m & \gamma^n & \gamma\mathrm{B}(\gamma) \end{vmatrix}, \quad \begin{vmatrix} \alpha^m & \alpha^n & \alpha^2\mathrm{B}(\alpha) \\ \beta^m & \beta^n & \beta^2\mathrm{B}(\beta) \\ \gamma^m & \gamma^n & \gamma^2\mathrm{B}(\gamma) \end{vmatrix}$$

are all equal to 0; so that, if we temporarily write

$$(m,\ n,\ p) \quad \text{for the alternant} \quad |\alpha^m\beta^n\gamma^p|,$$

we have

$$\begin{aligned}
&(mn5)a_0 + (mn4)a_1 + (mn3)a_2 + (mn2)a_3 + (mn1)a_4 + (mn0)a_5 = 0\\
n6)a_0 + &(mn5)a_1 + (mn4)a_2 + (mn3)a_3 + (mn2)a_4 + (mn1)a_5 = 0\\
&(mn4)b_0 + (mn3)b_1 + (mn2)b_2 + (mn1)b_3 + (mn0)b_4 = 0\\
&(mn5)b_0 + (mn4)b_1 + (mn3)b_2 + (mn2)b_3 + (mn1)b_4 = 0\\
n6)b_0 + &(mn5)b_1 + (mn4)b_2 + (mn3)b_3 + (mn2)b_4 = 0.
\end{aligned}$$

Here, however, by taking any two of the numbers 0, 1, 2, 3, 4, 5, 6 as values for m, n, two of the alternants will disappear, and we shall be able to eliminate the five others, the final and complete result thus being in Cayley's notation,

$$\left\| \begin{matrix} . & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & . \\ . & . & b_0 & b_1 & b_2 & b_3 & b_4 \\ . & b_0 & b_1 & b_2 & b_3 & b_4 & . \\ b_0 & b_1 & b_2 & b_3 & b_4 & . & . \end{matrix} \right| = 0.$$

For example, putting m, n equal to 0, 1, then equal to 0, 2, and finally equal to 1, 2, we should have the particular three results, which, in accordance with our usage under Trudi, we might write

$$\left\| \begin{matrix} (a_0, \ldots, a_5)_2 \\ (b_0, \ldots, b_4)_3 \end{matrix} \right\| = 0.$$

All this, however, is considerably modified from Mansion's exposition.

MALET, J. C. (1878).

[On a problem in algebra. *Annali di Mat.* (2), ix. pp. 306–313.]

Malet first ascertains the conditions to be satisfied by the coefficients in order that two equations of the n^{th} degree in x shall be such that to every root r in the one there is a root c/r in the other. He then proceeds to show that the equation of the $(mn)^{\text{th}}$ degree, whose every root is the product of a root of the equation

$$x^m - a_1 x^{m-1} + a_2 x^{m-2} - \ldots = 0,$$

and a root of the equation

$$x^n - b_1 x^{n-1} + b_2 x^{n-2} - \ldots = 0,$$

is

$$0 = \begin{vmatrix} b_{n-1}x^m - a_1b_nx^{m-1} & b_n & . & \dots & & & & & & & \\ b_{n-2}x^m - a_2b_nx^{m-2} & b_{n-1} & b_n & \dots & & & & & & & \\ b_{n-3}x^m - a_3b_nx^{m-3} & b_{n-2} & b_{n-1} & \dots & & & & & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & a_{m-1}x & a_{m-2}x^2 & a_{m-3}x^3 - b_3a_m \\ \dots & \dots & \dots & \dots & a_m & a_{m-1}x & a_{m-2}x^2 - b_2a_m \\ \dots & \dots & \dots & \dots & . & a_m & a_{m-1}x - b_1a_m \end{vmatrix},$$

where, after the first column of binomial elements, there follow $m-2$ columns having b_n for their first non-zero element, and in front of the last column of binomial elements there are $n-2$ columns having a_m for their last non-zero element. By mere change of b_n, b_{n-1}, ... into $1/b_n$, b_1/b_n, ... there is readily obtained a companion equation whose roots are quotients instead of products of the roots of the two given equations: and then, on putting x in this equal to 1 there emerges the eliminant of the said pair of equations. For example, the given equations being

$$\left.\begin{aligned} x^3 - a_1x^2 + a_2x - a_3 &= 0, \\ x^3 - b_1x^2 + b_2x - b_3 &= 0, \end{aligned}\right\}$$

the equation whose roots are the roots of these mated by multiplication is

$$\begin{vmatrix} b_2x^3 - a_1b_3x^2 & b_3 & x^3 & . \\ b_1x^3 - a_2b_3x & b_2 & a_1x^2 & x^3 - b_3a_3 \\ x^3 - a_3b_3 & b_1 & a_2x & a_1x^2 - b_2a_3 \\ . & 1 & a_3 & a_2x - b_1a_3 \end{vmatrix} = 0;$$

the corresponding equation whose roots are each mated by division is

$$\begin{vmatrix} b_1x^3 - a_1x^2 & 1 & x^3 & . \\ b_2x^3 - a_2x & b_1 & a_1x^2 & b_3x^3 - a_3 \\ b_3x^3 - a_3 & b_2 & a_2x & a_1b_3x^2 - a_3b_1 \\ . & b_3 & a_3 & a_2b_3x - a_3b_2 \end{vmatrix} = 0;$$

and the eliminant of the given equations is

$$\begin{vmatrix} b_1 - a_1 & 1 & 1 & \\ b_2 - a_2 & b_1 & a_1 & b_3 - a_3 \\ b_3 - a_3 & b_2 & a_2 & a_1b_2 - a_2b_1 \\ . & b_3 & a_3 & a_2b_3 - a_3b_2 \end{vmatrix} = 0.$$

Of these results it is the last that our subject requires us to be most interested in. Malet himself notes it as involving a simpler determinant than that usually given. It is necessary to point out, however, that the said determinant is easily derivable from Sylvester's bigradient

$$\begin{vmatrix} 1 & -a_1 & a_2 & -a_3 & . & . \\ . & 1 & -a_1 & a_2 & -a_3 & . \\ . & . & 1 & -a_1 & a_2 & -a_3 \\ . & . & 1 & -b_1 & b_2 & -b_3 \\ . & 1 & -b_1 & b_2 & -b_3 & . \\ 1 & -b_1 & b_2 & -b_3 & . & . \end{vmatrix}.$$

For, performing on the latter the operations

$$\text{row}_5 - \text{row}_2, \qquad \text{row}_6 - \text{row}_1 - (a_1 - b_1)\,\text{row}_2,$$

we obtain

$$\begin{vmatrix} 1 & -a_1 & a_2 & -a_3 \\ 1 & -b_1 & b_2 & -b_3 \\ a_1 - b_1 & b_2 - a_2 & a_3 - b_3 & . \\ b_2 - a_2 + a_1^2 - a_1 b_1 & a_3 - b_3 - a_1 a_2 + a_2 b_1 & a_3 a_1 - a_3 b_1 & . \end{vmatrix},$$

and the further operation

$$\text{row}_4 - a_1\,\text{row}_3 + a_2\,\text{row}_2 - b_2\,\text{row}_1$$

gives us substantially Malet's form.

GÜNTHER, S. (1879): PAIGE, C. LE (1880).

[Eine Relation zwischen Potenzen und Determinanten. *Zeitschrift f. Math. u. Phys.*, xxiv. pp. 244–248.]

[Question 566. *Nouv. Corresp. Math.*, vi. p. 333: Solution by C. Le Paige, pp. 382–383.]

The subject here, proposed by H. Brocard, is simply the evaluation of the bigradient which is the discriminant of $(x^{m+2}-1)/(x-1)$, the result being $(m+2)^m$.

For example, when m is 2,

$$\begin{vmatrix} 1 & 1 & 1 & 1 & . \\ . & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & . & . \\ . & 1 & 2 & 3 & . \\ . & . & 1 & 2 & 3 \end{vmatrix} = 4^2.$$

Günther's proof is unnecessarily lengthy. The determinant can be readily transformed into one whose diagonal has for its elements 1 repeated $m+1$ times and $m+2$ repeated m times, and whose other terms all vanish. For example, when m is 2, the requisite operations are

$$\text{row}_5 - \text{row}_4 + \text{row}_2,$$
$$\text{row}_4 - \text{row}_3 + \text{row}_1,$$
$$\text{row}_3 - \text{row}_2 - \text{row}_1.$$

Le Paige finds the resultant of the two implied equations without the help of determinants.

MANSION, P. (1879).

[On the equality of Sylvester's and Cauchy's eliminants. *Messenger of Math.*, ix. pp. 60–63.]

Mansion's proof, though very unlike, is not essentially different from the process of applying Trudi's condensation-theorem to Sylvester's bigradient. The additional fact, to which Mansion draws attention, namely, that many minors of the one eliminant have equivalents among the minors of the other, is also virtually included in Trudi. Thus, the four identities which Mansion indicates in the form (see his fig. 11)

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & . & . \\ . & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & . \\ . & . & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ . & . & . & . & . & b_0 & b_1 & b_2 & b_3 \\ . & . & . & . & b_0 & b_1 & b_2 & b_3 & . \\ . & . & . & b_0 & b_1 & b_2 & b_3 & . & . \\ . & . & b_0 & b_1 & b_2 & b_3 & . & . & . \\ . & b_0 & b_1 & b_2 & b_3 & . & . & . & . \\ b_0 & b_1 & b_2 & b_3 & . & . & . & . & . \end{vmatrix} = \begin{vmatrix} . & . & b_0 & b_1 & b_2 & b_3 \\ . & b_0 & b_1 & b_2 & b_3 & . \\ b_0 & b_1 & b_2 & b_3 & . & . \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 & \mu_6 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 & \nu_6 \end{vmatrix}$$

where the λ's, μ's, ν's stand for

$$\begin{array}{ccccccc}
a_0b_2+a_1b_1 & a_0b_3+a_1b_2+a_2b_1 & a_1b_3+a_2b_2+a_3b_1-a_4b_0 & a_2b_3+a_3b_2-a_5b_0 & a_3b_3-a_6b_0 \\
a_0b_3+a_1b_2 & a_1b_3+a_2b_2 & a_2b_3+a_3b_2+a_5b_0 & a_3b_3-a_6b_0+a_4b_2-a_5b_1 & a_4b_3-a_6b_1 \\
a_1b_3 & a_2b_3 & a_3b_3-a_6b_0 & a_4b_3-a_6b_1 & a_5b_3-a_6b_2
\end{array}$$

are only four of the ten noted by Trudi, the others being excluded, so to speak, by drawing three vertical dotted lines on the right of each determinant instead of continuing the horizontal dotted lines all the way towards the right.

From the foregoing there is probably no serious omission of papers dealing directly with bigradients, and in particular with the bigradient eliminant. But, as it is possible to study the subject of the common roots of two integral equations without direct reference to bigradients, and as the other determinants that may then be used can generally be transformed into bigradients, it will doubtless be of service to the student of determinants to give the following list of titles of papers on elimination. When taken together with the preceding papers on the same subject they will also be helpful to the student of the theory of equations :

1870. Gordan, P. Ueber die Bildung der Resultante zweier Gleichungen. *Math. Annalen*, iii. pp. 355–414.

1872. Naegelsbach, H. Ueber die Resultante zweier ganzen Functionen. *Zeitschrift f. Math. u. Phys.*, xvii. pp. 333–346.

1876. Darboux, G. Sur la théorie de l'élimination entre deux équations à une variable. *Bull. des Sci. Math.*, x. pp. 56–64 ; (2) i. pp. 54–64.

1877. Rouché, E. Sur l'élimination. *Nouv. Annales de Math.*, (2) xvi. pp. 105–113.

1877. Igel, B. Einige Sätze und Beweise zur Theorie der Resultante. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), lxxvi. pp. 145–168.

1877. Forestier, C. Exposition succincte de quelques méthodes d'élimination entre deux équations. *Mém. de l'Acad. des Sci.* (Toulouse), (7) ix. pp. 142–163.

1878. Baraniecki, M. A. On the determination of the common roots . . . (In Polish). *Pamiętnik Towarzystwa Nauk Ścisłych* (Paryż), x.

1879. Söderblom, A. Om algebr. equationer och equationscurver. 64 pp. Upsala.

1879. Biehler, C. Sur la théorie des équations. Dissert. 60 pp. Paris.

1879. Falk, M. Sur la méthode de l'élimination de Bezout et Cauchy. *Nova Acta Reg. Soc.* (Upsala), x. No. 15, 36 pp.

1879. Hioux, V. Note sur la méthode d'élimination Bezout-Cauchy. *Nouv. Annales de Math.*, (2) xviii. pp. 289–295.

1879. Mansion, P. Sur l'élimination. *Bull. . . . Acad. . . . Belgique*, (2) xlvii. pp. 532–541 ; xlviii. pp. 463–472, 473–490, 491–526. Also separately, 64 pp., Paris, 1884.

The papers of Falk and Mansion devote some little space to reviewing the work of their predecessors, and are therefore additionally helpful. They do not, however, mention Trudi, nor indeed does any one of the other writers of that period.

It may be noted as a significant fact in connection with the history of the subject that in 1876 the editors of the *Nouvelles Annales* found themselves called on to republish Cauchy's important paper of 1840 (see *Hist.*, i. pp. 240–243). Its reprint occupies pp. 385–416, 433–451 of vol. xv. of the second series. On this account, save for junior readers, the 'exposition succincte' above noted was quite unnecessary.

CHAPTER XIV.

HESSIANS, FROM 1862 TO 1879.

CAYLEY, A. (1862).

[On certain developable surfaces. *Quart. Journ. of Math.*, vi. pp. 108–126; or *Collected Math. Papers*, v. pp. 267–283.]

The matter here dealt with connects itself with Hessians through the fact that if $u = 0$ be the equation of a developable surface, $H(u)$ contains u as a factor. This fact also accounts for the introduction of the term 'pro-hessian,' the pro-hessian of u being defined as $H(u) \div u$. The paper contains the calculation of the Hessians and pro-hessians of the discriminants of

$$(a, b, c, d\not\!\!\not t, 1)^3,$$
$$(a, 2b, 3c, 0, -27e\not\!\!\not t, 1)^4,$$
$$(a, b, 0, d, e\not\!\!\not t, 1)^4.$$

Thus, in the case of the first of the three, as we already know from Cayley (*Hist.*, ii. p. 382), the discriminant is its own pro-hessian.

SALMON, G., AND FIEDLER, W. (1863).

[Vorlesungen zur Einführung in die Algebra der linearen Transformationen. viii+271 pp. Leipzig.]

At the close of Lesson VI. of the original Fiedler inserts on his own responsibility a very full account of Hessians (§§ 57–60, pp. 98–110). Unfortunately he reproduces Hesse's faulty evidence in

support of the doubtful proposition regarding the vanishing of the Hessian of a function; namely, that such evanescence necessitates among the first differential-coefficients of the function the existence of a linear relation with constant coefficients, and that therefore through a linear transformation it is possible to express the function in terms of one variable fewer.

In the second edition published in 1877 the account does not appear, and in a note (no. 37) the real facts are shortly stated and reference made to Gordan and Noether (see below, pp. 368–9).

TRUDI, N. (1862): BALTZER, R. (1864, -70, -75).

[Teoria dei Determinanti, ... ix+268 pp. Napoli.]

[Theorie und Anwendung der Determinanten. 2te Aufl. vi+224 pp.; 3te Aufl. vi+241 pp.; 4te Aufl. viii+236 pp. Leipzig.]

Trudi's treatment (p. 205, § 10; p. 208, § 3; pp. 218–228, §§ 1–9) is full, like Fiedler's, but is not in the same way faulty.

The successive editions of Baltzer show a number of changes, but as a rule they are merely for the improvement of the exposition.

In speaking formerly, under Baltzer (*Hist.*, ii. p. 399), of the vanishing determinant,

$$\begin{vmatrix} \frac{m}{m-1}u & u_1 & u_2 & \cdots & u_n \\ u_1 & u_{11} & u_{12} & \cdots & u_{1n} \\ u_2 & u_{21} & u_{22} & \cdots & u_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_n & u_{n1} & u_{n2} & \cdots & u_{nn} \end{vmatrix},$$

we might have noted that because of evanescence and axisymmetry every element of its adjugate will vanish if one of them vanishes, and that in particular this is true when the latter is in the $(1.1)^{\text{th}}$ place and is thus Δ—a fact which is naturally corroborated by the adjugate being known from (α) and (β) to have Δ for a factor of every one of its elements. It is there also seen that the said adjugate

$$= \frac{\Delta^{n+1}}{(1-m^2)^n} \begin{vmatrix} 1 & x_1 & x_2 & \dots & x_n \\ x_1 & x_1^2 & x_1x_2 & \dots & x_1x_n \\ x_2 & x_2x_1 & x_2^2 & \dots & x_2x_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_n & x_nx_1 & x_nx_2 & \dots & x_n^2 \end{vmatrix},$$

which, as it ought, manifestly vanishes, whatever Δ may be.*

STUDNIČKA, F. J. (1868).

[Ueber die Anwendung der Hesse'schen Determinanten in der Theorie der Maxima und Minima von Functionen mehrerer unabhängigen Variablen. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), Jahrg. 1868, pp. 67–72.]

As is known, from Lagrange (1797),† the expression whose sign decides whether an extreme value of

$$f(x_1,\ x_2,\ \dots,\ x_n)$$

is a maximum or minimum is

$$h_1^2 f_{11} + 2h_1h_2f_{12} + \dots + h_n^2 f_{n,n},$$

where the h's are infinitesimal increments of the x's, and

$\frac{\partial^2 f}{\partial x_p\, \partial x_q}$ is denoted by $f_{p,q}$.

The origin of Studnička's so-called application lies in the fact that the form of this expression is that of a quadric in the h's with the Hessian

$$|f_{11}\ f_{22}\ \dots\ f_{nn}|$$

for its discriminant. There is nothing fresh in his paper save the use of the determinant notation and of the name 'Hessian.'

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* It may also be noted that an identity, which in this connection we may appropriately write in the form

$$\begin{array}{c|c} x_1\ x_2\ \dots\ x_n & \\ \hline & x_1 \\ \mathrm{H}(u) & x_2 \\ & \vdots \\ & x_n \end{array} = m(m-1)u,$$

is attributed by Baltzer to Lacroix, the reference being '*Calc. diff.*, § 91' at first, and then later to '*Calc. diff.*, § 292.'

† Lagrange, J. L. *Théorie des Fonctions Analytiques*, 2e Partie, chap. iie; or *Œuvres*, ix. pp. 280–295.

GUNDELFINGER, S. (1872, 1876).

[Ueber die Ausartungen einer Curve dritter Ordnung. *Math. Annalen*, iv. pp. 559–572.]

[Intorno ad alcune formole della teoria delle curve di secondo e di terzo ordine. *Annali di Mat.*, (2) v. pp. 223–235.]

The subject here dealt with had already been examined by Sylvester in 1852 (see *Cambr. and Dubl. Math. Journ.*, vii. pp. 187–188). Much of it does not concern us. The most relevant part bears on Sylvester's fifth result, namely, that *a ternary cubic when it is proportional to its Hessian is resolvable into linear factors.* This Gundelfinger proves at length (pp. 227–230).

As an example we may note for ourselves the circulant

$$x^3 + y^3 + z^3 - 3xyz,$$

the Hessian of which is

$$-54(x^3 + y^3 + z^3 - 3xyz).$$

CAYLEY, A. (1872).

[Theorem in regard to the Hessian of a quaternary function. *Quart. Journ. of Math.*, xii. pp. 193–197 ; or *Collected Math. Papers*, ix. pp. 90–93.]

The function in question is

$$P^h + \lambda Q^k,$$

where P, Q are quaternary functions of the same variables, and λ is a constant. The result is lengthy and unattractive, and the demonstration confessedly tedious.

CASORATI, F. (1874).

[Sui determinanti di funzioni. *Mem. del R. Istituto Lombardo*, xiii. pp. 181–187.]

As already explained (chap. ix.), part of Casorati's object in this paper was to ascertain the effect produced on the Hessian by multiplying or dividing the basic function by another function of the same variables. In his quest (§ 2) no theorem analogous to those which hold in the case of the Jacobian, prejacobian and Wronskian

rewarded him. He reached, however, three results of minor importance; namely,

(1) *If an integral function of* n *variables be divisible by the* m^{th} *power of a function of the same variables, the Hessian of the former is divisible by the* $n(m-1)^{th}$ *power of the latter.*

(2) *If a function of* n *variables be divisible by a function of* μ *of these variables, the Hessian of the former is divisible by the* $(n-2\mu)^{th}$ *power of the latter.*

(2′) *If a function of* n *variables be divisible by a linear function of the same variables, the Hessian of the former is divisible by the* $(n-2)^{th}$ *power of the latter.*

The last section (§ 4) is also devoted to the Hessian, namely, to an extension of the theorem regarding its covariance, the number (ν) of new variables being no longer the same as the number (n) in the untransformed function. The case where $\nu < n$ is fully worked out.

PASCH, M. (1874).

[Zur Theorie der Hesse'schen Determinante. *Crelle's Journ.*, lxxx. pp. 169–176.]

Here again the subject is Hesse's contested proposition, the result being to show that the proposition holds in the case of ternary and quaternary cubics.

If u be a ternary cubic and $H(u)$ vanish identically, it is known that

$$\begin{vmatrix} u_{11} & u_{12} & u_{13} & u_1 \\ u_{21} & u_{22} & u_{23} & u_2 \\ u_{31} & u_{32} & u_{33} & u_3 \\ u_1 & u_2 & u_3 & . \end{vmatrix} = 0;$$

and the determinant here being axisymmetric, and having the complementary minor of the element in the $(4,4)^{th}$ place equal to 0, it follows that

$$\left(U_{11}^{\frac{1}{2}}u_1 + U_{22}^{\frac{1}{2}}u_2 + U_{33}^{\frac{1}{2}}u_3\right)^2 = 0,$$

where U_{rs} is the cofactor of u_{rs} in $H(u)$. If therefore it could be shown that the cofactors of u_1, u_2, u_3 in this last identity are replaceable by constants, the desired end would be reached.

This is what Pasch accomplishes. He begins with the much more general identity

$$\begin{vmatrix} u_{11} & u_{12} & \cdots & u_{1n} & \sum(y_r u_{1r}) \\ u_{21} & u_{22} & \cdots & u_{2n} & \sum(y_r u_{2r}) \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ u_{n1} & u_{n2} & \cdots & u_{nn} & \sum(y_r u_{nr}) \\ v_1 & v_2 & \cdots & v_n & \cdot \end{vmatrix} = 0,$$

and derives therefrom three others by operating thrice with

$$\xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \cdots + \xi_n \frac{\partial}{\partial x_n}.$$

From this set of four he obtains two other sets by making the substitutions

$$y_1, y_2, \ldots, y_n = x_1, x_2, \ldots, x_n,$$
$$y_1, y_2, \ldots, y_n = \xi_1, \xi_2, \ldots, \xi_n;$$

and then by combining the first, second, third of the third set with the second, third, fourth of the second set he reaches his general results. To bring these into close touch with Hesse's proposition it only remains to put

$$v_1, v_2, \ldots, v_n = u_1, u_2, \ldots, u_n.$$

The paper closes with matter analogous to Gundelfinger's, namely, with the enunciation of the conditions for a ternary cubic being resolvable in various special ways: the quaternary cubic is, however, also now dealt with.

GORDAN, P. (1875): NOETHER, M. (1876).

[Ueber einen Satz von Hesse. *Sitzungsb. . . . Soc. zu Erlangen*, viii. pp. 89–94.]

[Ueber die algebraischen Formen mit identisch verschwindender Hesse'sche Determinante. *Sitzungsb. . . . Soc. zu Erlangen*, viii. pp. 51–56.]

[Ueber die algebraischen Formen, deren Hesse'sche Determinante identisch verschwindet. *Math. Annalen*, x. pp. 547–568.]

In the first of these papers Gordan, going a little farther than Pasch, shows that Hesse's contested proposition holds for *all* ternary

forms; in the second Noether indicates a simpler way by which he and Gordan had been able to include all quaternaries also, and announces that Gordan had convinced himself of the impossibility of further extension. In the third the two writers give a careful and exhaustive exposition of their combined labours.

We may therefore now sum up by saying that the proposition holds only for binary, ternary and quaternary quantics, and that if we prefer to include also n-ary quadrics we have to bear in mind that the Hessian is then practically identical with the discriminant. Sylvester's dictum of 1853 has thus come to be justified.

GRAVELAAR, N. L. W. A. (1877).

[Eene stelling uit de theorie der lineare substitution. *Nieuw Archief v. Wisk.*, iii. pp. 193–202.]

Accepting without question Hesse's pair of theorems of 1851, and apparently being unaware of any subsequent investigations on the subject, Gravelaar enunciates and seeks to establish generalisations of both, the extension taking the direction of a greater reduction than 1 in the number of variables. The direct theorem, it will be remembered, had already received a slight extension of this kind (see *Hist.*, ii. p. 398 at bottom). In the case of the disputed converse theorem he satisfies himself of its truth by concluding too readily, from the axisymmetry of the adjugate of the given vanishing Hessian, that all the primary minors of the Hessian have a common factor of the highest possible degree in the variables, and that consequently all the cofactors are constants.

BARANIECKI, M. A. (1878–9).

[Teorya Wyznaczników. xxiii+595 pp. Paryź.]

To Hessians Baraniecki devotes as many as twenty-two pages (pp. 509–530), giving a full and well-informed exposition.

In the course of this he does not go beyond the binary quantic, in which case he points out that we then have

$$\left.\begin{aligned} u_{11}x_1 + u_{12}x_2 &= (m-1)u_1 \\ u_{21}x_1 + u_{22}x_2 &= (m-1)u_2 \end{aligned}\right\},$$

and that therefore, if the determinant $|u_{11}u_{22}|$ vanish,* we must have

$$\frac{u_{11}}{u_{21}} = \frac{u_{12}}{u_{22}} = \frac{u_1}{u_2},$$

and consequently $c_1u_1 + c_2u_2 = 0.$

By way of illustration he takes

$$u = 8x^3 - 36x^2y + 54xy^2 - 27y^3,$$

and, finding the Hessian to vanish, obtains the relation

$$3u_1 + 2u_2 = 0,$$

and $$u = 8(x - \tfrac{3}{2}y)^3.$$

To the work of Pasch, Gordan, and Noether he merely refers in a footnote.

SYLVESTER, J. J. (1878): SHARP, W. J. C. (1879): ELLIOTT, E. B. (1881).

[Questions 5762, 6077, 6756. *Educ. Times*, xxxii. p. 269, xxxiii. p. 60, xxxiv. p. 172; or *Math. from Educ. Times*, xxxiii. pp. 34–35, 92; xxxix. p. 41.]

If u be a binary n^{thic} it is known that

$$\left.\begin{aligned} x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} &= (n-1)\frac{\partial u}{\partial x}, \\ \text{and} \\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} &= n(n-1)u: \end{aligned}\right\}$$

and from these it readily follows that

$$(n-1)^2\left(\frac{\partial u}{\partial x}\right)^2 - n(n-1)u\frac{\partial^2 u}{\partial x^2} = -y^2\mathrm{H}(u).$$

This is the result numbered 5762 and attributed to Sylvester by Sharp when proving it.† It is, however, merely the simplest case

* It might have been noted before this that *the Hessian of* u could be defined *as the determinant of the set of equations got from applying to the first derivatives of* u *Euler's theorem regarding the differentiation of homogeneous functions.*

† The original 'question' bearing this number was proposed in September 1878, and is quite different from Sylvester's.

($r = 1$) of Brioschi's result of 1854 (*Hist.*, ii. pp. 394–396), which is the subject of 'question 6756.'

Should u have p roots each equal to α, we have only to substitute $(x-\alpha)^p v$ for u, when it is at once found that the Hessian must have $2(p-1)$ roots equal to α, as Sharp states (6077).

SYLVESTER, J. J. (1879).

[Question 6154. *Educ. Times*, xxxii. p. 341; or *Math. from Educ. Times*, xxxiv. pp. 108–109.]

When correctly stated the theorem here given is that if *the roots of a binary quantic be all real and different, the roots of its Hessian will be all imaginary*, and the proofs rest of course on the fact that the Hessian being always negative never changes its sign for any real value of x/y. What is more worth noting, however, is the fact on which one of the proofs is based, namely, that *if* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be the roots of a binary quantic, its Hessian is an arithmetical multiple of*

$$-\sum(\alpha_1-\alpha_2)^2(x-\alpha_3)^2(x-\alpha_4)^2 \ldots (x-\alpha_n)^2.$$

From this also, or from the result 6077 above, there comes the generalisation that *if* m *of the roots of the quantic be identical and all the rest different, the number of imaginary roots in the Hessian is* $2(n-m-1)$.

CHAPTER XV.

CIRCULANTS, FROM 1861 TO 1880.

In addition to the determinants previously classed under this heading there are now placed those in which it is not the elements that circulate but square arrays of elements ; for example,

$$\begin{vmatrix} a_1 & a_2 & a_3 & d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 & e_1 & e_2 & e_3 \\ c_1 & c_2 & c_3 & f_1 & f_2 & f_3 \\ d_1 & d_2 & d_3 & a_1 & a_2 & a_3 \\ e_1 & e_2 & e_3 & b_1 & b_2 & b_3 \\ f_1 & f_2 & f_3 & c_1 & c_2 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & c_1 & c_2 & e_1 & e_2 \\ b_1 & b_2 & d_1 & d_2 & f_1 & f_2 \\ e_1 & e_2 & a_1 & a_2 & c_1 & c_2 \\ f_1 & f_2 & b_1 & b_2 & d_1 & d_2 \\ c_1 & c_2 & e_1 & e_2 & a_1 & a_2 \\ d_1 & d_2 & f_1 & f_2 & b_1 & b_2 \end{vmatrix},$$

the circulating arrays in the former case being those of $|a_1b_2c_3|$, $|d_1e_2f_3|$, and in the latter those of $|a_1b_2|$, $|c_1d_2|$, $|e_1f_2|$. Probably the first example of such ' block ' circulants to appear was

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix},$$

spoken of originally as biaxisymmetric : its circulating arrays are themselves those of circulants, namely, of C(a, b), C(c, d), and consequently its rows all contain the same elements (*Hist.*, ii. pp. 145–146).

ROBERTS, M. (1861).

[Question 581. *Nouv. Annales de Math.*, xx. p. 139. Solution by E. Beltrami in (2) iii. (1864, February), pp. 64–66.]

Roberts' theorem concerns the circulant C whose elements are the terms of the expansion of

$$\frac{(t+1)^n-1}{t},$$

and is to the effect (1) that there are no odd powers of t in the development of the circulant, and (2) that if in the said development ξ be put for t^2, the equation in ξ

$$C_{t^2=\xi} = 0$$

has for its roots the squares of the differences of the roots of the equation

$$x^n-1 = 0.$$

If $\omega_1, \omega_2, \ldots$ be the n^{th} roots of 1 we have identically

$$(t-\omega_1)(t-\omega_2)\ldots(t-\omega_n) = t^n-1,$$

and therefore also

$$\{t-(\omega_1-\omega_r)\}\{t-(\omega_2-\omega_r)\}\ldots\{t-(\omega_n-\omega_r)\} \equiv (t+\omega_r)^n-1,$$

where the r^{th} factor on the left is simply t itself. Hence the expression

$$\frac{(t+\omega_1)^n-1}{t}\cdot\frac{(t+\omega_2)^n-1}{t}\cdots\frac{(t+\omega_n)^n-1}{t}$$

consists of $n(n-1)$ factors, which, if suitably combined in pairs, are replaceable by $\frac{1}{2}n(n-1)$ factors of the form $t^2-(\omega_r-\omega_s)^2$. But the said expression being equal to C by Spottiswoode's theorem (which, however, Beltrami does not assume*) the desired result at once follows.

ZEHFUSS, G. (1862).

[Anwendungen einer besonderen Determinante. *Zeitschrift f. Math. u. Phys.*, vii. pp. 439–445.]

Zehfuss proves Spottiswoode's result by multiplying the rows in order by $\theta_r^n, \theta_r^{n-1}, \ldots, \theta_r$ respectively, and the columns in order by $1, \theta_r, \theta_r^2, \ldots, \theta_r^{n-1}$, a procedure which amounts to multiplying the

* For his mode of proof see under Baltzer (1864).

determinant by $\theta_r^n(1 \cdot \theta_r \cdot \theta_r^2 \ldots \theta_r^{n-1})^2$, that is, by 1. Addition of the rows is then all that is wanted to reach the desired result. He notes the special case $C(x, y, 0, 0, \ldots)_n = x^n + y^n$.

The rest of the article (pp. 441–445) is devoted to the "Anwendungen," namely, (1) to Eisenstein's expression

$$x^3 + DD'y^3 + DD''z^3 - 3Dxyz,$$

where the letters denote complex numbers and $D = D'D''$; and (2) to a letter of Jacobi's on cyclotomy and the theory of integers.

BALTZER, R. (1864).

[Theorie und Anwendung der Determinanten. . . . . 2te vermehrte Aufl. viii+224 pp. Leipzig.]

With Baltzer (§ 11, $_{1,2,3}$) the determinant

$$\begin{vmatrix} a_0-y & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0-y & a_1 & \ldots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0-y & \ldots & a_{n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & a_2 & a_3 & \ldots & a_0-y \end{vmatrix},$$

or $C(a_0-y, a_1, a_2, \ldots, a_{n-1})$ say,

—which, of course, is not more general than

$$C(a_0, a_1, a_2, \ldots, a_{n-1}),$$

—is openly reached (p. 92) by eliminating x dialytically* from the equations

$$y = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} \quad \text{and} \quad 1 = x^n.$$

* Namely, by using on the first equation the multipliers x, $x^2, \ldots, x_{n-1}$, and substituting 1 for x^n wherever the latter turns up, exactly as Beltrami did. The same result, however, is reached by following Bezout's 'abridged method.' On the other hand, the application of Sylvester's dialytic method unmodified entails, as we know, the performance of multiplication on *both* equations, and gives an eliminant of a higher order. For example, in the case of $n = 3$ it gives

$$\begin{vmatrix} a_2 & a_1 & a_0 & . & . \\ . & a_2 & a_1 & a_0 & . \\ . & . & a_2 & a_1 & a_0 \\ 1 & . & . & -1 & . \\ . & 1 & . & . & -1 \end{vmatrix},$$

where we have to increase the 4th column by the 1st, and the 5th by the 2nd, before we can reach $C(a_0, a_2, a_1)$.

He does not, however, note, as Beltrami did in a special case early in the same year, that by using another method of elimination, namely, Euler's of 1748, the eliminant is found to be

$$\prod_{r=1}^{r=n}\left(a_0 - y + a_1\theta_r + a_2\theta_r^2 + \ldots + a_{n-1}\theta_r^{n-1}\right),$$

where θ_r is an n^{th} root of 1 : and thus he fails to bring out the fact that Spottiswoode's result of 1853 is nothing more or less than the statement of the equality of those two eliminants.* Instead of this he performs the operation

$$\mathrm{col}_1 + \theta_r \mathrm{col}_2 + \theta_r^2 \mathrm{col}_3 + \ldots + \theta_r^{n-1}\mathrm{col}_n$$

and so arrives at a practically equivalent result, namely, that the equation

$$\mathrm{C}(a_0 - y,\ a_1,\ a_2,\ \ldots,\ a_{n-1}) = 0$$

is satisfied by putting

$$y = a_0 + a_1\theta_r + a_2\theta_r^2 + \ldots + a_{n-1}\theta_r^{n-1}.$$

Two other points worth noting are (1) his calling

$$\mathrm{C}(a_0,\ a_1,\ a_2,\ \ldots,\ a_{n-1})$$

the *norm* of

$$a_0 + a_1\theta_r + a_2\theta_r^2 + \ldots + a_{n-1}\theta_r^{n-1},$$

in accordance with an extension of a usage of Gauss', and (2) his statement that of the n^n terms got by working out the product

$$\prod_{r=1}^{r=n}\left(a_0 + a_1\theta_r + a_2\theta_r^2 + \ldots + a_{n-1}\theta_r^{n-1}\right)$$

only the $1 \cdot 2 \cdot 3 \ldots n$ terms of the determinant remain. In regard to the former it has to be remarked that θ_r must then be restricted to stand for a *primitive* n^{th} root of 1, and in regard to the latter that even some of the $1 \cdot 2 \cdot 3 \ldots n$ terms of the determinant do not remain.

WOLSTENHOLME, J. (1867).

[A Book of Mathematical Problems, ... 344 pp. London.]

Result no. 920 is

$$\mathrm{C}(\cos\theta, \cos 2\theta, \ldots, \cos n\theta) = \frac{\{\cos\theta - \cos(n+1)\theta\}^n - \{1 - \cos n\theta\}^n}{2(1 - \cos n\theta)}$$

after deleting a sign-factor.

* See *Hist.*, ii. pp. 369–370.

BALTZER, R. (1870).

[Theorie und Anwendung der Determinanten . . . . 3te verbesserte Aufl. viii+242 pp. Leipzig.]

In addition to a few changes in phraseology this edition contains the fresh theorem that *in every term of* $C(a_0, a_1, a_2, \ldots, a_{n-1})$ *the sum of the suffixes is divisible by* n. The proof is based on the fact that the suffix of any element (i, k) is either $-i+k$ or $n-i+k$ according as i is less* or greater than k; for from this it follows that in the case of any term

$$(1, r)(2, s)(3, t) \ldots \ldots$$

the sum of the suffixes differs from a multiple of n by

$$-1-2-3-\ldots+r+s+t+\ldots,$$

that is by 0.

STERN, M. A. (1871).

[Einige Bemerkungen über eine Determinante. *Crelle's Journ.*, lxxiii. pp. 374–380.]

Stern opens with what he means to be merely a fresh proof of Spottiswoode's result: what he actually obtains, however, is something more important, namely, not merely the establishment of the fact that

$$a_1 + a_2\theta_r + a_3\theta_r^2 + \ldots + a_n\theta_r^{n-1}$$

is a factor of

$$C(a_1, a_2, \ldots, a_n),$$

but the further fact that the cofactor is

$$A_1\theta_r^n + A_2\theta_r^{n-1} + A_3\theta_r^{n-2} + \ldots + A_n\theta_r,$$

where $A_1, A_2, \ldots$ are the signed complementary minors of the elements of the first row of C. By way of proof he notes that if the multiplication of these two factors be performed, the coefficient of θ_r^n in the product is

$$A_1a_1 + A_2a_2 + \ldots + A_na_n, \quad i.e.\ C;$$

* He means 'not greater.'

and the coefficient of any other power of θ_r being equal to the product of $A_1, A_2, \ldots, A_n$ by a row of C other than the first must vanish. The proof is thus seen to be based on the identities

$$\left.\begin{aligned} a_1A_1 + a_2A_2 + \ldots + a_nA_n &= C \\ a_nA_1 + a_1A_2 + \ldots + a_{n-1}A_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ a_2A_1 + a_3A_2 + \ldots + a_1A_n &= 0 \end{aligned}\right\},$$

which Stern also uses to obtain by means of addition the special case

$$(a_1 + a_2 + \ldots + a_n)(A_1 + A_2 + \ldots + A_n) = C,$$
$$\text{or, say,} \quad \Sigma a \cdot \Sigma A = C.$$

Differentiating, he next obtains from this, with the help of a result of Cremona's,

$$\left.\begin{aligned} \Sigma A + \Sigma a \cdot \frac{\partial \Sigma A}{\partial a_h} &= nA_h, \\ \Sigma A + \Sigma a \cdot \frac{\partial \Sigma A}{\partial a_\kappa} &= nA_\kappa, \end{aligned}\right\}$$

$$\text{and} \quad \therefore \quad \Sigma a \cdot \left(\frac{\partial \Sigma A}{\partial a_h} - \frac{\partial \Sigma A}{\partial a_\kappa}\right) = n(A_h - A_\kappa),$$

from which he concludes that

$$A_h - A_\kappa \text{ contains } \Sigma a \text{ as a factor.}$$

Thenceforward he specialises. Taking the case where n is prime, and where, therefore, the n roots are $1, \theta, \theta^2, \ldots, \theta^{n-1}$, he denotes the product of all Spottiswoode's factors except

$$a_1 + a_2 + \ldots + a_n \quad \text{and} \quad \theta^n a_1 + \theta^{n-1} a_2 + \ldots + \theta a_n$$

by

$$b_1 + b_2\theta + b_3\theta^2 + \ldots + b_n\theta^{n-1}$$

and establishes the result

$$C = \Sigma a \cdot \frac{n\Sigma ab - (\Sigma a)^{n-1}}{n-1},$$

where Σab stands for $a_1b_1 + a_2b_2 + \ldots + a_nb_n$. The expressing of the b's in terms of the a's is not attempted. Further specialisation affects the a's, the investigation (pp. 377–380) becoming more intimately associated with the Theory of Integers.

GLAISHER, J. W. L. (1872).

[On functions with recurring derivatives. *Proceed. London Math. Soc.*, iv. pp. 113–116.]

The functions referred to are

$$1 + \frac{x^n}{n!} + \frac{x^{2n}}{(2n)!} + \ldots .$$

$$x + \frac{x^{n+1}}{(n+1)!} + \frac{x^{2n+1}}{(2n+1)!} + \ldots .$$

. . . . . .

$$\frac{x^{n-1}}{(n-1)!} + \frac{x^{2n-1}}{(2n-1)!} + \frac{x^{3n-1}}{(3n-1)!} + \ldots ,$$

which are readily seen to satisfy the differential equation

$$\frac{d^n u}{dx^n} = u,$$

the derivate of the first function being the last, the derivate of the second being the first, and so on. Denoting them by $\phi_0, \phi_1, \phi_2, \ldots, \phi_{n-1}$, Glaisher of course obtains at once

$$\left.\begin{aligned} \phi_0 + \phi_1 + \phi_2 + \ldots + \phi_{n-1} &= e^x \\ \phi_0 + \theta_1\phi_1 + \theta_1^2\phi_2 + \ldots + \theta_1^{n-1}\phi_{n-1} &= e^{\theta_1 x} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \phi_0 + \theta_{n-1}\phi_1 + \theta_{n-1}^2\phi_2 + \ldots + \theta_{n-1}^{n-1}\phi_{n-1} &= e^{\theta_{n-1} x} \end{aligned}\right\},$$

where $1, \theta_1, \theta_2, \ldots, \theta_{n-1}$ are the n^{th} roots of 1, and thence, by multiplication and the use of Spottiswoode's theorem,

$$\begin{vmatrix} \phi_0 & \phi_1 & \phi_2 & \ldots & \phi_{n-1} \\ \phi_{n-1} & \phi_0 & \phi_1 & \ldots & \phi_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \phi_1 & \phi_2 & \phi_3 & \ldots & \phi_0 \end{vmatrix} = e^{x(1+\theta_1+\ldots+\theta_{n-1})} = e^0 = 1.$$

Similarly $\psi_0, \psi_1, \psi_2, \ldots, \psi_{n-1}$ being the functions got from the ϕ's by making the even-numbered terms of the latter negative, and therefore being solutions of the equation

$$\frac{d^n u}{dx^n} = -u,$$

it is found that

$$\begin{vmatrix} \psi_0 & \psi_1 & \psi_2 & \dots & \psi_{n-1} \\ -\psi_{n-1} & \psi_0 & \psi_1 & \dots & \psi_{n-2} \\ -\psi_{n-2} & -\psi_{n-1} & \psi_0 & \dots & \psi_{n-3} \\ \dots & \dots & \dots & \dots & \dots \\ -\psi_1 & -\psi_2 & -\psi_3 & \dots & \psi_0 \end{vmatrix} = 1.$$

It is properly pointed out that the case of the ϕ's where $n = 3$ had been obtained in 1827 by Louis Olivier (see *Crelle's Journ.*, ii. pp. 243–251).

GÜNTHER, S. (1875).

[Lehrbuch der Determinanten-Theorie, . . . viii+236 pp. Erlangen.]

Günther devotes two pages (§ 11, pp. 93–95) to the subject, but they contain nothing fresh save the proposal to call the determinant 'doppeltorthosymmetrisch,' a quite unsuitable name which accurately describes a very different special form.*

BALTZER, R. (1875).

[Theorie und Anwendung der Determinanten. 4te verbesserte Aufl. . . . viii+247 pp. Leipzig.]

The only fresh matter in this edition is the statement that in the case of the evanescent determinant

$$C(a_0 - \phi, a_1, a_2, \dots, a_{n-1}),$$

where

$$\phi = a_0 + a_1\theta + a_2\theta^2 + \dots + a_{n-1}\theta^{n-1},$$

the signed complementary minors of the elements of any row form

* A determinant which is doubly 'orthosymmetric' can have only two different elements, and must have all its odd-numbered columns identical, and all its even-numbered columns identical. This is readily seen on starting with the first two elements and then carrying out the requirements of double 'orthosymmetry.' For example,

$$\begin{vmatrix} a & b & a & b \\ b & a & b & a \\ a & b & a & b \\ b & a & b & a \end{vmatrix}.$$

an equirational progression whose constant multiplier is θ. In illustration of this we may add that the adjugate of

$$C(-b\theta - c\theta^2,\ b,\ c),$$

if we write M for $b^2\theta^2 + bc + c^2\theta$, is

$$\begin{vmatrix} M & \theta M & \theta^2 M \\ \theta M & \theta^2 M & M \\ \theta^2 M & M & \theta M \end{vmatrix} \quad i.e. \quad M^3 \quad \begin{vmatrix} 1 & \theta & \theta^2 \\ \theta & \theta^2 & 1 \\ \theta^2 & 1 & \theta \end{vmatrix} \quad i.e. \quad 0.$$

NICODEMI, R. (1877).

[Intorno ad alcune funzioni piu generali delle funzioni iperboliche. *Giornale di Mat.*, xv. pp. 193–234.]

The section which concerns determinants (pp. 205–210) contains an already known proof of Spottiswoode's theorem, this theorem being thereupon used, as Glaisher had already done (1872), to obtain a generalization of

$$\begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} = 1.$$

SCOTT, R. F. (1878).

[On some theorems in determinants. *Messenger of Math.*, viii. pp. 33–37.]

In § 2 of this paper there are five fresh results, the first two being, if $\lambda = \log\{C(a_1, a_2, \ldots a_n)\}$,

$$C(a_1, a_2, \ldots, a_n) \times C\left(\frac{\partial\lambda}{\partial a_1}, \frac{\partial\lambda}{\partial a_2}, \ldots, \frac{\partial\lambda}{\partial a_n}\right) = n^n,$$

$$\text{and Hessian of } \lambda = (-1)^{\frac{1}{2}n(n-1)} \cdot n^n.$$

In establishing these, the requisite differentiations are performed on the linear factors of the circulant. The next two are merely stated, namely,

$$C(1^2, 2^2, \ldots, n^2) = (-1)^{n-1} \cdot \frac{(n+1)(2n+1)}{12} \cdot n^{n-2} \cdot \{(n+2)^n - n^n\},$$

$$C\left(1, n-1, \tfrac{1}{2}(n-1)(n-2), \ldots, 1\right) = \begin{cases} 2^{n-1} & \text{when } n \text{ is odd.} \\ 0 & \text{when } n \text{ is even.} \end{cases}$$

The last is an extension of Wolstenholme's result of 1867, giving the evaluation of a circulant in which the elements are the cosines (or sines) of the angles a, $a+b$, $a+2b$, . . . , $a+(n-1)b$, namely,

$$\frac{\{\cos a-\cos(a+nb)\}^n-\{\cos(a-b)-\cos(a+n-1.b)\}^n}{2(1-\cos nb)}.$$

This is reached by using the exponential expression for the cosine.

GLAISHER, J. W. L. (1877–78).

[Sur un déterminant. *Assoc. franç. pour l'avancem. des sci.*, vi. pp. 177–179.]

[On the values of a class of determinants. *Report . . . British Assoc.* . . . xlvii. p. 20.]

[On the factors of a special form of determinant. *Quart. Journ. of Math.*, xv. pp. 347–356.]

As has been already pointed out, the annexing of $-x$ to the diagonal elements of the circulant $C(a_1, a_2, \ldots, a_n)$, does not alter the determinant as regards generality. If, however, the order of the last $n-1$ rows be reversed, thus producing a determinant equal to $(-1)^{\frac{1}{2}(n-1)(n-2)}C$ and symmetric with respect to the primary diagonal, the annexing of $-x$ to the elements of the said diagonal produces a determinant requiring fresh investigation. This requirement Glaisher supplies.

Having found that

$$\begin{vmatrix} a-x & b & c \\ b & c-x & a \\ c & a & b-x \end{vmatrix} = -\{x-(a+b+c)\} \cdot \begin{Bmatrix} x^2-(a+\omega b+\omega^2 c) \\ \cdot(a+\omega^2 b+\omega c) \end{Bmatrix},$$

$$\begin{vmatrix} a-x & b & c & d \\ b & c-x & d & a \\ c & d & a-x & b \\ d & a & b & c-x \end{vmatrix} = \{x-(a+b+c+d)\} \cdot \{x-(a-b+c-d)\} \cdot \begin{Bmatrix} x^2-(a+bi+ci^2+di^3) \\ \cdot(a-bi+ci^2-di^3) \end{Bmatrix},$$

and that the cofactor of $x-(a+b+c+d+e)$ in the next case was a quadratic in x^2, he surmised the existence of a general proposition including the three, and stated his surmise at the meetings of the

two Associations mentioned above. The immediate result was that Professor J. C. Adams formally established the anticipated identity. Taking a set of equations whose eliminant is the determinant in question, for example

$$\left.\begin{aligned} \kappa x &= \kappa a+\lambda b+\mu c+\nu d+\xi e \\ \lambda x &= \kappa b+\lambda c+\mu d+\nu e+\xi a \\ \mu x &= \kappa c+\lambda d+\mu e+\nu a+\xi b \\ \nu x &= \kappa d+\lambda e+\mu a+\nu b+\xi c \\ \xi x &= \kappa e+\lambda a+\mu b+\nu c+\xi d \end{aligned}\right\},$$

Adams used on them the multipliers 1, θ, θ^2, θ^3, θ^4 (θ being an imaginary fifth root of 1), thus obtaining by addition

$$\begin{aligned} &x(\kappa+\theta\lambda+\theta^2\mu+\theta^3\nu+\theta^4\xi) \\ &\quad = (a+\theta b+\theta^2 c+\theta^3 d+\theta^4 e)(\kappa+\theta^{-1}\lambda+\theta^{-2}\mu+\theta^{-3}\nu+\theta^{-4}\xi). \end{aligned}$$

Similarly he obtained

$$\begin{aligned} &x(\kappa+\theta^{-1}\lambda+\theta^{-2}\mu+\theta^{-3}\nu+\theta^{-4}\xi) \\ &\quad = (a+\theta^{-1}b+\theta^{-2}c+\theta^{-3}d+\theta^{-4}e)(\kappa+\theta\lambda+\theta^2\mu+\theta^3\nu+\theta^4\xi), \end{aligned}$$

and then from the two by multiplication

$$x^2 = (a+\theta b+\theta^2 c+\theta^3 d+\theta^4 e)(a+\theta^{-1}b+\theta^{-2}c+\theta^{-3}d+\theta^{-4}e).$$

This last being free of κ, λ, μ, ν, ξ it followed that

$$\begin{aligned} &x^2-(a+\theta b+\theta^2 c+\theta^3 d+\theta^4 e) \\ &\quad\cdot(a+\theta^{-1}b+\theta^{-2}c+\theta^{-3}d+\theta^{-4}e) \end{aligned}$$

was a factor of the determinant with which he started, and therefore that the said determinant was equal to

$$\begin{aligned} &-\{x-(a+b+c+d+e)\} \\ &\cdot\{x^2-(a+b\theta+c\theta^2+d\theta^3+e\theta^4) \\ &\qquad\cdot(a+b\theta^4+c\theta^3+d\theta^2+e\theta)\} \\ &\cdot\{x^2-(a+b\theta^2+c\theta^4+d\theta+e\theta^3) \\ &\qquad\cdot(a+b\theta^3+c\theta+d\theta^4+e\theta^2)\}. \end{aligned}$$

Glaisher himself properly points out that since θ is of the form $\cos\frac{2}{5}m\pi+i\sin\frac{2}{5}m\pi$, Adams' quadratic factor must be of the form

$$x^2-(\mathrm{A}+\mathrm{B}i)(\mathrm{A}-\mathrm{B}i), \quad i.e. \quad x^2-(\mathrm{A}^2+\mathrm{B}^2),$$

a result which is in accordance with the fact of the reality of the roots of Lagrange's determinantal equation.

Glaisher's theorem, it should also be noted, is another instance of the assertion of the equivalence of two different forms of an eliminant.

Three results of computation made in the course of the paper are also worth recording, but in a shorter notation, namely,

$$\mathrm{C}(a, b, c, d, e) = \overset{0}{\sum} a^5 - 5\overset{0}{\sum} a^3(be+cd) + 5\overset{0}{\sum} a(b^2e^2+c^2d^2) - 5abcde,$$

$$\frac{\mathrm{C}(a, b, c, d, e)}{a+b+c+d+e} = \overset{0}{\sum} a^4 - \overset{0}{\sum} a^3(b+c+d+e) + \overset{0}{\sum} a^2(b^2+c^2) + \overset{0}{\sum} a^2(2bc+2bd-3be-3cd+2ce+2de) - \overset{0}{\sum} abcd$$

and

$$\begin{vmatrix} 1 & b & c & d & e \\ 1 & c-x & d & e & a \\ 1 & d & e-x & a & b \\ 1 & e & a & b-x & c \\ 1 & a & b & c & d-x \end{vmatrix} = x^4 - x^2\left(2\overset{0}{\sum} a^2 - \overset{0}{\sum} ab - \overset{0}{\sum} ac\right) + \frac{\mathrm{C}(a, b, c, d, e)}{a+b+c+d+e},$$

where, as before,

$$\overset{0}{\sum} a^3(be+cd) = a^3(be+cd) + b^3(ca+de) + c^3(db+ea) + \ldots.$$

GLAISHER, J. W. L. (1878).

[On a special form of determinant, and on certain functions of n variables analogous to the sine and cosine. *Quart. Journ. of Math.*, xvi. pp. 15–33.]

Part of this paper (§§ 8, 9, 19, 20) is a continuation of the paper of 1872, and the other part a continuation of that of 1877–78.

The proof that the product of two circulants is expressible as a circulant is effected by substituting Spottiswoode's equivalent for each. Similarly there is established, though somewhat imperfectly, the fresh theorem

$$\mathrm{C}(a_1, a_2, \ldots, a_{2n}) = \mathrm{C}(\mathrm{B}_1, \mathrm{B}_2, \ldots, \mathrm{B}_n),$$

where

$$\mathrm{B}_1 = (a_1, a_2, \ldots, a_{2n} \,|\!|\, a_1, -a_{2n}, a_{2n-1}, \ldots, -a_2)$$

$$\mathrm{B}_2 = (a_1, a_2, \ldots, a_{2n} \,|\!|\, a_3, -a_2, a_1, \ldots, -a_4)$$

. . . . . .

$$\mathrm{B}_n = (a_1, a_2, \ldots, a_{2n} \,|\!|\, a_{2n-1}, -a_{2n-2}, a_{2n-3}, \ldots, -a_{2n}),$$

the special point to be noted being that, since $x^{2n} = (x^2)^n$, the $(2n)^{\text{th}}$ roots of 1 are the square roots of the n^{th} roots of 1, and therefore may be grouped in pairs whose sums vanish. For example, such a pair of *sixth* roots of 1 being $\theta, -\theta$, the product of the corresponding pair of factors of $\mathrm{C}(a, b, c, d, e, f)$ is

$$(a+\theta b+\theta^2 c+\theta^3 d+\theta^4 e+\theta^5 f)(a-\theta b+\theta^2 c-\theta^3 d+\theta^4 e-\theta^5 f),$$

the development of which we might obtain by changing it into

$$(a+\theta^2 c+\theta^4 e)^2-(b+\theta^2 d+\theta^4 f)^2\theta^2$$

and then squaring, etc., but which is better investigated by seeking the cofactors of

$$\theta^0 \text{ or } \theta^6, \quad \theta^1 \text{ or } \theta^7, \quad \theta^2 \text{ or } \theta^8, \quad \theta^3 \text{ or } \theta^9, \quad \theta^4 \text{ or } \theta^{10}, \quad \theta^5$$

in the result of the multiplication. These are found to be

$$\begin{array}{lll}
(\ a, -f, \ e, -d, \ c, -b \,\{\, a, b, c, d, e, f) & \text{i.e. } a^2-2bf+2ce-d^2, \\
(-b, \ a, -f, \ e, -d, \ c \,\{ & \text{i.e. } 0, \\
(\ c, -b, \ a, -f, \ e, -d \,\{ & \text{i.e. } 2ac-b^2-2df+e^2, \\
(-d, \ c, -b, \ a, -f, \ e \,\{ & \text{i.e. } 0, \\
(\ e, -d, \ c, -b, \ a, -f \,\{ & \text{i.e. } 2ae-2bd+c^2-f^2, \\
(-f, \ e, -d, \ c, -b, \ a \,\{ & \text{i.e. } 0,
\end{array}$$

thus bringing clearly out their law of formation. The product itself is evidently

$$(a^2-2bf+2ce-d^2)+\theta^2(2ac-b^2-2df+e^2)+\theta^4(2ae-2bd+c^2-f^2),$$

where θ^2 is one of the third roots of 1, and we are consequently entitled to conclude that

$$\mathrm{C}(a, b, c, d, e, f) = \mathrm{C}(a^2-d^2-2bf+2ce, \quad -b^2+e^2+2ca-2df, \quad c^2-f^2-2db+2ea).$$

Glaisher actually uses this to compute the final development of the six-line circulant. After correcting three misprints* and effecting condensation by employing a symbol for 'alternating cyclic sums,' we find the said development to be

$$\begin{aligned}
&\overset{0}{\sum}(\pm a^6)-3\overset{0}{\sum}\{\pm a^4(bf+ce+d^2)\} \\
&\qquad +2\overset{0}{\sum}\{\pm a^3(c^3+3b^2e+3f^2c+6bcd+6def)\} \\
&\qquad +9\overset{0}{\sum}(\pm a^2c^2e^2)+9\overset{0}{\sum}\{\pm a^2(b^2f^2-2c^2df)\},
\end{aligned}$$

* $2c^3a^3$, $12c^3bcd$, $12a^3def$ should be $2c^3a^3$, $12a^3bcd$, $12c^3def$.

where, for example, $\sum^{0}\{\pm a^2(b^2f^2-2c^2df)\}$ stands for

$$a^2(b^2f^2-2c^2df) - b^2(c^2a^2-2d^2ea) + c^2(d^2b^2-2e^2fb) - \ldots,$$

but where, nevertheless, $\sum^{0}(\pm a^2c^2e^2)$ does not stand for

$$a^2c^2e^2-b^2d^2f^2+c^2e^2a^2-d^2f^2b^2+e^2a^2c^2-f^2b^2d^2$$

but merely for $a^2c^2e^2-b^2d^2f^2$.*

The rest of the paper (§§ 10–18) does not contain anything fresh so far as circulants are concerned.

MENESSON, . (1878).

[Solutions des questions proposées (Question 185). *Nouv. Correspondance Math.*, iv. pp. 185–187.]

Not only is Spottiswoode's result here explicitly obtained by equating the two forms of the eliminant of

$$\left.\begin{aligned} a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_1x+a_0 &= 0 \\ x^n-1 &= 0 \end{aligned}\right\},$$

which were referred to under Baltzer (1864), but the parallel result is also stated, namely, that

$$C(a_0, a_1, a_2, \ldots, a_{n-1}) = a_{n-1}^n(\beta_1^n-1)(\beta_2^n-1)\ldots(\beta_{n-1}^n-1),$$

where $\beta_1, \beta_2, \ldots, \beta_{n-1}$ are the roots of the first equation. The latter in a less pleasing form is given by Wolstenholme later in the same year (Problem 1632).

MINOZZI, A. (1878).

[Sopra un determinante. *Giornale di Mat.*, xvi. pp. 148–151.]

Minozzi's purpose is to find the final development of a circulant, not from the determinant form, but from Spottiswoode's product. His first point is that, multiplication of the n factors having been performed, the terms of the resulting expression must be of the form

$$Aa_0^{e_0}a^{e_1}a_2^{e_2}\ldots a_{n-1}^{e_{n-1}},$$

* This is the reason why we do not combine the two into

$$\sum^{0}\{\pm a^2(b^2f^2+c^2e^2-2c^2df)\}.$$

where the e's are positive integers whose sum is n. His next is that any a entering into this, say a_k, must be accompanied by one of the θ's raised to the power k, and that therefore the form of A is known. His third is that A, being thus of necessity a symmetric function of the roots of the equation $x^n - 1 = 0$, can be calculated, and this all the more readily by reason of the fact that so many of the simple symmetric functions are equal to 0. For example, the coefficient of $a_0^2 a_1 a_3$ in $C(a_0, a_1, a_2, a_3)$ is

$$\sum \theta_1^0 \theta_2^0 \theta_3^1 \theta_4^3,$$

i.e. $$\theta_3^1\theta_4^3 + \theta_3^3\theta_4^1 + \theta_2^1\theta_4^3 + \theta_2^1\theta_3^3 + \theta_2^3\theta_4^1 + \theta_2^3\theta_3^1 + \theta_1^1\theta_4^3 + \theta_1^1\theta_3^3 + \theta_1^1\theta_2^3 + \theta_1^3\theta_4^1 + \theta_1^3\theta_3^1 + \theta_1^3\theta_2^1,$$

i.e. $$(\theta_1 + \theta_2 + \theta_3 + \theta_4)(\theta_1^3 + \theta_2^3 + \theta_3^3 + \theta_4^3) - (\theta_1^4 + \theta_2^4 + \theta_3^4 + \theta_4^4),$$

i.e. $$0 \cdot 0 - 4.$$

Minozzi is careful to note that, in accordance with Baltzer's theorem of 1870, it is only necessary to calculate A when the sum of the suffixes of the a's is 0 or a multiple of n. For example, in the case of $C(a_0, a_1, a_2, a_3)$ the coefficients would have to be found for

$$a_0a_0a_0a_0, \quad a_0a_0a_1a_3 \quad a_0a_2a_3a_3 \quad a_3a_3a_3a_3.$$
$$a_0a_0a_2a_2 \quad a_1a_1a_3a_3$$
$$a_0a_1a_1a_2 \quad a_1a_2a_2a_3$$
$$a_1a_1a_1a_1, \quad a_2a_2a_2a_2,$$

LEMONNIER, H. (1879).

[Calcul d'un déterminant. *Bull. Soc. Math. de France*, vii. pp. 175–177; or, more fully, *Nouv. Annales de Math.*, (2) xviii. pp. 518–524.]

The determinant in question is the circulant in which the elements are in equidifferent progression. The two modes of procedure followed are like Baehr's of 1860, but neither is so good.

GLAISHER, J. W. L. (1879).

[Theorems in Algebra. *Messenger of Math.*, viii. pp. 140–144.]

One of the two theorems is concerned with determinants, and is to the effect that if the $2n$ quantities $a_1, a_2, \ldots, a_{2n}$ be connected by the n relations

$$\left.\begin{array}{llllll}
(a_1, & -a_{2n}, & a_{2n-1}, \ldots, & -a_2 \,\}\, a_1, a_2, a_3, \ldots, a_{2n}) & = & 1 \\
(a_3, & -a_2, & a_1, \ldots, & -a_4 \,\}\,) & = & 0 \\
(a_5, & -a_4, & a_3, \ldots, & -a_6 \,\}\,) & = & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(a_{2n-1}, & -a_{2n-2}, & a_{2n-3}, \ldots, & -a_{2n} \,\}\,) & = & 0
\end{array}\right\},$$

then

$$\mathrm{C}(a_1, a_2, a_3, \ldots, a_{2n}) = 1,$$

and the elements of the first row of C are respectively equal to the complementary minors of the elements of the first column.

The truth of the first part of the theorem is a direct consequence of the same author's theorem of the preceding year, our illustration of which makes clear that if

$$2ac-b^2-2df+e^2 = 0 \quad \text{and} \quad 2ae-2bd+c^2-f^2 = 0,$$

we must have

$$\mathrm{C}(a, b, c, d, e, f) = (a^2-d^2-2bf+2ce)^3,$$

and with a third condition

$$\mathrm{C}(a, b, c, d, e, f) = 1.$$

To establish the second part we have to take along with the n conditioning equations the n identical equations

$$\left.\begin{array}{llllll}
(a_2, & -a_1, & a_{2n}, \ldots, & -a_3 \,\}\, a_1, a_2, a_3, \ldots, a_{2n}) & = & 0 \\
(a_4, & -a_3, & a_2, \ldots, & -a_5 \,\}\,) & = & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
(a_{2n}, & -a_{2n-1}, & a_{2n-2}, \ldots, & -a_1 \,\}\,) & = & 0
\end{array}\right\},$$

thus obtaining a set of $2n$ equations having $a_1, a_2, a_3, \ldots, a_{2n}$ as quasi unknowns and $(-1)^n\mathrm{C}$ as the determinant of their coefficients. The usual expression for an unknown as the quotient of two determinants leads at once to the desired result. Thus, when $n = 3$ the set of equations is

$$\left(\begin{vmatrix}
a & -f & e & -d & c & -b \\
b & -a & f & -e & d & -c \\
c & -b & a & -f & e & -d \\
d & -c & b & -a & f & -e \\
e & -d & c & -b & a & -f \\
f & -e & d & -c & b & -a
\end{vmatrix} \,\right\}\, a, b, c, d, e, f) = 1, 0, 0, 0, 0, 0.$$

PUCHTA, A. (1877).

[Ein Determinantensatz und seine Umkehrung. *Denkschr.* . . . *Akad. d. Wiss.* (Wien), xxxviii. (2), pp. 215–221.]

The special determinant dealt with is that which was incorrectly factorized by Bellavitis in 1857 (*Hist.*, i. pp. 145–146) and correctly by Ferrers in 1861, namely,

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = (a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d).$$

Viewing the immediately preceding case to be

$$\begin{vmatrix} a & b \\ b & a \end{vmatrix} = (a+b)(a-b),$$

Puchta asserts the next higher case to be

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ b & a & d & c & f & e & h & g \\ c & d & a & b & g & h & e & f \\ d & c & b & a & h & g & f & e \\ e & f & g & h & a & b & c & d \\ f & e & h & g & b & a & d & c \\ g & h & e & f & c & d & a & b \\ h & g & f & e & d & c & b & a \end{vmatrix} = \begin{cases} (a+b+c+d+e+f+g+h)(a+b+c+d-e-f-g-h \\ \cdot(a+b-c-d+e+f-g-h)(a+b-c-d-e-f+g+h \\ \cdot(a-b+c-d+e-f+g-h)(a-b+c-d-e+f-g+h \\ \cdot(a-b-c+d+e-f-g+h)(a-b-c+d-e+f+g-h \end{cases}$$

and proves that, for example,

$$a-b+c-d+e-f+g-h$$

is a factor by performing the operation

$$\mathrm{col}_1-\mathrm{col}_2+\mathrm{col}_3-\mathrm{col}_4+\mathrm{col}_5-\mathrm{col}_6+\mathrm{col}_7-\mathrm{col}_8.$$

He then concludes that the order-number of the determinant is a positive integral power of 2, and that the matrix in any case is of the form

$$\begin{matrix} \mathrm{A} & \mathrm{B} \\ \mathrm{B} & \mathrm{A}, \end{matrix}$$

where A and B are similar matrices of the next lower order. Following this comes a somewhat tedious investigation (pp. 217–219) of the law of formation of the linear factors, the result being a not

very concise or clearly worded rule-of-signs,* which is practically to the effect that *The first sign is always +, the second sign is + or the opposite, the next set of two signs is the same as the first set of two or the opposite, the next set of four signs is the same as the first set of four or the opposite, and so on.* For example, if the order-number of the determinant be 2^4, there are four occasions on which we have to choose between same (s) or opposite (o), and if the register of our decisions be $s\ o\ s\ s$, the resulting factor is

$$a+b-c-d+e+f-g-h+i+j-k-l+m+n-o-p.$$

What Puchta calls in his title the 'converse' proposition (pp. 220–221) really concerns the finding of a determinant equal to a product of linear factors which differ only in the signs of their terms. The illustrative example given, namely,

$$(a+b-c)(a-b+c)(-a+b+c) = -\begin{vmatrix} a-b & -c & c \\ -b & a-c & b \\ c-b & b-c & a \end{vmatrix}$$

is not attractive.

DOSTOR, G. (1877).

[ÉLÉMENTS . . . DES DÉTERMINANTS, . . . xxxi+352. Paris.]

Dostor gives (p. 72) a very peculiar mode of factorizing

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$$

This depends on the fact, that if the given determinant be multiplied by

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{vmatrix},$$

* At the basis of this is a proposition which may also be used in practice for obtaining the factors, namely, *If α_r be one of the linear factors of the determinant of the array* A, *and β_r the corresponding factor of the determinant of the array* B, *then*

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix} = (\alpha_1+\beta_1)(\alpha_1-\beta_1)\cdot(\alpha_2+\beta_2)(\alpha_2-\beta_2)\cdot(\alpha_3+\beta_3)(\alpha_3-\beta_3)\cdot(\alpha_4+\beta_4)(\alpha_4-\beta_4).$$

we can remove from the columns of the product-determinant the factors

$$a+b+c+d, \quad a+b-c-d, \quad a-b+c-d, \quad a-b-c+d$$

respectively, and have left the determinant

$$\begin{vmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$$

—that is to say, the conjugate of the multiplying determinant.

NOETHER, M. (1879).

[Zur Theorie der Thetafunctionen von beliebig vielen Argumenten (§ 15, pp. 322–325). *Math. Annalen*, xvi. pp. 270–344.]

Noether arrives independently at Puchta's result of 1877, and in an interesting way. Multiplying the second row of each successive pair of rows by β_1, the last two rows of each successive quartet of rows by β_2, the last four rows of each successive octet by β_3, ... , and then treating the columns in exactly the same manner he obtains of course a determinant not differing in substance from the original if the β's be square roots of 1. But the columns and rows of the new form having on this assumption all the same sum, the desired resolution is effected. Thus, the eight-line determinant is found equal to

$$\prod_{\beta^2=1} (a+\beta_1 b+\beta_2 c+\beta_1\beta_2 d+\beta_3 e+\beta_1\beta_3 f+\beta_2\beta_3 g+\beta_1\beta_2\beta_3 h).$$

It should be carefully noted that if this be slightly changed in form, namely, into

$$\prod_{\beta^2=1} \{a+\beta_1 b+\beta_2(c+\beta_1 d)+\beta_3(e+\beta_1 f+\beta_2 g+\beta_1\beta_2 h)\},$$

it is seen to be but a symbolic representation of the verbal rule formulated under Puchta (1877).

Noether next points out that *the signed complementary minor of any one of the variables is the same for all positions which the variable occupies*, and that consequently *the adjugate determinant is of the same form as the original.* He is then able to profit by the fact that the factor

$$a+\beta_1 b+\beta_2 c+\beta_1\beta_2 d+ \ . \ . \ . \ .$$

might have been removed from the determinant in its original form after simply increasing the first column by β_1 times the second, β_2 times the third, $\beta_1\beta_2$ times the fourth, and so on. For this being done, the elements remaining in the first column are

$$1,\ \beta_1,\ \beta_2,\ \beta_1\beta_2,\ \ldots\ ,$$

showing that the determinant remaining is

$$A+\beta_1 B+\beta_2 C+\beta_1\beta_2 D+\ \ldots\ ,$$

and therefore that the original determinant is equal to

$$(a+\beta_1 b+\beta_2 c+\beta_1\beta_2 d+\ \ldots\ )(A+\beta_1 B+\beta_2 C+\beta_1\beta_2 D+\ \ldots\ ),$$

a proposition which at once recalls Stern's of 1871 regarding the simple circulant.

NOETHER, M. (1879).

[Notiz über eine Classe symmetrischer Determinanten. *Math. Annalen*, xvi. pp. 551–555.]

The resemblance between the properties established in the preceding paper and those of the simple circulant may have struck Noether and led him to go deeper into the matter, with the result that he found a further point of resemblance, namely, in connection with elimination. When he thus came to recognize that Bellavitis' four-line determinant and Puchta's eight-line determinant were the eliminants of

$$\left.\begin{aligned} a+bx+cy+dxy &= 0 \\ x^2 = y^2 &= 1 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} a+bx+cy+dxy+ez+fxz+gyz+hxyz &= 0 \\ x^2 = y^2 = z^2 &= 1 \end{aligned}\right\}$$

respectively, and to think at the same time of the circulant as the eliminant of

$$\left.\begin{aligned} a_1+a_2x+a_3x^2+\ \ldots\ +a_nx^{n-1} &= 0 \\ x^n &= 1 \end{aligned}\right\},$$

it was not difficult or unnatural to advance to the eliminant of

$$\left.\begin{aligned} \sum_{i_\rho < n_\rho} a_{i_1 i_2 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_\mu^{i_\mu} &= 0 \\ x_1^{n_1} = x_2^{n_2} = \ldots = x_\mu^{n_\mu} &= 1 \end{aligned}\right\},$$

and thereafter, apart from elimination, to specify it as a determinant, the mode of specification being by partitionment into squares exactly as in the particular case where the n's were all equal to 2. Taking, for example, $\mu = 3$, $n_1 = 2$, $n_2 = 2$, $n_3 = 3$, there results a twelve-line determinant whose matrix is of the form

$$\begin{matrix} P & Q & R \\ R & P & Q \\ Q & R & P, \end{matrix}$$

where P, Q, R are matrices of Bellavitis' form with the variables

$$\begin{matrix} a_{000} & a_{100} & a_{010} & a_{110}, \\ a_{001} & a_{101} & a_{011} & a_{111}, \\ a_{002} & a_{102} & a_{012} & a_{112}, \end{matrix}$$

respectively.* The determinant of the $(n_1 n_2 \ldots n_\mu)^{\text{th}}$ order thus reached is then shown to possess the properties of the less general determinant considered in his previous paper.

GEGENBAUR, L. (1880).

[Ueber eine specielle symmetrische Determinante. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), lxxxii. pp. 938–942.]

An unimportant alternative proof of Noether's general theorem of the preceding year, with a so-called property of the determinant.

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*The multiple-suffix notation is not very attractive. If we use

$(x, y, \ldots)(\xi, \eta, \zeta, \ldots)$ to stand for $(x\xi, y\xi, \ldots, x\eta, y\eta, \ldots, x\zeta, y\zeta, \ldots)$,

the set of equations in this special case is

$$\left.\begin{aligned} ((1, x)(1, y)(1, z, z^2) \between a, b, c, \ldots, l) &= 0 \\ x^2 = y^2 = z^2 &= 1 \end{aligned}\right\},$$

and the eleven other equations requisite for elimination are got from the first of these by multiplying it by

$$x,\ y,\ xy,\ z,\ xz,\ yz,\ xyz,\ z^2,\ xz^2,\ yz^2,\ xyz^2.$$

In the same notation the general set would he

$$\left.\begin{aligned} ((1, x, x^2, \ldots, x^{p-1})(1, y, y^2, \ldots, y^{q-1})(1, z, z^2, \ldots, z^{r-1}) \ldots \between a, b, c, \ldots) &= 0 \\ x^p = y^q = z^r = \ldots &= 1 \end{aligned}\right.$$

CHAPTER XVI.

CONTINUANTS, FROM 1850 TO 1880.

It has to be recalled that the corresponding chapter of the immediately preceding volume aimed at bringing the record up to the year 1870. On this account two of the papers here dealt with belong properly to that volume. The approximate date assigned to the first paper rests on the opinion of the editor of it. Jacobi died in 1851.

JACOBI, C. G. J. (1850 ?).

[Allgemeine Theorie der kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird. *Crelle's Journ.*, lxix. (1868), pp. 29–64 ; or *Gesammelte Werke*, pp. 385–426.]

In this paper, published seventeen years after his death, Jacobi constructs a series whose first three terms are

$$a_1, \quad a_2, \quad a_3,$$

and every other term a sum of multiples of the three terms preceding it ; thus, the multiples in question being

$$\begin{array}{ccc} 1 & l_1 & m_1, \\ 1 & l_2 & m_2, \\ \multicolumn{3}{c}{\ldots\ldots,} \end{array}$$

the 4^{th}, 5^{th}, 6^{th}, . . . terms are

$$a_1+l_1a_2+m_1a_3,$$

$$m_2a_1+(l_1m_2+1)a_2+(m_1m_2+l_2)a_3,$$

$$(m_2m_3+l_3)a_1+(l_1m_2m_3+l_1l_3+m_3)a_2+(m_1m_2m_3+m_1l_3+l_2m_3+1)a_3,$$

. . . . . .

The interest of the investigation mainly concerns the student of Continued Fractions; but one result it is important for us to note, namely, that if three consecutive terms a_h, a_{h+1}, a_{h+2} of the series be

$$p_1a_1+p_2a_2+p_3a_3,$$
$$q_1a_1+q_2a_2+q_3a_3,$$
$$r_1a_1+r_2a_2+r_3a_3,$$

then

$$|\,p_1q_2r_3\,| \;=\; 1;$$

and that if conversely a_1, a_2, a_3 be expressed in terms of a_h, a_{h+1}, a_{h+2}, the determinant of the coefficients of the latter is also 1.

We may note for ourselves that the first part of the theorem is readily proved by using on the determinant the law of formation of the series, and that the second part is a direct consequence of the fact that the determinant there is the adjugate of $|\,p_1q_2r_3\,|$. It is not noted that the p's, q's, r's are themselves expressible as determinants, the special form of which is first referred to by Sylvester in 1853 (*Hist.* II., p. 418) and does not again turn up until Fürstenau's related paper of 1872, our account of which begins on the page opposite this.

WOLSTENHOLME, J. (1867).

[A Book of Mathematical Problems, ... 344 pp. London.]

Problem 921 concerns the persymmetric and centrosymmetric continuant

$$\begin{vmatrix} x-2 & 1 & . & . & \dots \\ 1 & x-2 & 1 & . & \dots \\ . & 1 & x-2 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

the fresh point being that when $x = 0$, the value of the n-line determinant is $n+1$.

We may add that this is readily established by performing the operation

$$\mathrm{col}_1 + 2\,\mathrm{col}_2 + 3\,\mathrm{col}_3 + \dots .$$

ONOFRIO, P. (1872).

[Ad una funzione che entra nella composizione delle ridotte delle frazioni continue, ... *Giornale di Mat.*, x. pp. 37–46.]

The function referred to is Euler's algorithm of 1764, and the properties of it given are mainly Euler's also. We may note as being fresher the two,

$$(x_1, \ldots, x_n, \ldots, x_r) = x_n(x_1, \ldots, x_{n-1})(x_{n+1}, \ldots, x_r) + (x_1, \ldots, x_{n-1}, 0, x_{n+1}, \ldots, x_r),$$

$$\frac{\partial}{\partial x_n}(x_1, \ldots, x_n, \ldots, x_r) = (x_1, \ldots, x_{n-1})(x_{n+1}, \ldots, x_r).$$

FÜRSTENAU, E. (1872).

[Ueber Kettenbrüche höherer Ordnung. Sch. Progr. Wiesbaden.]

By a continued-fraction 'of higher order' is meant a fraction like

$$a + \cfrac{b + \cfrac{d + \ldots}{e + \ldots}}{c + \cfrac{f + \ldots}{g + \ldots}},$$

where every plus sign is in general followed by a fraction-line. Those considered by Fürstenau originate in the two sets of substitutions

$$X = a_0 + \frac{x_1}{y_1}, \quad y_1 = a_1 + \frac{x_2}{y_2}, \quad y_2 = a_2 + \frac{x_3}{y_3}, \quad \ldots,$$

$$Y = b_0 + \frac{1}{y_1}, \quad x_1 = b_1 + \frac{1}{y_2}, \quad x_2 = b_2 + \frac{1}{y_3}, \quad \ldots,$$

the successive approximations to X thus being

$$a_0, \quad a_0 + \frac{b_1}{a_1}, \quad a_0 + \cfrac{b_1 + \cfrac{1}{a_2}}{a_1 + \cfrac{b_2}{a_2}}, \quad \ldots.$$

and to Y

$$b_0, \quad b_0 + \frac{1}{a_1}, \quad b_0 + \cfrac{1}{a_1 + \cfrac{b_2}{a_2}}, \quad \ldots.$$

When each approximation is reduced so as to consist of only one numerator and denominator, the series of denominators in the one case is the same as in the other. Further, the p^{th} denominator, D_p say, of the series is expressible in the form

$$\begin{vmatrix} a_1 & b_2 & 1 & . & . & . & . \\ -1 & a_2 & b_3 & 1 & . & . & . \\ . & -1 & a_3 & b_4 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & a_{p-1} \end{vmatrix},$$

and the corresponding numerators X_p, Y_p as determinants obtainable by bordering this, namely,

$$\begin{vmatrix} b_0 & 1 & . & . & . & . \\ -1 & a_1 & b_2 & 1 & . & . \\ . & -1 & a_2 & b_3 & 1 & . \\ . & . & -1 & a_3 & b_4 & . \\ . & . & . & . & . & . \end{vmatrix}, \quad \begin{vmatrix} a_0 & b_1 & 1 & . & . & . \\ -1 & a_1 & b_2 & 1 & . & . \\ . & -1 & a_2 & b_3 & 1 & . \\ . & . & -1 & a_3 & b_4 & . \\ . & . & . & . & . & . \end{vmatrix}.$$

A glance at the last column of each determinant shows that the recurrence-formulae for the X's, the Y's, and the N's are all the same, namely, for example

$$N_{p+2} = a_{p+1}N_{p+1} + b_{p+1}N_p + N_{p-1}.$$

As a consequence of this it is easy to show that

$$\begin{vmatrix} X_{p+2} & X_{p+1} & X_p \\ Y_{p+2} & Y_{p+1} & Y_p \\ D_{p+2} & D_{p+1} & D_p \end{vmatrix} = 1;$$

for the application of the recurrence-formulae changes the determinant into

$$\begin{vmatrix} X_{p-1} & X_{p+1} & X_p \\ Y_{p-1} & Y_{p+1} & Y_p \\ D_{p-1} & D_{p+1} & D_p \end{vmatrix} \quad i.e. \quad \begin{vmatrix} X_{p+1} & X_p & X_{p-1} \\ Y_{p+1} & Y_p & Y_{p-1} \\ D_{p+1} & D_p & D_{p-1} \end{vmatrix},$$

and therefore into

$$\begin{vmatrix} X_3 & X_2 & X_1 \\ Y_3 & Y_2 & Y_1 \\ D_3 & D_2 & D_1 \end{vmatrix}, \text{ and thence finally } \begin{vmatrix} 1 & b_1 & a_0 \\ . & 1 & b_0 \\ . & . & 1 \end{vmatrix}.$$

The points of connection of this paper of Fürstenau's with the opening part of Jacobi's of 1850 (?), should be carefully noted. Sylvester's anticipation (1853) of such determinant forms as those for D_p, . . . should also be recalled (*Hist.*, ii. p. 418).

STUDNIČKA, F. J. (1872).

[Ueber eine besondere Art von symmetralen Determinanten und deren Verwendung in der Theorie der Kettenbrüche. *Sitzungsb.* . . . . *böhm. Ges. d. Wiss.* (Prag), Jahrg. 1872, pp. 74–78.]

The determinant referred to in the title is

$$\begin{vmatrix} a_1 & -1 & . & \ldots & . \\ 1 & a_2 & -1 & \ldots & . \\ . & 1 & a_3 & \ldots & . \\ . & . & . & . & . \\ . & . & . & \ldots & a_n \end{vmatrix},$$

and the fact of its skewness leads Studnička, unfortunately, to follow Cayley's method of 1847 in finding the ordinary expansion of it.

There is then formulated the familiar 'rule' referred to by Sylvester in his first paper of 1853.

CASORATI, F. (1872).

[Le proprietà cardinali degli strumenti ottici anche non centrati, ... *Rendiconti del Reale Istituto Lombardo*, v., fasc. iv. 13 pp.]

In the course of his investigation Casorati has to deal with a series of quantities $\beta_0, \beta_1, \beta_2, \ldots$, each of which, after the second, is dependent on the two preceding it in the manner indicated by the set of equations

$$\begin{aligned} \beta_2 &= u_1\beta_1 + \beta_0 \\ \beta_3 &= u_2\beta_2 + \beta_1 + \pi_1 \\ \beta_4 &= u_3\beta_3 + \beta_2 \\ \beta_5 &= u_4\beta_4 + \beta_3 + \pi_3 \\ \beta_6 &= u_5\beta_5 + \beta_4 \\ \beta_7 &= u_6\beta_6 + \beta_5 + \pi_5 \\ \beta_8 &= u_7\beta_7 + \beta_6 \\ &. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ , \end{aligned}$$

and his problem is to obtain for any one of the said series an expression involving no others of the series except the first two, the multipliers $u_1, u_2, \ldots$, and the addends $\pi_1, \pi_3, \ldots$. Taking the case of seven equations, he finds the result of solution to be

$$\beta_8 = \begin{vmatrix} 1 & . & . & . & . & . & u_1\beta_1+\beta_0 \\ u_2 & -1 & . & . & . & . & -\beta_1+\pi_1 \\ 1 & u_3 & -1 & . & . & . & . \\ . & 1 & u_4 & -1 & . & . & \pi_3 \\ . & . & 1 & u_5 & -1 & . & . \\ . & . & . & 1 & u_6 & -1 & \pi_5 \\ . & . & . & . & 1 & u_7 & . \end{vmatrix},$$

the determinant being next transformed so as to show the cofactors of β_1 and β_0 as continuants—a process which might have been extended to the cofactors of the π's.

BAUER, G. (1872).

[Von einem Kettenbruche Euler's und einem Theorem von Wallis. *Abhandl. d. k. bayer. Acad. d. Wiss.* (München), II. Cl. xi. (2), pp. 99–116.]

The continued-fraction referred to is

$$m + \frac{n}{m+1} + \frac{n+1}{m+2} + \ddots$$

and the theorem is that announced by Wallis in connection with the identity

$$\frac{4}{\pi} = 1 + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \ddots$$

of Brouncker, namely, that the product of

$$a-1+\frac{1^2}{2(a-1)} + \frac{3^2}{2(a-1)} + \ddots \quad \text{and} \quad a+1+\frac{1^2}{2(a+1)} + \frac{3^2}{2(a+1)} + \ddots$$

is a^2. The two things are linked together in the title because the main feature of the paper is the establishment of a relation between a continued-fraction of Euler's type and a continued-fraction of the very different type which appears in Wallis' theorem. If we denote by S the continued-fraction

$$\frac{n}{a_1}+\frac{n+a_1}{a_2}+\frac{n+a_2}{a_3}+\ddots+\frac{n+a_{r-1}}{a_r},$$

which includes Euler's, and by T the fraction

$$\frac{n+a_1}{a_1}+\frac{n+a_2}{a_2}+\ddots+\frac{n+a_{r-1}}{a_{r-1}}+\frac{n+a_r}{a_r+1},$$

the relation in question is

$$\frac{n(S+1)}{S+n} = T,$$

or, as Bauer puts it,

$$\frac{n(P_r+Q_r)}{P_r+nQ_r} = T,$$

where

$$Q_r = \begin{vmatrix} a_1 & 1 & . & \cdots & . & . \\ -n-a_1 & a_2 & 1 & \cdots & . & . \\ . & -n-a_2 & a_3 & \cdots & . & . \\ . & . & -n-a_3 & \cdots & . & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ . & . & . & \cdots & a_{r-1} & 1 \\ . & . & . & \cdots & -n-a_{r-1} & a_r \end{vmatrix}$$

and P_r is n times the complementary minor of the element in the place (1, 1) of Q_r. In the latter form it is a relation between continued-fraction determinants, and as such claims our attention.

By way of proof Bauer increases each row by all the rows following

it, and thereafter diminishes each column except the last by the columns immediately preceding it, and thus obtains

$$Q_r = \begin{vmatrix} -n & 1 & . & \dots & . & a_r+1 \\ -n-a_1 & a_1 & 1 & \dots & . & a_r+1 \\ . & -n-a_2 & a_2 & \dots & . & a_r+1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & . & \dots & a_{r-1} & a_r+1 \\ . & . & . & \dots & -n-a_{r-1} & a_r \end{vmatrix},$$

and therefore

$$= -nD_1 + (n+a_1)\cdot\{D_2 - (a_r+1)\cdot\Pi\},$$

if D_1 be put for the determinant got by deleting the first row and first column of Q_r, D_2 for the determinant got by deleting the first two rows and first two columns, and Π for

$$(n+a_2)(n+a_3)\dots(n+a_{r-1}).$$

Similarly, of course,

$$P_r = n\begin{vmatrix} -n & 1 & . & \dots & . & a_r+1 \\ -n-a_2 & a_2 & 1 & \dots & . & a_r+1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & . & \dots & a_{r-1} & a_r+1 \\ . & . & . & \dots & -n-a_{r-1} & a_r \end{vmatrix};$$

and thus, by altering the $(1,1)^{\text{th}}$ element into $a_1-(n+a_1)$, there is obtained $P_r = nD_1 - n(n+a_1)D_2$;

so that on the elimination first of D_1 and then of D_2 from these expressions for Q_r and P_r, there results

$$P_r + Q_r = (n+a_1)\{(1-n)D_2 - (a_r+1)\Pi\},$$

$$P_r + nQ_r = n\{(1-n)D_1 - (n+a_1)(a_r+1)\Pi\}.$$

These, on returning from D_1, D_2, Π to their lengthy equivalents, are changeable into

$$P_r + Q_r = (n+a_1)\begin{vmatrix} 1-n & . & . & \dots & . & a_r+1 \\ -n-a_2 & a_2 & 1 & \dots & . & a_r+1 \\ . & -n-a_3 & a_3 & \dots & . & a_r+1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & . & \dots & 1 & a_r+1 \\ . & . & . & \dots & a_{r-2} & a_r+1 \\ . & . & . & \dots & -n-a_{r-1} & a_r \end{vmatrix}$$

and

$$P_r + nQ_r = n \begin{vmatrix} 1-n & . & . & \dots & . & a_r+1 \\ -n-a_1 & a_1 & 1 & \dots & . & a_r+1 \\ . & -n-a_2 & a_2 & \dots & . & a_r+1 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & 1 & a_r+1 \\ . & . & . & \dots & a_{r-2} & a_r+1 \\ . & . & . & \dots & -n-a_{r-1} & a_r \end{vmatrix}$$

where the two determinants, being of like formation, may be suitably denoted by V_2 and V_1, and whence by division there is thus obtained

$$\frac{n(P_r + Q_r)}{P_r + nQ_r} = (n+a_1)\frac{V_2}{V_1}.$$

With the V notation the recurrence-formula

$$V_i = a_1 V_{i+1} + (n + a_{i+1})V_{i+1}$$

is next established,* from which it readily follows that

$$\frac{V_1}{V_2} = a_1 + \frac{n+a_2}{a_2} + \ddots + \frac{n+a_{r-1}}{a_{r-1}} + \frac{n+a_r}{a_r+1},$$

and therefore

$$= \frac{n+a_1}{T}.$$

The result desired, namely,

$$\frac{n(P_r + Q_r)}{P_r + nQ_r} = T,$$

is thus realised.

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* It would have seemed more direct and natural to have shown that V is transformable into the continued-fraction determinant

$$\begin{vmatrix} a_1 & n+a_2 & . & \dots & . & . \\ -1 & a_2 & n+a_3 & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & a_r & n+a_r \\ . & . & . & \dots & -1 & a_r+1 \end{vmatrix},$$

and therefore V_2 into the determinant got from this by deleting the first row and first column.

In a subsequent part of the paper it is stated that in a quite similar manner the following more general theorem may be proved, namely, that if

$$S = \frac{b_1c_0}{d+b_2-c_0} + \frac{b_2c_1}{d+b_3-c_1} + \ddots + \frac{b_rc_{r-1}}{d+b_{r+1}-c_{r-1}}$$

and

$$T = \frac{b_2c_0}{d+b_2-c_1} + \ddots + \frac{b_rc_{r-2}}{d+b_r-c_{r-1}} + \frac{b_{r+1}c_{r-1}}{d+b_{r+1}},$$

then

$$\frac{(c_0-d)S+b_1c_0}{S+b_1} = T.$$

Strictly speaking, this is the only result of Bauer's which concerns determinants, and the rest of his paper might with justice be passed over. The main application, however, is so easily deduced therefrom, and is so interesting, that space may well be given it. There is, indeed, little more to be done than to combine the case where $c_i = b_{i+1}$ with the case where $c_i = b_{i+2}$. For the one substitution gives us

$$\frac{(b_1-d)P+b_1^2Q}{P+b_1Q} = \frac{b_1b_2}{d} + \ddots + \frac{b_{r-1}b_r}{d} + \frac{b_rb_{r+1}}{d+b_{r+1}}$$

if

$$\frac{P}{Q} = \frac{b_1^2}{d+b_2-b_1} + \ddots + \frac{b_r^2}{d+b_{r+1}-b_r};$$

the other gives

$$\frac{(b_2-d)P'+b_1b_2Q'}{P'+b_1Q'} = \frac{b_2^2}{d+b_2-b_3} + \ddots + \frac{b_r^2}{d+b_r-b_{r+1}} + \frac{b_{r+1}^2}{d+b_{r+1}}$$

if

$$\frac{P'}{Q'}=\frac{b_1b_2}{d}+\ddots+\frac{b_rb_{r+1}}{d};$$

and, the first continued-fraction in the one case being the same as the second continued-fraction in the other case, save that the last denominator of the former is $d+b_{r+1}$ and of the latter d, it thus at once follows that *If the continued-fraction*

$$\frac{b_1b_2}{d}+\frac{b_2b_3}{d}+\ddots \qquad \textit{or } S, \textit{ say,}$$

be convergent, then

$$\left.\begin{aligned}\frac{b_1^2}{d+b_1-b_2}+\frac{b_2^2}{d+b_2-b_3}+\ddots &= b_1\frac{S+b_1}{S+d+b_1}\\ \textit{and}\\ \frac{b_1^2}{d+b_2-b_1}+\frac{b_2^2}{d+b_3-b_2}+\ddots &= -b_1\frac{S-b_1}{S+d+b_1}\end{aligned}\right\};$$

consequently

$$\left.\begin{aligned}\frac{d}{2}-b_1+\frac{b_1^2}{d+b_1-b_2}+\frac{b_2^2}{d+b_2-b_3}+\ddots &= \frac{d}{2}\cdot\frac{S+d-b_1}{S+d+b_1}\\ \frac{d}{2}+b_1+\frac{b_1^2}{d+b_2-b_1}+\frac{b_2^2}{d+b_3-b_2}+\ddots &= \frac{d}{2}\cdot\frac{S+d+b_1}{S+d-b_1}\end{aligned}\right\},$$

and therefore finally

$$\left\{\frac{d}{2}-b_1+\frac{b_1^2}{d+b_1-b_2}+\frac{b_2^2}{d+b_2-b_3}+\ddots\right\}\times\left\{\frac{d}{2}+b_1+\frac{b_1^2}{d+b_2-b_1}+\frac{b_2^2}{d+b_3-b_2}+\ddots\right\}=\left(\frac{d}{2}\right)^2.$$

NACHREINER, V. (1872).

[Beziehungen zwischen Determinanten und Kettenbrüchen. (Diss.) 24 pp. München.]

The starting-point here is Spottiswoode's short statement given on p. 374 of the 51st volume of *Crelle's Journal*. This having been quoted, pains are then taken to establish a so-called 'generalization,' viz.

$$\frac{1}{a_r}+\frac{b_{r+1}}{a_{r+1}}+\frac{b_{r+2}}{a_{r+2}}+\ddots+\frac{b_n}{a_n} \quad = \quad \frac{\partial}{\partial a_r}\log_e \nabla,$$

where

$$\nabla \equiv \begin{vmatrix} a_r & b_{r+1} & . & \dots & . & . \\ -1 & a_{r+1} & b_{r+2} & \dots & . & . \\ . & -1 & a_{r+2} & \dots & . & . \\ . & . & . & . & . & . \\ . & . & . & \dots & a_{n-1} & b_n \\ . & . & . & \dots & -1 & a_n \end{vmatrix};$$

but unfortunately the left-hand side of the identity is not given as here, $a_r+\frac{b_{r+1}}{a_{r+1}}+\dots$ being incorrectly put* instead of its reciprocal.

The identity

$$\begin{vmatrix} a_1 & b_1 & . & \dots \\ c_1 & a_2 & b_2 & \dots \\ . & c_2 & a_3 & \dots \\ . & . & . & . \end{vmatrix} = \begin{vmatrix} a_1 & 1 & . & \dots \\ b_1c_1 & a_2 & 1 & \dots \\ . & b_2c_2 & a_3 & \dots \\ . & . & . & . \end{vmatrix}$$

is next demonstrated, the procedure involving the division of the 2^{nd}, 3^{rd}, 4^{th}, ... columns by b_1, b_1b_2, $b_1b_2b_3$, ... respectively, and thereafter the multiplication of the corresponding rows by the same. This is used to deduce some variants of the above 'generalization,' and then the results thus far obtained are employed in dealing with known properties of continued-fractions.

* This oversight runs through sect. i., and should be corrected in the results marked (3), (4), (6), (7), (8).

The third section opens with the identity

$$\begin{vmatrix} 1 & . & \dots & . & . \\ a_1 & 1 & \dots & . & . \\ b_2 & a_2 & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & a_{n-1} & 1 \\ . & . & \dots & b_n & a_n \end{vmatrix} = \begin{vmatrix} a_0 & 1 & . & \dots & . & . \\ 1 & A_1 & 1 & \dots & . & . \\ . & 1 & A_2 & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & A_{n-1} & 1 \\ . & . & . & \dots & 1 & A_n \end{vmatrix} \cdot b_n b_{n-2} \dots$$

where $A_1 \equiv a_1 \frac{1}{b_1}$, $A_2 \equiv a_2 \frac{b_1}{b_2}$, $A_3 \equiv a_3 \frac{b_2}{b_1 b_3}$,

and generally, $A_r \equiv a_r \frac{b_{r-1} b_{r-3} b_{r-5} \dots}{b_r b_{r-2} b_{r-4} \dots}$.

This is reached by a rather curious process, of which only the first two steps need be indicated.* They are, in the case of the fifth order,—

$$\begin{vmatrix} a_0 & 1 & . & . & . \\ b_1 & a_1 & 1 & . & . \\ . & b_2 & a_2 & 1 & . \\ . & . & b_3 & a_3 & 1 \\ . & . & . & b_4 & a_4 \end{vmatrix} = b_4 \begin{vmatrix} a_0 & 1 & . & . & . \\ b_1 & a_1 & 1 & . & . \\ . & b_2 & a_2 & 1 & . \\ . & . & b_3 & a_3 & 1 \\ . & . & . & 1 & \frac{a_4}{b_4} \end{vmatrix}$$

$$= b_3 b_4 \begin{vmatrix} a_0 & 1 & . & . & . \\ b_1 & a_1 & 1 & . & . \\ . & b_2 & a_2 & 1 & . \\ . & . & 1 & \frac{a_3}{b_3} & 1 \\ . & . & . & \frac{1}{b_3} & \frac{a_4}{b_4} \end{vmatrix} = b_4 \begin{vmatrix} a_0 & 1 & . & . & . \\ b_1 & a_1 & 1 & . & . \\ . & b_2 & a_2 & 1 & . \\ . & . & 1 & \frac{a_3}{b_3} & 1 \\ . & . & . & 1 & \frac{a_4 b_3}{b_4} \end{vmatrix}.$$

$$= . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

*The transformation is probably most readily effected by dividing the 2nd, 3rd, 4th, ... rows by

$$b_1, \quad b_2, \quad b_1 b_3, \quad b_2 b_4, \quad b_1 b_3 b_5, \quad b_2 b_4 b_6, \quad b_1 b_3 b_5 b_7, \quad \dots$$

respectively, and multiplying the 3rd, 4th, 5th, ... columns by the same, the most instructive order of performing these operations being that in which each division is immediately followed by the corresponding multiplication. As the number of the divisions is necessarily one more than the number of multiplications the reason for the outside factor $b_n b_{n-2} b_{n-4} \dots$ on the left member of the identity is also thus made apparent.

The rest of the paper (pp. 13–24) is occupied mainly with continued-fractions, and contains nothing new in regard to the related determinants. Possibly the nearest approach to a fresh result is the use of the known identity

$$a_0-a_1+a_2-\ldots = \frac{a_0}{1}+\frac{a_1}{a_0-a_1}+\frac{a_0a_2}{a_1-a_2}+\ddots$$

to suggest (p. 16) that

$$\begin{vmatrix} a_0-a_1 & -1 & . & \ldots \\ a_0a_2 & a_1-a_2 & -1 & \ldots \\ . & a_1a_3 & a_2-a_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n = a_1a_2\ldots a_{n-1}\{a_0-a_1+a_2-a_3+\ldots\},$$

an equality easily established otherwise.

GÜNTHER, S. (1873).

[Darstellung der Näherungswerthe von Kettenbrüchen in independenter Form. iv+128 pp. Erlangen.]

When we come to Günther we have reached the first formal treatise on our subject. His booklet consists of three chapters, the second of which, with the title 'Darstellung der Zähler und Nenner jedes Näherungsbruches in Determinanten-Form,' is that which mainly concerns us. The first chapter (pp. 1–30) is of the nature of a historical review of the various modes proposed for the calculation of convergents, and the third is professedly an application of the results of the second chapter to 'Analysis, Algebra, und Physik.'

That portion (§ 6) of chapter i. which deals with the introduction of determinants into the treatment of continued-fractions attributes the first idea of the existence of the relation to Ramus (1855); notes that Heine (1859) discovered it independently; and nevertheless represents Heine as having used a result of Painvin's. All this, of course, stood at the time* in need of serious modification, which unfortunately was not forthcoming, with the result that some of

* For Heine's repudiation see his *Handbuch der Kugelfunction*, i. (1878), pp. 261–262.

the inaccuracies continue to be propagated in text-books to the present day.

In the second and third chapters neither pains nor space is spared to furnish an exposition of known results, with careful citation of the titles of all writings made use of, whether important or unimportant.* Throughout the two chapters, however, only one new and noteworthy property in regard to continued-fraction determinants is brought to light, the source of it being an identity attributed to Heilermann,† namely,

$$x+\cfrac{b_1}{1+\cfrac{b_2}{x+\cfrac{b_3}{1+\ddots}}} = x+b_1-\cfrac{b_1b_2}{x+b_2+b_3-\cfrac{b_3b_4}{x+b_4+b_5-\ddots}}$$

where the number of b's appearing on both sides is n. Clearly the numerator of each convergent on the left must have an intimate relation with the numerator of one of the convergents on the right, and this Günther doubtless sought to discover. The result which he established is

$$\begin{vmatrix} x & b_1 & . & . & .... & . & . \\ -1 & 1 & b_2 & . & .... & . & . \\ . & -1 & x & b_3 & .... & . & . \\ . & . & -1 & 1 & .... & . & . \\ . & . & . & . & .... & . & . \\ . & . & . & . & .... & x & b_{2n-1} \\ . & . & . & . & .... & -1 & 1 \end{vmatrix} = \begin{vmatrix} x+b_1 & -b_1b_2 & . & .... \\ -1 & x+b_2+b_3 & -b_3b_4 & .... \\ . & -1 & x+b_4+b_5 & .... \\ . & . & . & . \end{vmatrix},$$

where the order-number of the determinant on the right is half that of the determinant on the left. The demonstration occupies as many as five pages (pp. 70–74), and yet the case where the original determinant is of odd order is not referred to.

Similarly, the continued-fraction

$$1+\cfrac{b_1}{1-b_1+\cfrac{b_2}{1-b_2+\ddots}}$$

* The total number of footnote references in the book is 116.

† *Zeitschrift f. Math. u. Phys.*, v. (1860), pp. 362–363.

suggests the finding (pp. 80, 81) of the results

$$\begin{vmatrix} 1 & b_1 & . & \dots \\ -1 & 1-b_1 & b_2 & \dots \\ . & -1 & 1-b_2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{n+1} = 1,$$

and

$$\begin{vmatrix} 1-b_1 & b_2 & . & \dots \\ -1 & 1-b_2 & b_3 & \dots \\ . & -1 & 1-b_3 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_{n} = 1-b_1+b_1b_2-b_1b_2b_3+\dots,$$

the latter of which may be compared with a result obtained by Nachreiner when under the same influence.

GÜNTHER, S. (1873).

[Ueber die allgemeine Auflösung von Gleichungen durch Kettenbrüche. *Math. Annalen*, vii. pp. 262–268.]

The starting-point here is Fürstenau's expression of 1860 for the smallest root of the equation

$$x_n+a_{n-1}x^{n-1}+\dots+a_2x^2+a_1x+a_0 = 0.$$

All that is of interest in the paper from our present point of view is the fact that Fürstenau's two determinants are changed so as to have the zero elements on the under side of the main diagonal matched by like elements in the corresponding places on the upper side, with a view to expressing the root of the equation in the form of an interminate continued-fraction. The procedure need not be described at length; it will be understood at once by applying it in the case of the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ a_0 & a_1 & a_2 & a_3 \\ . & a_0 & a_1 & a_2 \\ . & . & a_0 & a_1 \end{vmatrix},$$

—a course which also enables us to test the accuracy of the illustrative example given later in the paper. In the first place, the element in the place (1, 4) is changed, the result being

$$\begin{vmatrix} a_1 & a_2 & a_3 & \\ a_0 & a_1 & a_2 & a_3{}^2-a_2 \\ . & a_0 & a_1 & a_3a_2-a_1 \\ . & . & a_0 & a_3a_1-a_0 \end{vmatrix} \div a_3\,;$$

secondly, the element in the place (1, 3) is changed, the result being

$$\begin{vmatrix} a_1 & a_2 & . & . \\ a_0 & a_1 & a_2{}^2-a_3a_1 & a_3{}^2-a_2 \\ . & a_0 & a_2a_1-a_3a_0 & a_3a_2-a_1 \\ . & . & a_2a_0 & a_3a_1-a_0 \end{vmatrix} \div a_3a_2\,;$$

and lastly, the element in the place (2, 4), with the result

$$\frac{\begin{vmatrix} a_1 & a_2 & . & . \\ a_0 & a_1 & a_2{}^2-a_3a_1 & . \\ . & a_0 & a_2a_1-a_3a_0 & (a_2{}^2-a_3a_1)(a_3a_2-a_1)-(a_2a_1-a_3a_0)(a_2{}^2-a_3) \\ . & . & a_2a_0 & (a_2{}^2-a_3a_1)(a_3a_1-a_0)-a_2a_0(a_3{}^2-a_2) \end{vmatrix}}{a_3a_2(a_2{}^2-a_3a_1)}.$$

It is thus seen that if the equation for solution be the quartic

$$x^4+a_3x^3+a_2x^2+a_1x+a_0 \;=\; 0,$$

a convergent to the smallest root is, according to Fürstenau,

$$\begin{vmatrix} -a_0 & a_2 & a_3 & 1 \\ . & a_1 & a_2 & a_3 \\ . & a_0 & a_1 & a_2 \\ . & . & a_0 & a_1 \end{vmatrix} \div \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ a_0 & a_1 & a_2 & a_3 \\ . & a_0 & a_1 & a_2 \\ . & . & a_0 & a_1 \end{vmatrix},$$

and therefore, by the preceding transformation, if we write U, V for the two lengthiest elements, is equal to

$$\begin{vmatrix} -a_0 & a_2 & . & . \\ . & a_1 & a_2{}^2-a_3a_1 & . \\ . & a_0 & a_2a_1-a_3a_0 & \mathrm{U} \\ . & . & a_2a_0 & \mathrm{V} \end{vmatrix} \div \begin{vmatrix} a_1 & a_2 & . & . \\ a_0 & a_1 & a_2{}^2-a_3a_1 & . \\ . & a_0 & a_2a_1-a_3a_0 & \mathrm{U} \\ . & . & a_2a_0 & \mathrm{V} \end{vmatrix},$$

and consequently

$$= \frac{-a_0}{a_1} - \frac{a_0a_2}{a_1} - \frac{a_0(a_2{}^2-a_3a_1)}{a_2a_1-a_3a_0} - \frac{a_2a_0\mathrm{U}}{\mathrm{V}}.$$

This does not agree with Günther's result, an awkward error appearing in the last column of both his determinants, and thus also in his continued-fraction.*

GÜNTHER, S. (1873).

[Beiträge zur Theorie der Kettenbrüche. *Archiv d. Math. u. Phys.*, lv. pp. 392–404.]

Only one of the sections of this paper—the third (pp. 397–401)—is connected with continued-fraction determinants, the subject dealt with being practically an extension of the result given in the immediately preceding paper by the same writer. Instead of the special persymmetric determinant which there forms the denominator of the fraction proposed for transformation, we have now a perfectly general determinant; that is to say, the given fraction now is

$$\begin{vmatrix} 1 & a_2 & a_3 & \dots \\ . & b_2 & b_3 & \dots \\ . & c_2 & c_3 & \dots \\ . & . & . & \dots \end{vmatrix}_n \div \begin{vmatrix} a_1 & a_2 & a_3 & \dots \\ b_1 & b_2 & b_3 & \dots \\ c_2 & c_2 & c_3 & \dots \\ . & . & . & \dots \end{vmatrix}_n .$$

The procedure followed is the same as before, a double application of it, however, being necessary.

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* With the correct values for U and V as here given, it is possible to take a step farther. For it will be found that

$$U = a_3 \begin{vmatrix} a_2 & a_3 & 1 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}, \qquad V = a_3 \begin{vmatrix} a_1 & a_3 & 1 \\ a_0 & a_2 & a_3 \\ . & a_1 & a_2 \end{vmatrix};$$

and consequently that the continued-fraction is

$$\frac{-a_0}{a_1} - \frac{a_0 a_2}{a_1} - \frac{a_0 \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}} - \frac{a_0 a_2 \begin{vmatrix} a_2 & a_3 & 1 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_3 & 1 \\ a_0 & a_2 & a_3 \\ . & a_1 & a_2 \end{vmatrix}},$$

where the three-line determinants are the complementary minors of the elements (4, 1), (4, 2) of the original determinant, and where the two-line determinants are the cofactors of the last elements in the three-line determinants, and so on.

MUIR, T. (1874).

[Continuants: a new special class of determinants. *Proceed. R. Soc. Edinburgh*, viii. pp. 229–236.]

Still another discoverer, twenty years after the publication of Sylvester's first paper on the subject! Fortunately the 'discoveries' are not expounded at inordinate length, the paper consisting of twenty-two very short paragraphs, almost every one of which contains a concisely stated property, accompanied, where necessary, by an indication of the mode of proof.

'Continuant' is formally defined as being a determinant which has the elements lying outside the principal diagonal and the two bordering minor diagonals each equal to zero, and which has the element of one of these minor diagonals each equal to negative unity. The continuant

$$\begin{vmatrix} a_1 & b_1 & . & \dots \\ -1 & a_2 & b_2 & \dots \\ . & -1 & a_3 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \text{ is denoted by } \mathrm{K}\begin{pmatrix} b_1 & b_2 & \dots \\ a_1 & a_2 & a_3 \dots \end{pmatrix},$$

$a_1, a_2, a_3, \dots$ being spoken of as the *main* diagonal, and $b_1, b_2, \dots$ as the *minor* diagonal. When the b's are each equal to unity, the continuant is called a *simple* continuant, and denoted by $\mathrm{K}(a_1, a_2, a_3, \dots)$.

Of results previously formulated it may be worth noting that the most important, namely,

$$\mathrm{K}\begin{pmatrix} b_1 \dots b_{n-1} \\ a_1 \quad a_2 \dots a_n \end{pmatrix} \cdot \mathrm{K}\begin{pmatrix} b_h \dots b_{p-1} \\ a_h \dots \dots a_p \end{pmatrix} \quad \text{where } h < p < n$$

$$= \mathrm{K}\begin{pmatrix} b_1 \dots b_{p-1} \\ a_1 \dots \dots a_p \end{pmatrix} \cdot \mathrm{K}\begin{pmatrix} b_h \dots b_{n-1} \\ a_h \dots \dots a_n \end{pmatrix}$$

$$\pm (-)^{p-h+1} b_{h-1} b_h \dots b_p \cdot \mathrm{K}\begin{pmatrix} b_1 \dots b_{h-3} \\ a_1 \dots \dots a_{h-2} \end{pmatrix} \cdot \mathrm{K}\begin{pmatrix} b_{p+2} \dots b_{n-1} \\ a_{p+2} \dots \dots a_n \end{pmatrix}$$

is not obtained in the same manner as in Thiele's paper of 1870, but from a general theorem regarding the product of *any* determinant by one of its own minors.*

* This theorem was not published until five years afterwards. See above, p. 79.

Leaving aside other results of this kind, we have remaining on the first three pages:

$$\mathrm{K}\begin{pmatrix} b_1 \ldots . b_{n-1} \\ -a_1, -a_2, \ldots . -a_n \end{pmatrix} = (-)^n \mathrm{K}\begin{pmatrix} b_1 \ldots . b_{n-1} \\ a_1, a_2, \ldots, a_n \end{pmatrix},$$

$$\mathrm{K}(1, a_1, a_2, \ldots, a_n) = \mathrm{K}(a_1+1, a_2, \ldots, a_n),$$

$$\mathrm{K}(\ldots a, b, c, 0, e, f, g, \ldots) = \mathrm{K}(\ldots a, b, c+e, f, g, \ldots),$$

$$\mathrm{K}(\ldots a, b, c, 0, 0, 0, e, f, \ldots) = \mathrm{K}(\ldots a, b, c+e, f, \ldots),$$

$$\ldots \ldots \ldots = \ldots \ldots \ldots$$

$$\mathrm{K}(\ldots a, b, 0, 0, e, f, \ldots) = \mathrm{K}(\ldots a, b, e, f, \ldots),$$

$$\ldots \ldots \ldots = \ldots \ldots \ldots$$

Then there follows the theorem that *whatever* a_1, a_2, *may be, the continuant* $\mathrm{K}(a_1, a_2 \ldots, a_n)$ *is prime to* $\mathrm{K}(a_1, a_2, \ldots a_{n-1})$, $\mathrm{K}(a_2, \ldots, a_n)$, $\mathrm{K}(a_1-1, a_2, \ldots, a_n)$, $\mathrm{K}(a_1, a_2, \ldots, a_n-1)$, the foundation for this being the ordinary division-process of finding the greatest common measure of two integers.

Symmetric continuants are next defined and four properties stated, the two last of the four being

$$a_n \cdot \mathrm{K}(a_1, \ldots, a_{n-1}, a_n, a_{n-1}, \ldots, a_1) = \mathrm{K}(a_1, \ldots, a_n)^2 - \mathrm{K}(a_1, \ldots, a_{n-1})^2$$

$$\mathrm{K}(a_1, , \ldots, a_{n-1}, 2a_n, a_{n-1}, \ldots, a_1) = 2\mathrm{K}(a_1, \ldots, a_{n-1})\mathrm{K}(a_1, \ldots, a_n).$$

The fundamental relation with continued-fractions having been established, and thence the theorem

$$\mathrm{A} + \underset{*}{\frac{b_1}{a_1}} + \frac{b_2}{a_2} + \ldots + \frac{b_3}{a_2} + \frac{b_2}{a_1} + \underset{*}{\frac{b_1}{2\mathrm{A}}} + \ldots = \sqrt{\frac{\mathrm{K}\begin{pmatrix} b_1, & b_2, \ldots\ldots . b_2, & b_1 \\ \mathrm{A}, & a_1, & a_2, \ldots . a_2, & a_1, & \mathrm{A} \end{pmatrix}}{\mathrm{K}\begin{pmatrix} b_2 \ldots\ldots . b_2 \\ a_1, & a_2 \ldots . a_2, & a_1 \end{pmatrix}}},$$

the deduction is made that

$$\frac{\mathrm{K}\begin{pmatrix} b_1, & b_2, \ldots\ldots . b_2, & b_1 \\ \mathrm{A}, & a_1, & a_2, \ldots, a_2, & a_1, & \mathrm{A} \end{pmatrix}}{\mathrm{K}\begin{pmatrix} b_2 \ldots\ldots . b_2 \\ a_1, & a_2, \ldots, a_2, & a_1 \end{pmatrix}}$$

$$= \frac{\mathrm{K}\begin{pmatrix} b_1, & b_2, \ldots\ldots, b_2, & b_1 & b_1, & b_2, \ldots\ldots, b_2, & b_1 \\ \mathrm{A}, & a_1, & a_2, \ldots, a_2, & a_1, & 2\mathrm{A}, & a_1, & a_2 \ldots, a_2, & a_1, & \mathrm{A} \end{pmatrix}}{\mathrm{K}\begin{pmatrix} b_2 \ldots\ldots . b_2, & b_1 & b_1, & b_2, \ldots\ldots, b_2 \\ a_1, & a_2, \ldots, a_2, & a_1, & 2\mathrm{A}, & a_1, & a_2, \ldots, a_2, & a_1 \end{pmatrix}}$$

$$= \ldots\ldots$$

Any other results given belong properly to the theory of continued-fractions.*

MUIR, T. (1874).

[Further note on continuants. *Proceed. R. Soc. Edinburgh*, viii. pp. 380–382.]

Proceeding from the fact, obtained in the previous paper, that the order of a continuant may be depressed if the first element of the main diagonal be unity, namely, that

$$\mathrm{K}\begin{pmatrix} b_1 & b_2 & b_3 \ldots . \\ 1 & a_1 & a_2 & a_3 \ldots . \end{pmatrix} = \mathrm{K}\begin{pmatrix} & b_2 & b_3 \ldots . \\ a_1+b_1 & a_2 & a_3 \ldots . \end{pmatrix},$$

and that

$$\mathrm{K}\begin{pmatrix} & mb_1 & b_2 \ldots . \\ ma_1 & & a_2 & a_3 \ldots . \end{pmatrix} = m \cdot \mathrm{K}\begin{pmatrix} b_1 & b_2 \ldots . \\ a_1 & a_2 & a_3 \ldots . \end{pmatrix},$$

the author shows that

$$\mathrm{K}\begin{pmatrix} b_1 & (b_1+a_1)b_2 & (b_2+a_2)b_3 \ldots . \\ 1 & a_1 & a_2 & a_3 \ldots . \end{pmatrix} = (b_1+a_1)(b_2+a_2) \ldots ,$$

and this is at once used to obtain as an equivalent for

$$\frac{d_1 d_2 d_3 \ldots .}{e_1 e_2 e_3 \ldots .},$$

the continued-fraction given in Stern's monograph of 1833 (*Crelle's Journal*, x. p. 267).

The fact communicated by Cayley, but now known to be found in Nachreiner's paper of 1872, viz., that any continuant is expressible by means of a simple continuant, is noted; likewise the identities

$$\mathrm{K}(a_1 x^{-1}, a_2 x, a_3 x^{-1}, a_4 x, \ldots)_n = \mathrm{K}(a_1, a_2, \ldots, a_n) \text{ if } n \text{ be even}$$
$$\text{and} \qquad = x^{-1}\mathrm{K}(a_1, a_2, \ldots, a_n) \text{ if } n \text{ be odd.}$$

WOLSTENHOLME, J. (1874).

[(Evaluation of a simple continuant whose diagonal is univarial.) Problem 4391. *Educ. Times*, xxvii. pp. 45–67; *Math. from Educ. Times*, xxi. pp. 83–85.]

The problem is to prove that

$$\mathrm{K}(a, {}^{-1}\, a, {}^{-1} \ldots . , a)_n = \sin(n+1)\theta / \sin\theta, \text{ if } a = 2\cos\theta.$$

* Some of these results and others are dealt with more fully in a pamphlet of 32 pages printed about the same time at the University Press in Glasgow, and entitled 'The Expression of a Quadratic Surd as a Continued-Fraction.'

Two proofs are given, the more interesting depending on the fact that

$$K_n = aK_{n-1} - K_{n-2},$$

and the fact that

$$\sin n\theta = a \sin (n-1)\theta - \sin (n-2)\theta;$$

in other words, that K_n and $\sin n\theta$ satisfy the same difference-equation.*

STUDNIČKA, F. J. (1874, 1877).

[Příspěvek k nauce o zlomcich řetězových neb řetézcich. *Časopis pro pestov. math. a fys.*, iii. pp. 61–70.]

[Beitrag zur Determinanten-Theorie. *Sitzungsb. d. k. böhm. Ges. d. Wiss.* (Prag), pp. 120–125.]

The first of these is merely a fuller and better exposition than his paper of March, 1872.

The second has been sufficiently noticed under general determinants.

GÜNTHER, S. (1874).

[Lehrbuch der Determinanten-Theorie, ... viii+236 pp. Erlangen.]

In view of the attention which Günther had previously given to the subject, it was natural that when he came to write his text-book on Determinants he should assign space to the consideration of the special form connected with continued-fractions. We are not surprised, therefore, to find a whole chapter (Kap. vi.) with the heading 'Kettenbrüchdeterminanten.' It extends to 31 pages, and is mainly a clear and detailed reproduction of his own and other previous work.

Other text-books of the period which deal with continuants in

* In connection herewith it may be noted that corresponding problems in the theory of continued-fractions date much earlier: for example, problem No. 40 of *Crelle's Journal*, ii. (1827), p. 193, a solution of which is given by Th. Clausen in a paper with the title 'Die Function $\frac{1}{a} + \frac{1}{a} + \frac{1}{a} + \ldots$ durch die Anzahl der a ausgedrückt,' published in *Crelle's Journal*, iii. pp. 87–88. In our analysis of Ramus' paper of 1856 the first of the two values there given for the continued-fraction is that obtained by Clausen.

a way calling for no special note are those of Hattendorf (1872), Diekmann (1875), Guldberg (1876), Baraniecki (1878), and Mansion (1878).

GÜNTHER, S. (1875).

[Das independente Bildungsgesetz der Kettenbrüche. *Denkschr. d. k. Akad. d. Wiss. in Wien*: *Math.-Nat. Cl.*, xxxvi. pp. 187–194.]

Of this paper the portion which is not introductory and semi-historical concerns the development of

$$\begin{vmatrix} x & -a_1 & . & \dots & . & . \\ a_1 & x & -a_2 & \dots & . & . \\ . & a_2 & x & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & x & -a_{n-1} \\ . & . & . & \dots & a_{n-1} & x \end{vmatrix},$$

or Δ say,

in a series arranged according to powers of x.

The result obtained is, as might have been expected, that to which he would have been led by the use of the combinatorial 'rule' more than once already referred to, and found fully enunciated by Stern in 1833 (*Crelle's Journal*, x. p. 6).

HELMLING, P. (1876).

[Anwendung der Determinanten zur Darstellung transcendenter Funktionen. (Title also in Latin.) Dissert. 43 pp. Dorpat.]

The subject here is the evaluation of certain definite-integrals by establishing for each a trinomial or quadrinomial recurrence-formula after the manner of Euler* (1777), and then making use of continuants or the analogous special determinants noted by Sylvester in 1853 as arising in this way (*Hist.*, ii. p. 416). No new property is brought to light.

SALMON, G. (1876).

[. . . . Modern Higher Algebra. Third Edition. xx+318 pp. Dublin.]

In this edition the sketch of the theory of determinants with which the book opens is extended to fifty pages, and half a page—

* *Acta Petrop.*, i. p. 3.

a model of compact exposition—is assigned to the special form we are now considering. The name 'continuant' is adopted.

MUIR, T. (1877).

[A theorem in continuants. *Philos. Magazine*, (5) iii. pp. 137–138.]

The theorem in question is

$$\mathrm{K}\begin{pmatrix} \omega+a_1x, & \omega+a_2x, & \\ a_1+x, & a_2, & \dots a_n \end{pmatrix} = \mathrm{K}\begin{pmatrix} \omega+a_2x, & \omega+a_3x, & \\ a_1, & a_2, & \dots a_n+x \end{pmatrix}.$$

The proof need not be reproduced, as it is included in another to be immediately given.

The writer having become acquainted with Sylvester's early papers on the subject, takes the opportunity to withdraw, in favour of the latter, the claim implied in the title of the paper of 1874.

MUIR, T. (1877).

[Extension of a theorem in continuants, with an important application. *Philos. Magazine* (5), iii. pp. 360–366.]

The theorem referred to being, as we have seen, a relation (of equality) between two continuants, it was natural to expect the existence of a relation between the two continued-fractions corresponding to those continuants. This expectation led to an acquaintance with Bauer's paper of 1872, and thereafter to the extension here intimated.

The extension for the case of the 5th order is

$$\begin{vmatrix} \delta+\frac{b_1}{r} & b_1 & . & . & . \\ -c_1 & \delta+\frac{b_2}{r}-c_1r & b_2 & . & . \\ . & -c_2 & \delta+\frac{b_3}{r}-c_2r & b_3 & . \\ . & . & -c_3 & \delta+\frac{b_4}{r}-c_3r & b_4 \\ . & . & . & -c_4 & \delta+\frac{b_5}{r}-c_4r \end{vmatrix}$$

$$= \begin{vmatrix} \delta + \frac{b_1}{r} - c_1 r & b_2 & . & . & . \\ -c_1 & \delta + \frac{b_2}{r} - c_2 r & b_3 & . & . \\ . & -c_2 & \delta + \frac{b_3}{r} - c_3 r & b_4 & . \\ . & . & -c_3 & \delta + \frac{b_4}{r} - c_4 r & b_5 \\ . & . & . & -c_4 & \delta + \frac{b_5}{r} \end{vmatrix},$$

the two sets of operations necessary to establish it being

$$\text{row}_1 + r\,\text{row}_2, \qquad \text{row}_2 + r\,\text{row}_3, \quad \ldots .$$

and

$$\text{col}_2 - r\,\text{col}_1, \qquad \text{col}_3 - r\,\text{col}_2, \quad \ldots .$$

The case previously announced is that where

$$c_1 = c_2 = c_3 = \ldots = 1 \quad \text{and} \quad \delta - r = \beta.$$

Bauer's result regarding the product of two continued-fractions having been noted in reviewing his paper, the generalization of it here obtained may also be shortly alluded to. Denoting by F the continued-fraction

$$\frac{b_1(b_2-\alpha)}{\beta} + \frac{b_2(b_3-\alpha)}{\beta} + \ddots + \frac{b_{n-1}(b_n-\alpha)}{\beta}$$

the author first shows, with the help of his theorem in continuants, that

$$\frac{\beta+\alpha}{2} - b_2 + \frac{b_2(b_2-\alpha)}{\beta+b_2-b_3} + \frac{b_3(b_3-\alpha)}{\beta+b_3-b_4} + \ddots + \frac{b_n(b_n-\alpha)}{\beta+b_n-\alpha} = \frac{\alpha-\beta}{2}\cdot\frac{\mathrm{F}-b_1}{\mathrm{F}+b_1};$$

thence, by merely changing the sign of a and the signs of all the b's, that

$$\frac{\beta-a}{2}+b_2+\frac{b_2(b_2-a)}{\beta-b_2+b_3}+\frac{b_3(b_3-a)}{\beta-b_3+b_4+}\cdot\cdot\cdot\cdot\cdot+\frac{b_n(b_n-a)}{\beta-b_n+a}=\frac{-a-\beta}{2}\cdot\frac{F+b_1}{F-b_1};$$

and therefore, by multiplication, that *the product of the two continued-fractions on the left is equal to*

$$\frac{\beta^2-a^2}{4}.$$

The paper concludes with two other curious theorems of similar character.*

* The rather noteworthy theorem in continuants which is the basis of this striking result in continued-fractions can be still more widely generalized as follows :

The continuant

$$\begin{vmatrix} \frac{b_1}{s_1}+A & b_1 & . & . & . \\ c_1 & \frac{b_2}{s_2}+\frac{s_1}{r_1}A+s_1c_1 & b_2 & . & . \\ . & c_2 & \frac{b_3}{s_3}+\frac{s_2}{r_2}A+s_2c_2 & b_3 & . \\ . & . & c_3 & \frac{b_4}{s_4}+\frac{s_3}{r_3}A+s_3c_3 & b_4 \\ . & . & . & c_4 & \frac{b_5}{s_5}+\frac{s_4}{r_4}A+s_4c_4 \end{vmatrix}$$

is unaltered in value by adding

$$r_1c_1,\quad r_2c_2-s_1c_1,\quad r_3c_3-s_2c_2,\quad r_4c_4-s_3c_3,\quad -s_4c_4$$

to the elements of the main diagonal, and changing the elements of the upper minor diagonal into

$$\frac{r_1}{s_2}b_2,\quad \frac{r_2}{s_3}b_3+s_2\left(1-\frac{s_1}{r_1}\right)A,\quad \ldots,\quad \frac{r_4}{s_5}b_5+s_4\left(1-\frac{s_3}{r_3}\right)A.$$

Putting in this $r_1, r_2, \ldots = s_1, s_2, \ldots$ we have the identity

$$\begin{vmatrix} \frac{b_1}{s_1}+A & b_1 & . & . & . \\ c_1 & \frac{b_2}{s_2}+A+s_1c_1 & b_2 & . & . \\ . & c_2 & \frac{b_3}{s_3}+A+s_2c_2 & b_3 & . \\ . & . & c_3 & \frac{b_4}{s_4}+A+s_3c_3 & b_4 \\ . & . & . & c_4 & \frac{b_5}{s_5}+A+s_4c_4 \end{vmatrix}$$

MUIR, T. (1877).

[Note on determinant expressions for the sum of a harmonical progression. *Proceed. R. Soc. Edinburgh*, ix. pp. 361–363.]

The progression is transformed by Euler's process into a continued-fraction, and thence into the quotient of one continuant by another. The second of these, in view of previously obtained factorizable continuants (Sylvester 1854, Painvin 1858) is worth noting, namely.

$$\begin{vmatrix} a & a^2 & . & . & \dots \\ 1 & 2a+b & (a+b)^2 & . & \dots \\ . & 1 & 2a+3b & (a+2b)^2 & \dots \\ . & . & 1 & 2a+5b & \dots \\ . & . & . & . & \dots \end{vmatrix} = a(a+b)(a+2b)(a+3b) \dots$$

MARTIN, A. (1877).

[Question 5484. *Educ. Times*, xxx. p. 197; or *Math. from Educ. Times*, xxix. pp. 90–91.]

The theorem here involved, though stated differently, is that the product of any two of the expressions

$$1 + \mathrm{K}(b,a,a,b), \quad 1 + \mathrm{K}(c,b,a,a,b,c), \quad 1 + \mathrm{K}(1,c,b,a,a,b,c,1)$$

diminished by the sum of the same two is a square. It is not pointed out, however, that the three squares in question are the squares of

$$\mathrm{K}(b,a,a,b,c), \quad \mathrm{K}(b,a,a,b,c,1), \quad \mathrm{K}(c,b,a,a,b,c,1).$$

$$= \begin{vmatrix} \frac{b_1}{s_1} + \mathrm{A} + s_1c_1 & \frac{s_1}{s_2}b_2 & . & . & . \\ c_1 & \frac{b_2}{s_2} + \mathrm{A} + s_2c_2 & \frac{s_2}{s_3}b_3 & . & . \\ . & c_2 & \frac{b_3}{s_3} + \mathrm{A} + s_3c_3 & \frac{s_3}{s_4}b_4 & . \\ . & . & c_3 & \frac{b_4}{s_4} + \mathrm{A} + s_4c_4 & \frac{s_4}{s_5}b_5 \\ . & . & . & c_4 & \frac{b_5}{s_5} + \mathrm{A} \end{vmatrix};$$

and specializing further by putting each of the s's equal to s we come back to the theorem of the paper above reviewed.

LUCAS, E. (1877).

[Théorie des fonctions numériques simplement périodiques. Sect. IV. *Nouv. Corresp. Math.*, iii. pp. 401–402; or *American Journ. of Math.*, i. pp. 192–193.]

Here there is nothing fresh save the writing of

$$\begin{vmatrix} p & -q & . & \dots \\ -1 & p & -q & \dots \\ . & -1 & p & \dots \\ . & . & . & \dots \end{vmatrix}$$ in the form $$\begin{vmatrix} p & \sqrt{q} & . & \dots \\ \sqrt{q} & p & \sqrt{q} & \dots \\ . & \sqrt{q} & p & \dots \\ . & . & . & \dots \end{vmatrix},$$

which latter, we may note, is axisymmetric with respect to the one diagonal, and persymmetric with respect to the other.

MUIR, T. (1878).

[On the word 'Continuant.' *American Journ. of Math.*, i. pp. 344.]

Sylvester having, without knowledge of the circumstances, complained of the introduction of the word 'continuant,' the introducer had to explain that the word had been proposed in ignorance of Sylvester's papers of 1853, and that, as soon as this ignorance was removed, he had drawn pointed attention to those papers,* persistently adhering, however, to the use of the new word for four reasons, which are shortly stated.

In a note, Sylvester agrees that the second and third of the reasons afford 'quite a sufficient justification for the use of the word in question.'

SCOTT, R. F. (1878).

[On some symmetrical forms of determinants. *Messenger of Math.*, viii. pp. 131–138.]

The first of the forms referred to is

$$\begin{vmatrix} c & a & . & \dots \\ b & c & a & \dots \\ . & b & c & \dots \\ . & . & . & \dots \end{vmatrix}_n,$$

*See above, p. 416.

which by means of its difference-equation is once more found to be equal to

$$\frac{\left\{c^2+\sqrt{c^2-4ab}\right\}^{n+1}-\left\{c^2-\sqrt{c^2-4ab}\right\}^{n+1}}{2^{n+1}\sqrt{c^2-4ab}}.$$

Five special cases, more or less known, are mentioned, the last being that in which $c^2 = 4ab$ and the value of the determinant $(ab)^{\frac{n}{2}}(n+1)$.

In the same way he finds (p. 182) that the determinant formed from the above by prefixing 0, 1, 1, . . . , 1 as a row and as a column is equal to

$$\frac{u^{n+1}+v^{n+1}}{(a+b+c)^2}-\frac{(n+1)c(u^{n+1}+v^{n+1})}{(a+b+c)(u-v)^2}$$
$$+\frac{2ab(n+1)}{a+b+c}\cdot\frac{u^n+v^n}{(u-v)^2}-\frac{(-a)^{n+1}+(-b)^{n+1}}{(a+b+c)^2};$$

also, that the determinant formed from this by deleting the first column and the last row is equal to

$$\frac{a^n}{a+b+c}+(-1)^n\frac{u^{n+1}-v^{n+1}+b(u^n-v^n)}{(a+b+c)(u-v)};$$

where u, v are the roots of the equation

$$x^2-cx+ab=0.$$

SYLVESTER, J. J. (1879).

[Notes on continuants. *Messenger of Math.*, viii. pp. 187–189.]

Here the main point of interest is the use of the old 'rule' referred to in his paper of May 1853 to provide a proof that the number of terms in the simple continuant $(a_1, a_2, \ldots, a_n)$ is

$$1+(n-1)+\frac{(n-2)(n-3)}{1\cdot 2}+\frac{(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3}+\ldots,$$

the various parts of this expression being shown to correspond with the various kinds of terms obtained in following the 'rule,' viz., 1 term containing *all* the elements, $n-1$ terms containing $n-2$ elements, $\frac{1}{2}(n-2)(n-3)$ terms containing $n-4$ elements, and

so on. In stating some related results he uses the word 'pro-continuant' for a determinant of the form

$$\begin{vmatrix} a & 1 & . & . & . & . \\ 1 & b & 1 & . & . & . \\ . & 1 & c & . & . & . \\ . & . & . & . & . & . \end{vmatrix},$$

that is to say, for the continuant

$$(a,^{-1},\ b,^{-1},\ c,^{-1},\ \ldots).$$

SCOTT, R. F. (1880).

[. . . . Theory of Determinants, and their applications in analysis and geometry. xii+251 pp. Cambridge.]

The title of one of Scott's chapters (chap. xiii. pp. 169–179) is 'Applications to the theory of continued-fractions,' and it is accurately descriptive of the contents. As in Günther's text-book, both ascending and descending continued-fractions are dealt with by means of determinants, the numerator of the last convergent to

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} + \ldots + \frac{b_n}{a_n},$$

viz., the determinant

$$\begin{vmatrix} b_1 & -1 & . & . & . & . \\ b_2 & a_2 & -1 & . & . & . \\ b_3 & . & a_3 & . & . & . \\ . & . & . & . & . & . \\ b_n & . & . & . & . & a_n \end{vmatrix},$$

being actually spoken of as 'the continuant for an ascending fraction.' No new property of continuants is given.

CHAPTER XVII.

MULTILINEANTS, UP TO 1877.

ANY student of determinants who has had to occupy himself with a problem involving the solution of a series of increasingly lengthy sets of linear equations must have had forced on him the idea of determinants of infinite order. Probably, however, the passage from order n to order ∞ is for the first time openly suggested in Wronski's statement (1812) of his 'loi générale,' where the r^{th} term of an ostensibly unending series involves a determinant of the r^{th} order (*Hist.*, i. pp. 472–478). With like probability the first actual mention of determinants with 'unendlich vielen Elementen,' and the first actual appearance of ∞ as the symbol of the order of a determinant date from 1825, being found in Schweins (*Hist.*, i. pp. 173–174). Between the latter date and that now reached the idea seems to have remained entirely below the surface.

FÜRSTENAU, E. (1860).

[Darstellung der reellen Wurzeln algebraischer Gleichungen durch Determinanten der Coefficienten. Sch.-Progr. 35 pp. Marburg.]

The given equation with real roots $x_1, x_2, \ldots, x_n$ being

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0,$$

Fürstenau uses the multipliers $1, x, x^2, \ldots, x^{p-1}$, and viewing the set of derived equations as being linear in $x, x^2, x^3, \ldots$, he eliminates $x^2, x^3, \ldots, x^p$ and obtains

$$\mathrm{R}_p x + \text{terms in } x^{p+1}, \ldots, x^{p+n-1} = -a\mathrm{R}_{p-1},$$

where R_p is the persymmetric recurrent

$$\begin{vmatrix} a_1 & a_0 & . & . & \cdots \\ a_2 & a_1 & a_0 & . & \cdots \\ a_3 & a_2 & a_1 & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & a_{n-4} & \cdots \\ 1 & a_{n-1} & a_{n-2} & a_{n-3} & \cdots \\ . & 1 & a_{n-1} & a_{n-2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_p .$$

Then, putting for x in this in succession $x_1, x_2, \ldots, x_n$, he has a set of equations from which he finds

$$R_p = (-1)^{np+p} \frac{\left| x_1^0 x_2^{p+1} x_3^{p+2} \cdots x_n^{p+n-1} \right|}{\left| x_1^0 x_2^1 x_3^2 \cdots x_n^{n-1} \right|},$$

and $\therefore$

$$= (-1)^{np+p} \frac{\left| x_1^{-p-1} x_2^0 x_3^1 \cdots x_n^{n-2} \right|}{\left| x_1^0 x_2^1 x_3^2 \cdots x_n^{n-1} \right|} \cdot (x_1 x_2 \ldots x_n)^{p+1},$$

$$= (-1)^{n+p} a_0^{p+1} \frac{\left| x_1^{-p-1} x_2^0 x_3^1 \cdots x_n^{n-2} \right|}{\left| x_1^0 x_2^1 x_3^2 \cdots x_n^{n-1} \right|}$$

$$= (-a_0)^{p+1} \left\{ \frac{1}{x_1^{p+1} f'(x_1)} + \frac{1}{x_2^{p+1} f'(x_2)} + \ldots + \frac{1}{x_n^{p+1} f'(x_n)} \right\}.$$

It thus follows that if x_1 be the arithmetically smallest root,

$$\lim \left(-a_0 \frac{R_{p-1}}{R_p} \right)_{p \to \infty} = -a_0 \frac{(-a_0)^p \dfrac{1}{x_1^p f'(x_1)}}{(-a_0)^{p+1} \dfrac{1}{x_1^{p+1} f'(x_1)}}$$

$$= x_1.$$

Returning then to the first derived set of equations, Fürstenau next eliminates $x^3, x^4, \ldots$ and obtains a quadratic from which may be found the two smallest roots x_1, x_2, and from which in particular it is seen that

$$x_1 x_2 = \lim (a_0 R_{2,p-1} / R_{2,p})_{p \to \infty},$$

where $R_{2,p}$ is the persymmetric determinant similar to R_p but beginning with the row a_2, a_1, a_0. In like fashion there is obtained

$$x_1x_2x_3 = \lim(- a_0R_{3,p-1}/R_{3,p})_{p\to\infty},$$

and so on generally.

Naturally it is pointed out at the beginning that

$$R_p = a_1R_{p-1} - a_0a_2R_{p-2} + a_0a_3R_{p-3} - \ldots.$$

and in the course of the work sufficient evidence is given that

$$R_pR_{p-q} - R_{p+1}R_{p-q-1} = a_0^{p-q}\Re_p,$$

where $\Re_p$ is the determinant got from R_{p+1} by deleting the $(q+1)^{th}$ row and last column.*

FÜRSTENAU, E. (1867).

[Neue Methode zur Darstellung und Berechnung der imaginären Wurzeln der Gleichungen durch Determinanten der Coefficienten. *Schriften d. Ges. zur Beford. . . . Naturw.* (Marburg), ix. pp. 19–48 ; also, separately, 32 pp. Marburg.]

Fürstenau does not confine himself here to the consideration of imaginary roots : he repeats in shorter form his exposition of 1860, merely taking up the new question when ready to do so. Thus, in the finding of x_1 it is only when he arrives at the expression for R_p as the quotient of two alternants that he introduces pairs of complex numbers,

$$\rho_1(\cos\phi_1 \pm i\sin\phi_1),$$
$$\rho_2(\cos\phi_2 \pm i\sin\phi_2),$$
$$\ldots\ldots\ldots$$

among the roots $x_1, x_2, \ldots, x_n$. So far as determinants are concerned, no fresh point is raised : and we may add that the extension of the method to imaginaries does not make it more attractive.

If we recall Faure's determinant form of 1855 (*Hist.*, ii. pp. 212–213) for the coefficients in the quotient of one power-series by another, it is not difficult to see that Fürstenau's first and main result may be formulated thus : *If the roots of the equation*

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0 = 0$$

* A fresh sketch of Fürstenau's pamphlet is given by Baltzer in the *Zeitschrift f. Math. u. Phys.*, vi. *Literaturz.* pp. 9–11.

be real, the arithmetically smallest of them is the quotient of the coefficient of x^{ν} *by the coefficient of* $x^{\nu+1}$ *in the expansion of*

$$(a_0+a_1x+a_2x^2+\ldots+x^n)^{-1}$$

when $p\to\infty$. It is thus not essentially different from that made known by D. Bernoulli * in 1728 and elaborated later by Euler and others.† Those interested in it may profitably consult portions of a lengthy paper published by Naegelsbach in 1876.‡

KÖTTERITZSCH, T. (1870).

[Ueber die Auflösung eines Systemes von unendlich vielen linearen Gleichungen. *Zeitschrift f. Math. u. Phys.*, xv. pp. 1–15, 229–268.]

In his first paper Kötteritzsch starts with the assertion that when n is infinite a solution of the set of equations

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = u_r \Big\}_{r=1}^{r=n}$$

can only be expected if a_{rs} and u_r be known functions of r, s and r respectively: and his proposed mode of aiding in the solution is to substitute for the given set of equations a derived set in the same variables but such that the r^{th} equation of the set has each of its first $r-1$ coefficients equal to 0. This is the same as to say that his standard set of equations is

$$\left.\begin{aligned} x_1 + b_{12}x_2 + b_{13}x_3 + \ldots + b_{1n}x_n + \ldots &= v_1 \\ x_2 + b_{23}x_3 + \ldots + b_{2n}x_n + \ldots &= v_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\}.$$

Unfortunately, the advantages of the transformation are not made apparent, nor is any light thrown on the subject by the example which is used for an illustration, namely, that in which

$$a_{rs} = a_s^r \quad \text{and} \quad u_r = u^r,$$

* Bernoulli, D. Observationes de seriebus, quae. . . . *Comment. acad. Petrop.*, iii. p. 85.

† 1748. Euler. Introd. in anal. infin. Cap. 17.
1798. Lagrange. Résolution des équat. num. Note 6.
1798. Legendre. . . . Théorie des Nombres. Art. 113.
1831. Fourier. Anal. des Équat.: Part I. Exposé synop., p. 68.
1834. Jacobi. Observatiunculae. . . . *Crelle's Journ.*, xiii. p. 349.

‡ Naegelsbach, H. Studien zu Fürstenau's neuer Methode. . . . *Archiv d. Math. u. Phys.*, lix. pp. 147–192, lxi. pp. 19–85.

the known solution (*Hist.*, ii. p. 155)

$$x^p = \frac{u}{a_p} \cdot \frac{(u-a_1)(u-a_2) \ldots (u-a_{p-1})(u-a_{p+1}) \ldots}{(a_p-a_1)(a_p-a_2) \ldots (a_p-a_{p-1})(a_p-a_{p+1}) \ldots}$$

being reached, and there being merely appended the remark that its usefulness depends on whether the infinite product involved has, or has not, a finite value for all positive integral values of p.

The second paper is very lengthy and almost equally disappointing, the new example being that in which $a_{rs} = 1/(a_r+b_s)$, as is the case in the set of n equations already solved by Binet in 1837 (*Hist.*, ii. pp. 155–158).

ZEIPEL, V. v. (1871).

We recall the special determinant, Z_{r+1} say, noted below (p. 458), which is such that, as r approaches ∞, the limit

$$(2r+1)Z_{r+1} = \frac{\pi}{2}.$$

HILL, G. W. (1877, 1886).

[On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon. *Acta Math.*, viii. (1886), pp. 1–36. Original edition published separately at Cambridge, U.S.A., in 1877.]

In the course of his investigation, Hill arrives at a differential equation of the second order; and, knowing the integral to be expressible in the form of an infinite series of multiples of cosines he substitutes such an expression in the equation, and, as the condition of satisfaction, obtains an infinite set of linear homogeneous equations connecting the unknown coefficients of the integral. From this he deduces by elimination the equation

$$\begin{vmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & (\xi-4)^2-\theta_0 & -\theta_1 & -\theta_2 & -\theta_3 & -\theta_4 & \ldots \\
\ldots & -\theta_1 & (\xi-2)^2-\theta_0 & -\theta_1 & -\theta_2 & -\theta_3 & \ldots \\
\ldots & -\theta_2 & -\theta_1 & \xi^2-\theta_0 & -\theta_1 & -\theta_2 & \ldots \\
\ldots & -\theta_3 & -\theta_2 & -\theta_1 & (\xi+2)^2-\theta_0 & -\theta_1 & \ldots \\
\ldots & -\theta_4 & -\theta_3 & -\theta_2 & -\theta_1 & (\xi+4)^2-\theta_0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix} = 0$$

for the determination of ξ, the main object of his quest. Calling the determinant here $D(\xi)$, he takes note of such properties of it as

$$D(-\xi) = D(\xi),$$
$$D(\xi+2\nu) = D(\xi),$$

and of the close resemblance which they bear to those of $\cos \pi\xi$. In this way he reaches without apparent difficulty an important substitute for his original equation, namely,

$$\cos \pi\xi = 1-\nabla(0),$$

where ∇ is the determinant obtained from D by multiplying the central row by -4, the first row on either side of the central row by $4/(4^2-1)$, the second row on either side by $4/(8^2-1)$, and so on. What remains then is the evaluation of $\nabla(0)$. This is accomplished by removing as a factor from each row the diagonal element of the row, and then calculating directly the value of the resulting determinant. The question of its convergence as the order tends to infinity is not forgotten, but no strict investigation is entered on.

It will be observed that this paper of Hill's is very different in character from those which precede it, and that with it we seem to pass from matters of languid though legitimate curiosity to one of serious importance.

ADAMS J. C. (1877).

[On the motion of the moon's node . . . *Monthly Notices R. Astron. Soc.*, xxxviii. pp. 43–49.]

Adams opens with an appreciative account of Hill's work, relating how in his own researches he had followed "in some repsect, a parallel course, sed longo intervallo." What is interesting for us to note is that the same infinite determinant had been reached, and the calculation connected therewith made in 1868 and to a greater degree of approximation in 1875.

In view of this statement attention may be drawn to a posthumously published paper of Adams' entitled "Development of a certain infinite determinant arising in relation to a motion of the node of the moon's orbit." (*Scientific Papers*, ii. pp. 85–103.)

CHAPTER XVIII.

CUBIC AND N-DIMENSIONAL DETERMINANTS UP TO 1880.

As our knowledge of the domain of a special subject is necessarily improved by an acquaintance with the surrounding country, we have considered it desirable throughout the foregoing to take a passing note of any attempts that may have been made to generalize the definition of a determinant. The first recorded step in this direction was taken by Cayley in 1843, the generalization consisting in passing from two sets of suffixes to m sets. (See *Hist.*, ii. p. 22.*) The subject with related matters was continued by him in 1845, in 1847, and in 1851. (See the papers referred to in *Hist.*, ii. pp. 33–34, 41–42, 63–68.) During the last of these years Sylvester became interested in the generalization as bearing on the discovery of invariants, and in that year and the year following devoted considerable attention to the subject. (See the papers referred to in *Hist.*, ii. pp. 61–63, 68–72.) We might also have noted that in 1853 Spottiswoode had been brought to think of something wider in scope than an ordinary determinant; for, when denoting the literal coefficients of the ternary cubic

$$ax^3+by^3+cz^3+3(fy^2z+gz^2x+hx^2y+f'yz^2+g'zx^2+h'xy^2)+6kxyz$$

by $\quad 111,\ 222,\ 333,\ 223,\ \ldots\ldots,\ 123,$

he says,† "The arrangement of the coefficients . . . . consequently involves space of three dimensions, and is beyond the cognizance of determinants . . . ." During the remainder of the period 1841–1860 no one else seems to have been attracted by the subject.

* Also Cayley's note of 1889 on it in his *Collected Math. Papers*, i. p. 584.

† *Crelle's Journ.*, li. p. 333.

At the beginning, however, of the period with which we are now concerned it again turned up, and was brought shyly into notice by its new discoverer. Fortunately this time the title of the little tract was free of all vagueness or ambiguity ; and, as a consequence, the functions which Cayley had originally called "functions resolvable into a series of determinants," and which now, on the other hand, were spoken of as 'determinants' themselves, were never afterwards for any length of time without a votary. The titles of the resulting writings up to 1880 are herewith given :

1861. [Gasparis, A. de]. Sur les déterminants dont les éléments ont plusieurs indices. Par Jean Blaise Grandpas.* 7 pp. 16 mo.

1863. Dahlander, G. R. Om en klass funktioner, hvilka ega flera egenskaper analoga med determinanternes. *Öfversigt af k. Vet.-Akad. Förhandl.* (Stockholm). Årg. xx. pp. 295–304.

1866. Armenante, A. Sui determinanti cubici. *Giornale di Mat.*, vi. pp. 175–181.

1868. Padova, E. Sui determinanti cubici. *Giornale di Mat.*, vi. pp. 182–189.

1868. Gasparis, A. de. Sopra due teoremi dei determinanti a tre indici ed un altra maniera di formazione degli elementi di un determinante ad m indici. *Rendic. dell' Accad. . . . di Napoli*, vii. pp. 118–121.

1868. Zehfuss, G. Ueber eine Erweiterung des Begriffes der Determinanten. 9 pp. Frankfurt a. M.

1876. Garbieri, G. Determinanti formati di elementi con un numero qualunque d'indici. *Giornale di Mat.*, xv. pp. 89–100.

1877. Bellavitis, G. Determinanti formati di n^1 elementi. *Atti . . . . Istituto Veneto.* An. 1877–78, pp. 251–273.

1878. Braasch, J. H. Determinanten höheren Ranges. Sch. Prog. 17 pp. Hamburg.

* Name composed of the same letters as the author's real name, Annibale de Gasparis. The paper was reprinted in the *Giornale di Matematiche*, xlviii. (1910).

1878. Gasparis, A. de. Prodotto di due determinanti a tre indici espresso con un determinante ordinario. *Atti . . . Accad. dei Lincei*, (3) *Transunti*, iii. pp. 44–45.

1879. Tanner, H. W. Lloyd. Notes on determinants of n dimensions. *Proceed. London Math. Soc.*, x. pp. 167–180.

1879. Scott, R. F. On cubic determinants and other determinants of higher class, and on determinants of alternate numbers. *Proceed. London Math. Soc.*, xi. pp. 17–29.

Of the many text-books published during the period, only Günther's (1875, 1877) and Baraniecki's (1878) gave the new 'determinants' a place (see above, pp. 54–55, 75); and the latter, not without reason, relegated them to an appendix.

CHAPTER XIX.

BORDERED DETERMINANTS, UP TO 1880.

To Sylvester, apparently, it first occurred to use the word 'border' in connection with determinants: we find him in 1852 speaking of 'a matrix *bordered* by the two given matrices' (*Hist.*, ii. p. 77). Such a matrix, however, had already presented itself to him in 1850 (*Hist.*, ii. pp. 118–119) and had been explicitly referred to in 1851 under a different designation. Thus, in his fretful 'Reply to Professor Boole's observations* . . .' the following passage occurs:

"Let the quadratic function be

$$ax^2+by^2+cz^2+dt^2+2exy+2\epsilon zt+2gxz+2\gamma yt+2hyz+2\eta xt$$

"and the linear functions (taken two in number)

$$lx+my+nz+pt,$$
$$l'x+m'y+n'z+p't.$$

"My numerator will be the determinant (hereinafter cited as the "*extended* determinant)

$$\begin{vmatrix} a & e & g & \eta & l & l' \\ e & b & h & \gamma & m & m' \\ g & h & c & \epsilon & n & n' \\ \eta & \gamma & \epsilon & d & p & p' \\ l & m & n & p & . & . \\ l' & m' & n' & p' & . & . \end{vmatrix}.$$

"To find the numerator of Mr. Boole's fraction we must form . . . ." A reference to the paper which Boole had commented on will

* *Cambridge and Dublin Math. Journ.*, vi. pp. 171–174; or *Collected Math. Papers*, i. pp. 181–183.

show that 'extended determinant' as here used would have been suitably replaced a year later by 'bordered discriminant.'

In Cayley's work the type in question appears as early as 1846 (*Hist.*, ii. p. 113), and probably he was ready to accept a name for it if the need again arose; at any rate in 1854, when dealing with skew determinants, he employed the expression 'un déterminant gauche bordé' for the case where the border is a single row and column.

With Hesse the type appears in 1853 (*Hist.*, ii. p. 129); and in the year following Brioschi published his theorem regarding what the English mathematicians called the 'bordered Hessian,' and thenceforward the usage tended to become more common. Certain it is that the determinants themselves soon came to be of very frequent occurrence, especially in geometrical investigations; good examples are Painvin's articles on the 'Application de la nouvelle analyse aux surfaces du second ordre' in the *Nouv. Annales de Math.* for 1859, C. W. Baur's in the *Zeitschrift f. Math. u. Phys.* for 1860 and 1861, and Clebsch's papers in *Crelle's Journ.* for 1863 and 1864.

A bordered determinant is in this sense *a determinant got by bordering another*, there being some reason for keeping that other prominently in view. Were this not the case it might be convenient to define it quite differently, namely, as *a determinant of the* $(n+r)^{th}$ *order which has nothing but zero elements in the complementary of its first* n*-line minor*. Theorems concerning such determinants may thus take two very different-looking forms. The property, for instance, which has already appeared in connection with the vanishing of one of the primary minors of a determinant may with at least equal convenience be put as follows: *If a null determinant be bordered, the resulting determinant is divisible by a linear homogeneous function of either set of bordering elements*; thus, if $|a_1b_2c_3| = 0$, then

$$\begin{vmatrix} . & x & y & z \\ \xi & a_1 & a_2 & a_3 \\ \eta & b_1 & b_2 & b_3 \\ \zeta & c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} x & y & z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot \begin{vmatrix} \xi & a_1 & a_2 \\ \eta & b_1 & b_2 \\ \zeta & c_1 & c_2 \end{vmatrix} \div |a_1b_2|$$

$$= - (xC_1+yC_2+zC_3)(\xi A_3+\eta B_3+\zeta C_3) \div C_3.$$

CLEBSCH, A. (1860).

[Ueber eine Classe von Eliminationsprobleme, und . . . . *Crelle's Journ.*, lviii. pp. 273–277.]

Clebsch's problem is the elimination of n unknowns from a special set of homogeneous equations, one of them a quadric, $n-2$ of them linear, and one of them an r^{th}-ic. The resultant in the case where r is 1 is a determinant formable, as in Sylvester's theorem just referred to, by bordering the discriminant of the quadric. When r is greater than 1, the resultant is more complicated, being in fact a function of such determinants. To the former case no reference is made by Clebsch, it naturally being from his point of view comparatively unimportant.

ROUTH, E. J. (1864).

(See under this heading in Chap. I.)

SALMON, G. (1866).

[. . . Modern Higher Algebra. 2nd ed. xv+296 pp. Dublin.]

Denoting the Hessian of u by $|u_{11}u_{22}u_{33}|$, Salmon draws attention (pp. 15–16) to the bordered Hessian

$$\begin{vmatrix} u_{11} & u_{12} & u_{13} & u_1 & a_1 \\ u_{21} & u_{22} & u_{23} & u_2 & a_2 \\ u_{31} & u_{32} & u_{33} & u_3 & a_3 \\ u_1 & u_2 & u_3 & . & . \\ a_1 & a_2 & a_3 & . & . \end{vmatrix},$$

where u_1, u_2, u_3 are the first differential-quotients of u, and the a's are constants. This five-line determinant he denotes by

$$\begin{pmatrix} ua \\ ua \end{pmatrix};$$

and performing on it the operation

$$(n-1)\,\text{col}_4 - x_1\,\text{col}_1 - x_2\,\text{col}_2 - x_3\,\text{col}_3,$$

followed by the operation

$$(n-1)\,\text{row}_4 - x_1\,\text{row}_1 - x_2\,\text{row}_2 - x_3\,\text{row}_3,$$

he readily deduces the result

$$\begin{pmatrix} ua \\ ua \end{pmatrix} = -\frac{n}{n-1}u\begin{pmatrix} a \\ a \end{pmatrix} - \frac{1}{(n-1)^2}(\Sigma ax)^2|u_{11}u_{22}u_{33}|.$$

He also points out that if there were four variables, and the matrix of the Hessian were triply bordered, we should in similar manner obtain

$$\begin{pmatrix} ua\beta \\ ua\beta \end{pmatrix} = -\frac{n}{n-1}u\begin{pmatrix} a\beta \\ a\beta \end{pmatrix} - \frac{1}{(n-1)^2}(\Sigma ax)^2\begin{pmatrix} \beta \\ \beta \end{pmatrix} - \frac{1}{(n-1)^2}(\Sigma \beta x)^2\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} + \frac{2}{(n-1)^2}\Sigma ax \cdot \Sigma \beta x\begin{pmatrix} a \\ \beta \end{pmatrix}.$$

There is also given at the close of the volume (pp. 248–252) a very helpful exposition of Clebsch's work of 1860 above referred to.

VERSLUYS, J. (1871).

[Applications des déterminants à l'algèbre et à gèometrie analytique. *Archiv d. Math. u. Phys.*, liii. pp. 137–187.]

Versluys' long series of 'applications' * at once calls to mind Painvin's of ten years earlier. The series is preceded by a page or so of purely algebraical introduction containing the result to which we have referred under Clebsch.

The problem, though not so stated, is to eliminate the variables from an n-ary quadric and $n-1$ linear homogeneous equations; for example, to eliminate x, y, z from

$$\left.\begin{aligned} ax^2+by^2+cz^2+2fyz+2gzx+2hxy &= 0 \quad (1) \\ \alpha_1 x+\beta_1 y+\gamma_1 z &= 0 \quad (2) \\ \alpha_2 x+\beta_2 y+\gamma_2 z &= 0 \quad (3) \end{aligned}\right\}.$$

As (1) can be written in the form

$$(ax+hy+gz)x+(hx+by+fz)y+(gx+fy+cz)z = 0,$$

Versluys asserts that it is necessary and sufficient that values for λ and μ can be found for which

$$\left.\begin{aligned} ax+hy+gz &= \lambda\alpha_1+\mu\alpha_2 \\ hx+by+fz &= \lambda\beta_1+\mu\beta_2 \\ gx+fy+cz &= \lambda\gamma_1+\mu\gamma_2 \end{aligned}\right\} \qquad (I)$$

* For others involving bordered determinants see *Archiv*, l. pp. 157–175; li. pp. 49–71.

when x, y, z have values determinable from (2) and (3); and that therefore it is necessary and sufficient that one set of values for x, y, z, λ, μ shall satisfy (I), (2), (3). The condition

$$\begin{vmatrix} a & h & g & \alpha_1 & \alpha_2 \\ h & b & f & \beta_1 & \beta_2 \\ g & f & c & \gamma_1 & \gamma_2 \\ \alpha_1 & \beta_1 & \gamma_1 & \cdot & \cdot \\ \alpha_2 & \beta_2 & \gamma_2 & \cdot & \cdot \end{vmatrix} = 0$$

is thus reached. A similar seven-line determinant is obtained for the next case.

We note for ourselves that if we write the given quadric as a bilinear, namely,

$$\begin{array}{ccc|c} x & y & z & \\ \hline a & h & g & x \\ h & b & f & y \\ g & f & c & z, \end{array}$$

and substitute therein the proportionate values of x, y, z deduced from (2) and (3), a quite different form of the eliminant is obtained. We thus arrive at the interesting identity

$$\begin{vmatrix} a & h & g & \alpha_1 & \alpha_1 \\ h & b & f & \beta_1 & \beta_2 \\ g & f & c & \gamma_1 & \gamma_2 \\ \alpha_1 & \beta_1 & \gamma_1 & \cdot & \cdot \\ \alpha_2 & \beta_2 & \gamma_2 & \cdot & \cdot \end{vmatrix} = \begin{array}{ccc|c} |\beta_1\gamma_2| & |\gamma_1\alpha_2| & |\alpha_1\beta_2| & \\ \hline a & h & g & |\beta_1\gamma_2| \\ h & b & f & |\gamma_1\alpha_2| \\ g & f & c & |\alpha_1\beta_2|, \end{array}$$

which is readily verifiable. In the second place we note that the unknowns might have been eliminated in two instalments, first, λ, μ from (I), and then x, y, z from the result and (2) and (3). The outcome in that case would have been

$$\begin{vmatrix} \begin{vmatrix} a & \alpha_1 & \alpha_2 \\ h & \beta_1 & \beta_2 \\ g & \gamma_1 & \gamma_2 \end{vmatrix} & \begin{vmatrix} h & \alpha_1 & \alpha_2 \\ b & \beta_1 & \beta_2 \\ f. & \gamma_1 & \gamma_2 \end{vmatrix} & \begin{vmatrix} g & \alpha_1 & \alpha_2 \\ f & \beta_1 & \beta_2 \\ c & \gamma_1 & \gamma_2 \end{vmatrix} \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix} = 0,$$

the determinant of which is again readily seen to be equal to Versluys' five-line form.

HESSE, O. (1872).

[Ein Cyclus von Determinanten-Gleichungen. *Crelle's Journ.*, lxxv. pp. 1–12 ; or *Abhandl. . . . . Akad. d. Wiss.* (München), xi. pp. 177–192 ; or (in Italian) *Giornale di Mat.*, xi. pp. 309–317 ; or *Werke*, pp. 585–598.]

Hesse slightly modifies Salmon's notation, writing

$$[u\alpha, u\alpha] \quad \text{for} \quad \begin{pmatrix} u\alpha \\ u\alpha \end{pmatrix},$$

and, more generally, putting

$$[\alpha\gamma, \beta\delta]$$

to stand for the determinant obtained by bordering $|u_{1n}|$ vertically by

$$\alpha_1, \alpha_2, \ldots, \alpha_n \quad \text{and} \quad \gamma_1, \gamma_2, \ldots, \gamma_n,$$

and horizontally by

$$\beta_1, \beta_2, \ldots, \beta_n \quad \text{and} \quad \delta_1, \delta_2, \ldots, \delta_n.$$

His starting point is Jacobi's theorem regarding a two-line minor of the adjugate, as applied to the adjugate of $[\alpha\gamma, \beta\delta]$; the result of this is

$$|u_{1n}| \cdot [\alpha\gamma, \beta\delta] = [\alpha, \beta][\gamma, \delta] - [\alpha, \delta][\gamma, \beta]. \qquad \text{(A)}$$

By inserting therein ξ and ϵ for β and γ respectively, a similar identity is obtained ; and the elimination of $[\alpha, \delta]$ between the two gives

$$|u_{1n}| \{[\epsilon, \xi][\alpha\gamma, \beta\delta] - [\gamma, \beta][\alpha\epsilon, \xi\delta]\} = [\alpha, \beta][\gamma, \delta][\epsilon, \xi] - [\gamma, \beta][\epsilon, \delta][\alpha, \xi]. \qquad \text{(B)}$$

Next, seeing that only the cofactor of $|u_{1n}|$ in this is affected by changing $\alpha, \beta, \gamma, \delta, \epsilon, \xi$ into $\gamma, \delta, \epsilon, \xi, \alpha, \beta$, and that the like happens when $\alpha, \beta, \gamma, \delta, \epsilon, \xi$ is changed* into $\epsilon, \xi, \alpha, \beta, \gamma, \delta$, it follows that the three cofactors of $|u_{1n}|$ must be equal : that is to say,

$$\left.\begin{aligned} [\epsilon, \xi][\alpha\gamma, \beta\delta] - [\gamma, \beta][\alpha\epsilon, \xi\delta] &= [\alpha, \beta][\gamma\epsilon, \delta\xi] - [\epsilon, \delta][\gamma\alpha, \beta\xi] \\ &= [\gamma, \delta][\epsilon\alpha, \xi\beta] - [\alpha, \xi][\epsilon\gamma, \delta\beta] \end{aligned}\right\}. \qquad \text{(C)}$$

From (B) another set of three cofactors of $|u_{1n}|$ is obtained by the

* We may note that these changes are equivalent to the performance of cyclical substitution on α, γ, ϵ and β, δ, ξ simultaneously.

performance of cyclical substitution on α, γ, ϵ; but the deduction in regard to them is that their sum vanishes: that is to say,

$$\left.\begin{array}{l} [\epsilon, \xi][\alpha\gamma, \beta\delta] - [\gamma, \delta][\alpha\epsilon, \xi\delta] \\ + [\alpha, \xi][\gamma\epsilon, \beta\delta] - [\epsilon, \delta][\gamma\alpha, \xi\delta] \\ + [\gamma, \xi][\epsilon\alpha, \beta\delta] - [\alpha, \delta][\epsilon\gamma, \xi\delta] \end{array}\right\} = 0. \qquad \text{(D)}$$

Lastly, by returning to (A) and performing therein cyclical substitution on β, δ, ξ, there is obtained another triad of equalities; and, if on these we use the multipliers $[\epsilon, \xi]$, $[\epsilon, \beta]$, $[\epsilon, \delta]$ and thereafter add, there results

$$|\bar{u}_{1n}|\{[\epsilon, \xi][\alpha\gamma, \beta\delta] + [\epsilon, \beta][\alpha\gamma, \delta\xi] + [\epsilon, \delta][\alpha\gamma, \xi\beta]\} = \begin{vmatrix} [\alpha, \beta] & [\alpha, \delta] & [\alpha, \xi] \\ [\gamma, \beta] & [\gamma, \delta] & [\gamma, \xi] \\ [\epsilon, \beta] & [\epsilon, \delta] & [\epsilon, \xi] \end{vmatrix}. \qquad \text{(E)}$$

The four foregoing identities (B), (C), (D), (E) are types of all the sets of equations which Hesse uses for the main purpose of his paper, namely, the generalization of Pascal's theorem.* The student who is interested in this latter subject would be much helped towards clearness and effective grasp of it if at the outset he would arrange for himself in three arrays the quantities, other than $|u_{1n}|$, involved in the investigation. These are

$$\begin{array}{lll} [\alpha, \beta] & [\gamma, \delta] & [\epsilon, \xi] \\ [\epsilon, \beta] & [\alpha, \delta] & [\gamma, \xi] \\ [\gamma, \beta] & [\epsilon, \delta] & [\alpha, \xi], \end{array} \qquad \begin{array}{lll} [\gamma\epsilon, \delta\xi] & [\epsilon\alpha, \xi\beta] & [\alpha\gamma, \beta\delta] \\ [\alpha\gamma, \delta\xi] & [\gamma\epsilon, \xi\beta] & [\epsilon\alpha, \beta\delta] \\ [\epsilon\alpha, \delta\xi] & [\alpha\gamma, \xi\beta] & [\gamma\epsilon, \beta\delta], \end{array}$$

$$\begin{array}{lll} [\alpha, \beta] & [\gamma\epsilon, \delta\xi] & [\gamma, \delta][\epsilon\alpha, \xi\beta] \\ [\epsilon, \beta] & [\alpha\gamma, \delta\xi] & [\alpha, \delta][\gamma\epsilon, \xi\beta] \\ [\gamma, \beta] & [\epsilon\alpha, \delta\xi] & [\epsilon, \delta][\alpha\gamma, \xi\beta] \end{array} \qquad \begin{array}{lll} [\epsilon, \xi][\alpha\gamma, \beta\delta] \\ [\gamma, \xi][\epsilon\alpha, \beta\delta] \\ [\alpha, \xi][\gamma\epsilon, \beta\delta]. \end{array}$$

The letter-symbols that go to the making of all three he would of course recognize to be

$$\alpha, \gamma, \epsilon \quad \text{and} \quad \beta, \delta, \xi.$$

Then he would note that, exactly as the first array is constructed with these fundamental components, the second array is constructed with binary combinations of them, namely,

$$\gamma\epsilon, \epsilon\alpha, \alpha\gamma \quad \text{and} \quad \delta\xi, \xi\beta, \beta\delta;$$

* Already in Cayley's case (1843) the study of Pascal's theorem had led to results in determinants (*Hist.*, ii. pp. 10–14).

and that the third array has for each of its elements the product of the corresponding elements in the two other arrays.

GUNDELFINGER, S. (1874).

[Auflösung eines Systems von Gleichungen, worunter zwei quadratisch und die übrigen linear sind. *Zeitschrift f. Math. u. Phys.*, xviii. pp. 541–551.]

What is interesting here from our point of view is the introductory proposition regarding the solution of a set of equations of which only *one* is quadratic and the others linear. In the case of three unknowns it is to the effect that *if*

$$\frac{x_1}{z_1}, \frac{y_1}{z_1} \quad \textit{and} \quad \frac{x_2}{z_2}, \frac{y_2}{z_2}$$

be the two sets of values of

$$\frac{x}{z}, \frac{y}{z}$$

which satisfy the equations

$$\left.\begin{aligned} ax^2+by^2+cz^2+2fyz+2gzx+2hxy &= 0 \\ px+qy+rz &= 0 \end{aligned}\right\},$$

then, whatever u, v, w *may be,*

$$(x_1u+y_1v+z_1w)(x_2u+y_2v+z_2w)$$

is equal to

$$\begin{vmatrix} a & h & g & p & u \\ h & b & f & q & v \\ g & f & c & r & w \\ p & q & r & . & . \\ u & v & w & . & . \end{vmatrix}.$$

No proof is given, reference being merely made to a lengthy paper by C. W. Baur in the *Zeitschrift*, xiv. pp. 129–140, 426–435.

For ourselves we may add that by equating coefficients and denoting the adjugate of

$$\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & . \end{vmatrix} \quad \text{by} \quad \begin{vmatrix} A & H & G & P \\ H & B & F & Q \\ G & F & C & R \\ P & Q & R & . \end{vmatrix}$$

the asserted identity gives

$$\left.\begin{aligned} x_1x_2 &= -A, & y_1z_2+y_2z_1 &= -2F,\\ y_1y_2 &= -B, & z_1x_2+z_2x_1 &= -2G,\\ z_1z_2 &= -C, & x_1y_2+x_2y_1 &= -2H, \end{aligned}\right\}$$

from which we have, for example,

$$\left.\begin{aligned} \frac{x_1}{y_1}+\frac{x_2}{y_2} &= \frac{2H}{B}\\ \frac{y_1}{x_1}+\frac{y_2}{x_2} &= \frac{2H}{A} \end{aligned}\right\},$$

and thence

$$\frac{x_1}{y_1},\ \frac{x_2}{y_2} = \frac{H+\sqrt{H^2-AB}}{B},\ \frac{H-\sqrt{H^2-AB}}{B};$$

x_1/y_1 and x_2/y_2 being thus the roots of the quadratic

$$Bx^2-2Hxy+Ay^2 = 0.$$

DARBOUX, G. (1874).

[Mémoire sur la théorie algébrique des formes quadratiques. *Journ. (de Liouville) de Math.*, (2) xix. pp. 347–396.]

Near the outset (p. 356) of this important memoir Darboux formulates as a lemma the theorem that *if* $|a_{in}|$ *be axisymmetric, the determinant*

$$\begin{vmatrix} a_{11} & \dots & a_{1n} & \xi_{11} & \dots & \xi_{1p}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ a_{n1} & \dots & a_{nn} & \xi_{n1} & \dots & \xi_{np}\\ \eta_{11} & \dots & \eta_{n1} & \cdot & \dots & \cdot\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \eta_{1p} & \dots & \eta_{np} & \cdot & \dots & \cdot \end{vmatrix}$$

can only be identically zero if all the (n−p)*-line minors of* $|a_{in}|$ *be zero, and then all the like bordered determinants in which* p *has lower values are also identically zero.* His proof is appropriate enough to his main subject; for another, a paper by Stickelberger 'Ueber Schaaren von bilinearen und quadratischen Formen' in *Crelle's Journ.*, lxxxvi. (1878) p. 21, may be consulted.

VERSLUYS, J. (1876): MANSION, P. (1877).

[Résolution d'un système d'équations, dont une est du second degré tandis que les autres sont linéaires. *Archiv d. Math. u. Phys.*, lx. pp. 128–137.]

[Résolution d'un système de n equations à n inconnues dont . . . *Nouv. Corresp. Math.*, iii. pp. 376–381.]

Versluys' solution of the set of equations

$$\left.\begin{aligned} ax^2+by^2+cz^2+2fyz+2gzx+2hxy &= 0 \\ px+qy+rz &= 0 \end{aligned}\right\}$$

rests on the fact that by using with the first equation the multiplier

$$\begin{vmatrix} a & h & g & p \\ h & b & f & q \\ g & f & c & r \\ p & q & r & . \end{vmatrix}, \quad \text{or } \mu \text{ say,}$$

he obtains as a substitute the equation

$$-\begin{vmatrix} \mathrm{A} & \mathrm{H} & \mathrm{G} & \mathrm{P} & x \\ \mathrm{H} & \mathrm{B} & \mathrm{F} & \mathrm{Q} & y \\ \mathrm{G} & \mathrm{F} & \mathrm{C} & \mathrm{R} & z \\ \mathrm{P} & \mathrm{Q} & \mathrm{R} & . & . \\ x & y & z & . & . \end{vmatrix} = 0$$

in accordance with an identity first foreshadowed by Sylvester in 1850 (*Hist.*, ii. p. 118). Performing on the five-line determinant here the operation

$$r \cdot \mathrm{row}_3 + q \cdot \mathrm{row}_2 + p \cdot \mathrm{row}_1,$$

followed by the operation

$$r \cdot \mathrm{col}_3 + q \cdot \mathrm{col}_2 + p \cdot \mathrm{col}_1,$$

and utilizing the second of the given equations, he thus readily eliminates z, the result being

$$-\frac{1}{r^2}\begin{vmatrix} \mathrm{A} & \mathrm{H} & . & \mathrm{P} & x \\ \mathrm{H} & \mathrm{B} & . & \mathrm{Q} & y \\ . & . & . & \mu & . \\ \mathrm{P} & \mathrm{Q} & \mu & . & . \\ x & y & . & . & . \end{vmatrix} = 0,$$

whence there comes, on leaving out the factor μ^2/r^2

$$\begin{vmatrix} \mathrm{A} & \mathrm{H} & x \\ \mathrm{H} & \mathrm{B} & y \\ x & y & . \end{vmatrix} = 0,$$

which of course carries with it

$$\begin{vmatrix} \mathrm{B} & \mathrm{F} & y \\ \mathrm{F} & \mathrm{C} & z \\ y & z & . \end{vmatrix} = 0, \qquad \begin{vmatrix} \mathrm{C} & \mathrm{G} & z \\ \mathrm{G} & \mathrm{A} & x \\ z & x & . \end{vmatrix} = 0.$$

The required values of the three ratios, x/y, y/z, z/x are thus to hand, the given ternary quadric and linear having given place to two (or three) binary quadrics. Similar procedure leads to the solution of

$$\left.\begin{array}{rcl} \begin{array}{cccc|c} x & y & z & w & \\ \hline a & h & g & k & x \\ h & b & f & j & y \\ g & f & c & i & z \\ k & j & i & d & w \end{array} & = & 0 \\ l_1x+l_2y+l_3z+l_4w & = & 0 \\ m_1x+m_2y+m_3z+m_4w & = & 0 \end{array}\right\},$$

the first step being the multiplication of both sides of the first equation by the bordered discriminant

$$\begin{vmatrix} a & h & g & k & l_1 & m_1 \\ h & b & f & j & l_2 & m_2 \\ g & f & c & i & l_3 & m_3 \\ k & j & i & d & l_4 & m_4 \\ l_1 & l_2 & l_3 & l_4 & . & . \\ m_1 & m_2 & m_3 & m_4 & . & . \end{vmatrix}.$$

In this case the given quaternary quadric and two linears are replaced by three (or six) binary quadrics. Versluys also points out that on solving any one of the said quadrics the quantity that remains under the root-sign after simplification is the bordered discriminant preceded by + or − according as the number of given linear equations is odd or even. For example, the first quadric reached above, namely,

$$\mathrm{B}x^2-2\mathrm{H}xy+\mathrm{A}y^2 = 0,$$

gives us

$$\frac{x}{y} = \frac{H \pm \sqrt{H^2 - AB}}{B},$$

where the negative of the quantity under the root-sign is a two-line minor of the adjugate of the bordered discriminant, and therefore is equal to $r^2\mu$, so that

$$\frac{x}{y} = \frac{H \pm r\sqrt{\mu}}{B},$$

and the reality of the roots is dependent on the sign of μ.

VOSS, A. (1877).

[Ueber gewisse Determinanten. *Math. Annalen*, xiii. pp. 161–167.]

The determinant which Voss borders is that whose matrix is unity, namely,

$$\begin{vmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n,$$

but the reason which he gives for restricting himself to this is itself instructive. Taking in the first instance any determinant $|a_{1n}|$ and giving its matrix an r-line border, he had multiplied the resulting determinant, V say, by $|A_{1n}|$ in the form of a determinant of $n+r$ lines. For example, with $n = 4$ and $r = 2$ his procedure would have been :

$$V \cdot |a_1b_2c_3d_4|^3 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & m_1 & n_1 \\ b_1 & b_2 & b_3 & b_4 & m_2 & n_2 \\ c_1 & c_2 & c_3 & c_4 & m_3 & n_3 \\ d_1 & d_2 & d_3 & d_4 & m_4 & n_4 \\ r_1 & r_2 & r_3 & r_4 & . & . \\ s_1 & s_2 & s_3 & s_4 & . & . \end{vmatrix} \cdot \begin{vmatrix} A_1 & A_2 & A_3 & A_4 & . & . \\ B_1 & B_2 & B_3 & B_4 & . & . \\ C_1 & C_2 & C_3 & C_4 & . & . \\ D_1 & D_2 & D_3 & D_4 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{vmatrix}$$

$$= \begin{vmatrix} |a_1b_2c_3d_4| & . & . & . & m_1 & n_1 \\ . & |a_1b_2c_3d_4| & . & . & m_2 & n_2 \\ . & . & |a_1b_2c_3d_4| & . & m_3 & n_3 \\ . & . & . & |a_1b_2c_3d_4| & m_4 & n_4 \\ \Sigma rA & \Sigma rB & \Sigma rC & \Sigma rD & . & . \\ \Sigma sA & \Sigma sB & \Sigma sC & \Sigma sD & . & . \end{vmatrix}.$$

He had then noted that on multiplying each of the last r rows of the product-determinant by $|a_{1n}|$ the said determinant as thus altered was divisible by $|a_{1n}|^n$, and that thus he was entitled to replace $|a_{1n}|$ in the diagonal on the right by 1, provided he altered the left-hand member to

$$V \cdot |a_{1n}|^{r-1}.$$

A less important result formulated by Voss is to the effect that *if a unit matrix receive an* r*-line border, the determinant of the resulting matrix can be expressed as an* r*-line determinant*: for example,

$$\begin{vmatrix} 1 & . & . & . & . & \lambda_1 & \lambda_2 & \lambda_3 \\ . & 1 & . & . & . & \mu_1 & \mu_2 & \mu_3 \\ . & . & 1 & . & . & \nu_1 & \nu_2 & \nu_3 \\ . & . & . & 1 & . & \rho_1 & \rho^2 & \rho_3 \\ . & . & . & . & 1 & \sigma_1 & \sigma_2 & \sigma_3 \\ l_1 & m_1 & n_1 & r_1 & s_1 & . & . & . \\ l_2 & m_2 & n_2 & r_2 & s_2 & . & . & . \\ l_3 & m_3 & n_3 & r_3 & s_3 & . & . & . \end{vmatrix} = \begin{vmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{vmatrix},$$

where

$$\begin{aligned} P_{11} &= (l_1 m_1 n_1 r_1 s_1 \,\S\, \lambda_1 \mu_1 \nu_1 \rho_1 \sigma_1), \\ P_{12} &= (l_2 m_2 n_2 r_2 s_2 \,\S\, \quad\quad), \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ P_{33} &= (l_3 m_3 n_3 r_3 s_3 \,\S\, \lambda_3 \mu_3 \nu_3 \rho_3 \sigma_3). \end{aligned}$$

Nothing by way of proof is given, the result in all probability being viewed merely as a variant of Sylvester's of 1852. (*Hist.*, ii. p. 200. See also our footnote to p. 57.)

PAIGE, C. LE (1880).

[Note sur les déterminants bordés. *Bull. . . . Soc. Math. de France*, viii. pp. 128–132.]

Le Paige's procedure is quite similar to that of Voss, but it is given in fuller and clearer detail, and is carried farther. At the outset he deals with *axisymmetric* determinants, and then afterwards passes on to determinants in general: it is better, however, to take the latter at once.

Multiplying his ' déterminant bordé '

$$\begin{vmatrix} a_1 & a_2 & \dots & a_5 & \lambda_1 & \mu_1 & \nu_1 \\ b_1 & b_2 & \dots & b_5 & \lambda_2 & \mu_2 & \nu_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ e_1 & e_2 & \dots & e_5 & \lambda_5 & \mu_5 & \nu_5 \\ l_1 & l_2 & \dots & l_5 & . & . & . \\ m_1 & m_2 & \dots & m_5 & . & . & . \\ n_1 & n_2 & \dots & n_5 & . & . & . \end{vmatrix},$$ or V say,

by $|a_1b_2c_3d_4e_5|^4$ in the form

$$\begin{vmatrix} \mathrm{A}_1 & \mathrm{A}_2 & \dots & \mathrm{A}_5 & . & . & . \\ \mathrm{B}_1 & \mathrm{B}_2 & \dots & \mathrm{B}_5 & . & . & . \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mathrm{E}_1 & \mathrm{E}_2 & \dots & \mathrm{E}_5 & . & . & . \\ . & . & \dots & . & 1 & . & . \\ . & . & \dots & . & . & 1 & . \\ . & . & \dots & . & . & . & 1 \end{vmatrix},$$

and dividing by $|a_1b_2c_3d_4e_5|^2$, he obtains

$$\mathrm{V}\cdot|a_1b_2c_3d_4e_5|^2, = \begin{vmatrix} 1 & . & \dots & . & \lambda_1 & \mu_1 & \nu_1 \\ . & 1 & \dots & . & \lambda_2 & \mu_2 & \nu_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ . & . & \dots & 1 & \lambda_5 & \mu_5 & \nu_5 \\ \Sigma l\mathrm{A} & \Sigma l\mathrm{B} & \dots & \Sigma l\mathrm{E} & . & . & . \\ \Sigma m\mathrm{A} & \Sigma m\mathrm{B} & \dots & \Sigma m\mathrm{E} & . & . & . \\ \Sigma n\mathrm{A} & \Sigma n\mathrm{B} & \dots & \Sigma n\mathrm{E} & . & . & . \end{vmatrix},$$

where

$$\Sigma l\mathrm{A} = l_1\mathrm{A}_1 + l_2\mathrm{A}_2 + \dots + l_5\mathrm{A}_5,$$

$$\Sigma m\mathrm{B} = m_1\mathrm{B}_1 + m_2\mathrm{B}_2 + \dots + m_5\mathrm{B}_5.$$

He then substitutes (*Hist.*, ii. pp. 199–200) for this eight-line determinant the quasi-product

$$\begin{Vmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_5 \\ \mu_1 & \mu_2 & \dots & \mu_5 \\ \nu_1 & \nu_2 & \dots & \nu_5 \end{Vmatrix} \cdot \begin{Vmatrix} \Sigma l\mathrm{A} & \Sigma l\mathrm{B} & \dots & \Sigma l\mathrm{E} \\ \Sigma m\mathrm{A} & \Sigma m\mathrm{B} & \dots & \Sigma m\mathrm{E} \\ \Sigma n\mathrm{A} & \Sigma n\mathrm{B} & \dots & \Sigma n\mathrm{E} \end{Vmatrix}$$

which, by the extended multiplication-theorem, is known to be

$$= \begin{vmatrix} \lambda_1\Sigma lA+\ldots+\lambda_5\Sigma lE & \lambda_1\Sigma mA+\ldots+\lambda_5\Sigma mE & \lambda_1\Sigma nA+\ldots+\lambda_5\Sigma nE \\ \mu_1\Sigma lA+\ldots+\mu_5\Sigma lE & \mu_1\Sigma mA+\ldots+\mu_5\Sigma mE & \mu_1\Sigma nA+\ldots+\mu_5\Sigma nE \\ \nu_1\Sigma lA+\ldots+\nu_5\Sigma lE & \nu_1\Sigma mA+\ldots+\nu_5\Sigma mE & \nu_1\Sigma nA+\ldots+\nu_5\Sigma nE \end{vmatrix}.$$

In the next place, he has the insight to recognize the elements of this last determinant as being 'formes bilinéaires,' and themselves expressible as bordered determinants. The general theorem thus reached he formulates as follows: *Le produit du déterminant que l'on obtient en bordant un déterminant du* n^{ième} *ordre*, Δ, *de* r *colonnes de* n *éléments* $\lambda, \mu, \ldots, \omega$ *et de* r *rangées* $\lambda', \mu', \ldots, \omega'$ *par la puissance* r-1 *de* Δ, *est égal á un déterminant du* r^{ième} *ordre dont les éléments sont les formes bilinéaires obtenues en bordant* Δ *d'une colonne et d'une rangée prises dans les* 2r *lignes*

$$\lambda, \mu, \ldots, \omega, \quad \lambda', \mu', \ldots, \omega'.$$

This will be seen to mean that the first element in the final three-line determinant above is

$$= \begin{vmatrix} . & l_1 & l_2 & \ldots & l_5 \\ \lambda_1 & a_1 & a_2 & \ldots & a_5 \\ \lambda_2 & b_1 & b_2 & \ldots & b_5 \\ . & . & . & . & . \\ \lambda_5 & e_1 & e_2 & \ldots & e_5 \end{vmatrix} \quad \text{or} \quad \begin{array}{cccc|c} l_1 & l_2 & \ldots & l_5 & \\ \hline \mathrm{A}_1 & \mathrm{A}_2 & \ldots & \mathrm{A}_5 & \lambda_1 \\ \mathrm{B}_1 & \mathrm{B}_2 & \ldots & \mathrm{B}_5 & \lambda_2 \\ . & . & . & . & . \\ \mathrm{E}_1 & \mathrm{E}_2 & \ldots & \mathrm{E}_5 & \lambda_5, \end{array}$$

according to our choice of notation for bilinear forms.

It is necessary to point out, however, that Le Paige's equality, like Muir's of 1881, is merely a case of Sylvester's of 1851, greater attention being directed to the second member of the equality by making it the first.

CHAPTER XX.

DETERMINANTS WHOSE ELEMENTS ARE COMBINATORY NUMBERS. UP TO 1880.

HERE again there arises a slight difficulty of classification, for, manifestly, there is no reason why determinants with elements of the form $m(m-1)\ldots(m-r+1)/r!$ should not be quite properly placed in one or other of the preceding chapters. As a matter of fact, for example, a not unimportant subsection of those that come up for treatment are in form persymmetric. We have been finally guided in the matter by considerations of mere convenience,—the convenience of the reader, whether he use the volume as a book of exposition or a book of reference. It will of course be fully recognized that a classification of special forms originating, so to say, at random could never be strictly logical.

BALTZER, R. (1864).

[THEORIE UND ANWENDUNG DER DETERMINANTEN. . . . 2^{te} verm. Auflage, viii+224 pp. Leipzig.]

Two instances of the evaluation of such determinants are given (pp. 16, 19), namely,

$$\begin{vmatrix} 1 & (c+m)_1 & (c+m+1)_2 & \ldots & (c+2m-1)_m \\ 1 & (c+m+1)_1 & (c+m+2)_2 & \ldots & (c+2m)_m \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & (c+2m)_1 & (c+2m+1)_2 & \ldots & (c+3m-1)_m \end{vmatrix}_{m+1} = 1,$$

and

$$\begin{vmatrix} (c+m)_m & (c+m+1)_m & \ldots & (c+2m)_m \\ (c+m+1)_m & (c+m+2)_m & \ldots & (c+2m+1)_m \\ \ldots & \ldots & \ldots & \ldots \\ (c+2m)_m & (c+2m+1)_m & \ldots & (c+3m)_m \end{vmatrix}_{m+1} = (-1)^{\frac{1}{2}m(m+1)}.$$

The first result follows from diminishing each row in backward order by the row immediately preceding, and the second from doing this repeatedly and then treating the columns in like fashion. The former may also be viewed as a corollary to a theorem of Stern's (1865) in regard to alternants, and the latter is a simplification of an instance given (1861) by Hankel in illustration of his theorem regarding persymmetric determinants, namely that, for example, when $c = 3$ and $m = 2$,

$$\mathrm{P}(10, 15, 21, 28, 36) = \mathrm{P}(10, 5, 1, 0, 0).$$

ZEIPEL, V. v. (1865).

[Om Determinanter, hvars elementer äro Binomialkoefficienter. *Lunds Univ. Årsskrift*, ii. pp. 1–68.]

The main subject of this painstaking and methodical investigation would be more definitely described as being various cases of the determinant

$$\left| (m)_p (m+d)_{p+e} (m+2d)_{p+2e} \ldots (m+rd)_{p+re} \right|,$$

that is to say, the determinant which not only has for its elements numbers of the type

$$r(r-1) \ldots (r-s+1)/s!,$$

but which has the bases of these numbers identical throughout one and the same row and regularly increasing throughout each column, and has the suffixes identical throughout one and the same column and regularly increasing throughout each row ; for example,

$$\begin{vmatrix} (5)_0 & (5)_1 & (5)_2 \\ (6)_0 & (6)_1 & (6)_2 \\ (7)_0 & (7)_1 & (7)_2 \end{vmatrix}, \quad \begin{vmatrix} 4 & 6 & 4 \\ 6 & 15 & 20 \\ 8 & 28 & 56 \end{vmatrix}.$$

A beginning is made with the case

$$d = 1, \quad p = 0, \quad e = 1,$$

when it is readily shown by reduction of the columns in order that the value of the determinant is the same as if m were 0, and therefore is equal to 1. Passing to the case where $d = 1$, $e = 1$ and p is not specialized, Zeipel removes from the rows the factors

$$m, \quad m+1, \quad m+2, \quad \ldots\ldots, \quad m+r,$$

and from the columns

$$\frac{1}{p}, \quad \frac{1}{p+1}, \quad \frac{1}{p+2}, \quad \ldots, \quad \frac{1}{p+r},$$

and obtains

$$\left|(m)_p(m+1)_{p+1} \ldots (m+r)_{p+r}\right| = \frac{(m+r)_{r+1}}{(p+r)_{r+1}}\left|(m-1)_{p-1}(m)_p \ldots (m+r-1)_{p+r-1}\right|,$$

repeated use of which gives him finally

$$\left|(m)_p(m+1)_{p+1} \ldots (m+r)_{p+r}\right| = \frac{(m+r)_{r+1}(m+r-1)_{r+1} \ldots (m+r-p+1)_{r+1}}{(p+r)_{r+1}(p+r-1)_{r+1} \ldots (r+1)_{r+1}}.$$

Another procedure consists in removing from the rows the factors

$$(m)_p, \quad (m+1)_p, \quad \ldots, \quad (m+r)_p,$$

and from the columns

$$\frac{1}{(p)_0}, \quad \frac{1}{(p+1)_1}, \quad \ldots, \quad \frac{1}{(p+r)_r},$$

the result reached in this way being

$$\frac{(m+r)_p \ldots (m+1)_p(m)_p}{(p+r)_p \ldots (p+1)_p(p)_p}.$$

A comparison of the two expressions shows that the procedures applied in like manner to

$$\left|(m+r-p+1)_{r+1}(m+r-p+2)_{r+2} \ldots (m+r)_{r+p}\right|$$

will give the same results in the reverse order; consequently

$$\left|(m)_p(m+1)_{p+1} \ldots (m+r)_{p+r}\right| = \left|(m+r-p+1)_{r+1}(m+r-p+2)_{r+2} \ldots (m+r)_{r+p}\right|.$$

A third form for the same determinant is got by repeatedly performing the operation of diminishing the rows in backward order each by the immediately preceding row, namely, the form

$$\begin{vmatrix} (m)_p & (m)_{p+1} & (m)_{p+2} & \ldots & (m)_{p+r} \\ (m)_{p-1} & (m)_p & (m)_{p+1} & \ldots & (m)_{p+r-1} \\ (m)_{p-2} & (m)_{p-1} & (m)_p & \ldots & (m)_{p+r-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (m)_{p-r} & (m)_{p-r+1} & (m)_{p-r+2} & \ldots & (m)_p \end{vmatrix},$$

which, it should be observed, is persymmetric with respect to the secondary diagonal, the base being constant and the suffixes decreasing from $p+r$ to $p-r$. By taking columns instead of rows and performing addition instead of subtraction, there is obtained a fourth form, namely,

$$\begin{vmatrix} (m)_p & (m+1)_{p+1} & (m+2)_{p+2} & \dots & (m+r)_{p+r} \\ (m+1)_p & (m+2)_{p+1} & (m+3)_{p+2} & \dots & (m+r+1)_{p+r} \\ (m+2)_p & (m+3)_{p+1} & (m+4)_{p+2} & \dots & (m+r+2)_{p+r} \\ \dots & \dots & \dots & \dots & \dots \\ (m+r)_p & (m+r+1)_{p+1} & (m+r+2)_{p+2} & \dots & (m+2r)_{p+r} \end{vmatrix},$$

in which the suffixes follow the same law as in the original, while the bases are identical throughout each north-east diagonal and increase regularly by 1 in each column.* An interesting special case, first noted in its recurrent form by Trudi (see above, p. 219), is that in which $p = 1$, the value of each form of determinant being then

$$(m+r)_{r+1};$$

this also, it should be noted, is their value when $p = m-1$. We may add by way of illustration that the first process of evaluation followed by Zeipel gives

$$\begin{vmatrix} (5)_2 & (5)_3 & (5)_4 \\ (6)_2 & (6)_3 & (6)_4 \\ (7)_2 & (7)_3 & (7)_4 \end{vmatrix} = \frac{(7)_3 \cdot (6)_3}{(4)_3 \cdot (3)_3} = \frac{35 \cdot 20}{4 \cdot 1} = 175;$$

the second gives the same determinant

$$= \frac{(5)_2(6)_2(7)_2}{(2)_2(3)_2(4)_2} = \frac{10 \cdot 15 \cdot 21}{1 \cdot 3 \cdot 6} = 175:$$

both give the same value for

$$\begin{vmatrix} (6)_3 & (6)_4 \\ (7)_3 & (7)_4 \end{vmatrix};$$

* A fifth form is

$$\begin{vmatrix} (m+r)_p & (m+r+1)_{p+1} & (m+r+2)_{p+2} & \dots & (m+2r)_{p+r} \\ (m+r-1)_{p-1} & (m+r)_p & (m+r+1)_{p+1} & \dots & (m+2r-1)_{p+r-1} \\ (m+r-2)_{p-2} & (m+r-1)_{p-1} & (m+r)_p & \dots & (m+2r-2)_{p+r-2} \\ \dots & \dots & \dots & \dots & \dots \\ (m)_{p-r} & (m+1)_{p-r+1} & (m+2)_{p-r+2} & \dots & (m+r)_p \end{vmatrix},$$

from which, it being persymmetric, the third form is obtainable by Hankel's transformation. Further, since in this fifth form the excess of each base over its suffix is $m+r-p$, we can have, as a counterpart to the third form, a variant in which the suffix is constant and the bases decrease from $m+2r$ to m.

and

$$\begin{vmatrix} (5)_2 & (5)_3 & (5)_4 \\ (5)_1 & (5)_2 & (5)_3 \\ (5)_0 & (5)_1 & (5)_2 \end{vmatrix}, \quad \begin{vmatrix} (5)_2 & (6)_3 & (7)_4 \\ (6)_2 & (7)_3 & (8)_4 \\ (7)_2 & (8)_3 & (9)_4 \end{vmatrix}$$

are additional equivalent forms.

The next Section (pp. 13–41) of the memoir concerns the cases where the bases or the suffixes do not proceed by the common increment 1. The simplest case is where $e = 1$ and d is not specialized, the result being

$$\left|(m)_p(m+d)_{p+1} \ . \ . \ . \ (m+rd)_{p+r}\right| = \frac{(m+rd)_p \ . \ . \ . \ (m+d)_p(m)_p}{(p+r)_p \ . \ . \ . \ (p+1)_p(p)_p} \cdot d^{\frac{1}{2}r(r+1)}.$$

This is obtained (pp. 30–35) by removing factors as in the second procedure above indicated, and then showing that the determinant so reduced,

$$\left|(m-p)_0(m-p+d)_1 \ . \ . \ . \ (m-p+rd)_r\right| = d^{\frac{1}{2}r(r+1)}.$$

The corresponding case where $d = 1$ and e is not specialized is treated in like fashion (pp. 35–41), the product of the removed factors being

$$\frac{(m)_p(m+1)_p \ . \ . \ . \ (m+r)_p}{(p)_p(p+e)_p \ . \ . \ . \ (p+re)_p},$$

and the reduced determinant

$$\left|(m-p)_0(m-p+1)_e \ . \ . \ . \ (m-p+r)_{re}\right|.$$

For the latter there is obtained the lengthy expression

$$\frac{\mathrm{P}(m-p, r) \cdot \mathrm{P}(m-p-e+1, r-1) \ . \ . \ . \ \mathrm{P}(m-p-\overline{r-1} \cdot \overline{e-1}, 1)}{\left\{(e-1)_{e-1}\right\}^r \left\{(2e-1)_{e-1}\right\}^{r-1} \left\{(re-1)_{e-1}\right\}}$$

where $\mathrm{P}(m, r) \equiv (m)_{e-1}(m+1)_{e-1} \ . \ . \ . \ (m+r-1)_{e-1}$.

On changing the left-hand member so as to have the base of the elements constant, and taking $p = 1$, $e = 2$, $r = m-1$, the outcome is

$$\begin{vmatrix} (m)_1 & (m)_3 & (m)_5 & \ . \ . \ . \ . \\ (m)_0 & (m)_2 & (m)_4 & \ . \ . \ . \ . \\ \ . & (m)_1 & (m)_3 & \ . \ . \ . \ . \\ \ . \ . & \ . \ . & \ . \ . & \ . \ . \ . \ . \end{vmatrix}_{m-1} = 2^{\frac{1}{2}m(m-1)},$$

where the equivalence of $(m)_k$ and $(m)_{m-k}$ makes the determinant centrosymmetric. Of less interest are the cases (pp. 13–21) in which the bases increase regularly by 1 save at one place where the step is from $m+r$ to $m+r+s$, and the cases (pp. 24–30) in which the suffixes increase regularly by 1 save at one place where the step is from $p+u$ to $p+u+v$. For such cases the modes of treatment are as far as possible the same as before. Interest, however, is reawakened when we come to the determinant

$$\left|(m)_p(n)_{p+1} \ . \ . \ . \ (y)_{p+\omega-1}(z)_{p+\omega}\right|,$$

where the $\omega+1$ bases $m, n, \ldots, y, z$ are any integers whatever. Taking first the case where $p = 0$, Zeipel removes the factors,

$$\frac{1}{1!}, \ \frac{1}{2!}, \ \frac{1}{3!}, \ \ldots .$$

from the columns, and shows that the resulting determinant is equal to the difference-product of $m, n, \ldots, y, z$, thus obtaining

$$\left|(m)_0(n)_1 \ . \ . \ . \ (y)_{\omega-1}(z)_\omega\right| \ = \ \frac{\zeta^{\frac{1}{2}}(m, n, \ldots, y, z)}{1^\omega \cdot 2^{\omega-1} \ . \ . \ . \ (\omega-1)^2\omega^1},$$

a result obtained the same year by Stern when studying alternants. Then proceeding in the same way, he obtains for

$$\left|(m)_p(n)_{p+1} \ . \ . \ . \ (y)_{p+\omega-1}(z)_{p+\omega}\right|$$

the expression

$$\frac{(m)_p(n)_p \ . \ . \ . \ (y)_p(z)_p}{(p)_p(p+1)_p \ . \ . \ . \ (p+\omega)_p} \cdot \left|(m-p)_0(n-p)_1 \ . \ . \ . \ (y-p)_{\omega-1}(z-p)_\omega\right|,$$

and thence, with the help of the previous case,

$$\frac{(m)_p(n)_p \ . \ . \ . \ (y)_p(z)_p}{(p)_p(p+1)_p \ . \ . \ . \ (p+\omega)_p} \cdot \frac{\zeta^{\frac{1}{2}}(m, n, \ldots, y, z)}{1^\omega \cdot 2^{\omega-1} \ . \ . \ . \ (\omega-1)^2\omega^1}.$$

The third Section (pp. 41–57) concerns the principal minors of arrays like

$$\left\|\begin{matrix} (m)_p & (m)_{p+1} & (m)_{p+2} & (m)_{p+3} \\ (m+d)_p & (m+d)_{p+1} & (m+d)_{p+2} & (m+d)_{p+3} \\ (m+2d)_p & (m+2d)_{p+1} & (m+2d)_{p+2} & (m+2d)_{p+3} \end{matrix}\right\|,$$

$$\left\|\begin{matrix} (m)_p & (m)_{p+1} & (m)_{p+2} \\ (m+d)_p & (m+d)_{p+1} & (m+d)_{p+2} \\ (m+2d)_p & (m+2d)_{p+1} & (m+2d)_{p+2} \\ (m+3d)_p & (m+3d)_{p+1} & (m+3d)_{p+2} \end{matrix}\right\|,$$

where the number of rows is 1 less or 1 more than the number of columns. Such minors are of course determinants which have been studied in the preceding Section, so that the nature of the relations existing among them may be readily obtained when wanted.

The fourth Section (pp. 57–68) is occupied in considering the determinants which arise from replacing the last column in

$$\left|(m)_0\ (m+1)_1\ \ldots\ (m+r)_r\right|,$$

$$\left|(m)_1\ (m+1)_2\ \ldots\ (m+r)_{r+1}\right|,$$

in each case by

$$x^n,\quad (x+1)^n,\quad \ldots\ .\quad (x+r)^n.$$

The first is found to be equal to

$$0 \quad \text{when} \quad n < r,$$

$$r! \quad \text{when} \quad n = r;$$

the second to be equal to

$$(-1)^r(x-m)^n \quad \text{when} \quad n \leqq r;$$

and both of them to be complicated in the remaining case.

Zeipel, we may remark in passing, seldom uses any other property of the combinatory numbers $(n)_r$ or $C_{n,r}$ than

$$(n)_r = (n)_{n-r} \quad \text{and} \quad (n)_r = (n-1)_r + (n-1)_{r-1};$$

and consequently his work is sometimes less compact than it might have been. Thus, where the bases of a row are constant and the suffixes are consecutive integers, the identity

$$(a-b)_s = (a)_s - (a)_{s-1}(b)_1 + (a)_{s-2}(b+1)_2 - (a)_{s-3}(b+2)_3 + \ldots,$$

which is obtained from equating coefficients of x^s in the expansions of $(1-x)^a(1-x)^{-b}$ and $(1-x)^{a-b}$, may be used with good effect. Taking, for example, a determinant of the type dealt with in his second Section, namely,

$$\begin{vmatrix} (m)_0 & (m)_1 & (m)_2 & (m)_3 & (m)_4 \\ (m+1)_0 & (m+1)_1 & (m+1)_2 & (m+1)_3 & (m+1)_4 \\ (m+2)_0 & (m+2)_1 & (m+2)_2 & (m+2)_3 & (m+2)_4 \\ (m+4)_0 & (m+4)_1 & (m+4)_2 & (m+4)_3 & (m+4)_4 \\ (m+5)_0 & (m+5)_1 & (m+5)_2 & (m+5)_3 & (m+5)_4 \end{vmatrix},$$

where at the same place in every column there is a break in the uniformity of the increment of the bases, and multiplying it rowwise by 1 in the form

$$\begin{vmatrix} 1 & . & . & . & . \\ -(m)_1 & 1 & . & . & . \\ (m+1)_2 & -(m)_1 & 1 & . & . \\ -(m+2)_3 & (m+1)_2 & -(m)_1 & 1 & . \\ (m+3)_4 & -(m+2)_3 & (m+1)_2 & -(m)_1 & 1 \end{vmatrix},$$

we obtain by means of the said identity

$$\begin{vmatrix} 1 & . & . & . & . \\ 1 & 1 & . & . & . \\ 1 & (2)_1 & 1 & . & . \\ 1 & (4)_1 & (4)_2 & (4)_3 & (4)_4 \\ 1 & (5)_1 & (5)_2 & (5)_3 & (5)_4 \end{vmatrix},$$

which

$$= |(4)_3(5)_4)| = 10.$$

Again, the companion identity

$$(a+b)_s = (a)_s + (a)_{s-1}(b)_1 + (a)_{s-2}(b)_2 + \ldots + (b)_s$$

may be employed with equal success. Thus, by reason of it, the determinant

$$\begin{vmatrix} (m+m')_3 & (m+n')_3 & (m+p')_3 & (m+q')_3 \\ (n+m')_3 & (n+n')_3 & (n+p')_3 & (n+q')_3 \\ (p+m')_3 & (p+n')_3 & (p+p')_3 & (p+q')_3 \\ (q+m')_3 & (q+n')_3 & (q+p')_3 & (q+q')_3 \end{vmatrix}$$

is seen to be equal to

$$\begin{vmatrix} (m)_0 & (m)_1 & (m)_2 & (m)_3 \\ (n)_0 & (n)_1 & (n)_2 & (n)_3 \\ (p)_0 & (p)_1 & (p)_2 & (p)_3 \\ (q)_0 & (q)_1 & (q)_2 & (q)_3 \end{vmatrix} \cdot \begin{vmatrix} (m')_3 & (m')_2 & (m')_1 & (m')_0 \\ (n')_3 & (n')_2 & (n')_1 & (n')_0 \\ (p')_3 & (p')_2 & (p')_1 & (p')_0 \\ (q')_3 & (q')_2 & (q')_1 & (q')_0 \end{vmatrix},$$

and therefore to be equal to

$$\frac{\xi^{\frac{1}{2}}(m, n, p, q) \cdot \xi^{\frac{1}{2}}(m', n', p', q')}{\{1^3 2^2 3^1\}^2}.$$

ZEIPEL, V. v. (1871).

[Om determinanter, hvilkas elementer äro binomialkoefficienter, multiplicerade med vissa faktorer. *Lunds Universitets Årsskrift*, viii. 36 pp.]

Here Zeipel takes as his groundwork determinants similar to those of the memoir we have just dealt with, the subjects of his new investigation being got by annexing to each of the elements a factor taken from a systematically-formed square array. For the purpose of increased clearness we shall diverge a little from his mode of exposition, keeping the basic determinant and the array of introduced factors separately in view.

Having established in § 1 of his previous memoir the identity

$$\left|(m)_p(m+1)_{p+1}\ldots(m+r)_{p+r}\right| = \frac{(m)_p(m+1)_p\ldots(m+r)_p}{(p)_p(p+1)_p\ldots(p+r)_p},$$

he now devotes the corresponding Section here (pp. 1–9) to the establishment of the companion result: *If each element of the determinant*

$$\begin{vmatrix} (\mathrm{m})_{\mathrm{p}} & (\mathrm{m})_{\mathrm{p+1}} & \ldots. & (\mathrm{m})_{\mathrm{p+r}} \\ (\mathrm{m+1})_{\mathrm{p}} & (\mathrm{m+1})_{\mathrm{p+1}} & \ldots. & (\mathrm{m+1})_{\mathrm{p+r}} \\ \ldots & \ldots & \ldots & \ldots \\ (\mathrm{m+r})_{\mathrm{p}} & (\mathrm{m+r})_{\mathrm{p+1}} & \ldots. & (\mathrm{m+r})_{\mathrm{p+r}} \end{vmatrix}$$

be multiplied by the corresponding element of the array

$$\begin{array}{cccccc} \mathrm{m-p} & \alpha & \beta & \ldots. & \rho \\ \mathrm{m-p+1} & \alpha+1 & \beta+1 & \ldots. & \rho+1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathrm{m-p+r} & \alpha+\mathrm{r} & \beta+\mathrm{r} & \ldots. & \rho+\mathrm{r}, \end{array}$$

the resulting determinant is equal to the original multiplied by

$$(\mathrm{m-p})(\mathrm{m-p+1})\ldots(\mathrm{m-p+r}).$$

By way of proof he follows his old procedure, removing from the rows in order the factors

$$(m-p)(m)_p,\quad (m-p+1)(m+1)_p,\quad \ldots.,\quad (m-p+\mathrm{r})(m+r)_p,$$

and from the columns the factors

$$\frac{1}{1},\quad \frac{1}{p+1},\quad \frac{1}{(p+1)(p+2)},\quad \ldots.,\quad \frac{1\cdot 2\ldots(r-1)}{(p+1)(p+2)\ldots(p+r)},$$

and then showing that the determinant as thus reduced is equal to $r!$. Attention is of course drawn to the fact that the r quantities $\alpha, \beta, \ldots, \rho$ introduced through the distributed multiplication do not appear in the expression of the final value.

The main result of the next two Sections (pp. 9–17, 18–24) is that *If each element of the determinant*

$$\begin{vmatrix} (m)_k & (m)_p & (m)_{p+1} & \ldots & (m)_{p+r-1} \\ (m+1)_k & (m+1)_p & (m+1)_{p+1} & \ldots & (m+1)_{p+r-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (m+r)_k & (m+r)_p & (m+r)_{p+1} & \ldots & (m+r)_{p+r-1} \end{vmatrix},$$

where k *is equal to one of the other suffixes, be multiplied by the corresponding element of the array*

$$\begin{vmatrix} 1 & \alpha & \beta & \ldots & \rho \\ 1 & \alpha+1 & \beta+1 & \ldots & \rho+1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \alpha+r & \beta+r & \ldots & \rho+r \end{vmatrix},$$

the resulting determinant is equal to

$$(k-p+1)(k-p+2)\ldots r \cdot \left\{ \frac{(m)_p (m+1)_p \ldots (m+r-1)_p (m+r)_p}{(p)_p (p+1)_p \ldots (p+r-1)_p (k)_p} \right. \\ \left. \cdot (m-p-\alpha)(m-p-\beta-1)(m-p-\gamma-2)\ldots \right\},$$

where the number of factors following the fraction is k−p. The proof is accomplished in instalments, the case where $p = 0$ being taken first and being made dependent on the case where $p = 0$ and $r = k$. The mode of showing that in this latter case the value is

$$(m-\alpha)(m-\beta-1)(m-\gamma-2)\ldots$$

is to prove that the determinant then vanishes if m be put equal to α, $\beta+1$, $\gamma+2, \ldots$; some shorter mode ought to be devisable. The other result worthy of note in the third Section is that *If each element of the determinant*

$$\begin{vmatrix} 1 & (m)_0 & (m)_1 & \ldots & (m)_{r-1} \\ 1 & (m+d)_0 & (m+d)_1 & \ldots & (m+d)_{r-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & (m+rd)_0 & (m+rd)_1 & \ldots & (m+rd)_{r-1} \end{vmatrix},$$

be multiplied by the corresponding element of the array

$$\begin{array}{ccccccc} 1 & \alpha & \beta & . & . & . & \rho \\ 1 & \alpha+d & \beta+d & . & . & . & \rho+d \\ . & . & . & . & . & . & . \\ 1 & \alpha+rd & \beta+rd & . & . & . & \rho+rd, \end{array}$$

the resulting determinant is equal to

$$r!\, d^{\frac{1}{2}r(r+1)}.$$

It is not at all difficult to establish, and indeed the case where $d = 1$ has already been made use of in the first Section as an auxiliary. We note for ourselves that in this theorem and in the preceding the value of the basic determinant is 0, there being in both cases two columns alike.

A similar pair of theorems in the fourth Section may be taken out of their order for the sake of comparison. The first is that, *If each element of the determinant*

$$\begin{vmatrix} (m)_k & (m)_0 & (m)_1 & . & . & . & (m)_{r-1} \\ (m+1)_k & (m+1)_0 & (m+1)_1 & . & . & . & (m+1)_{r-1} \\ . & . & . & . & . & . & . \\ (m+r)_k & (m+r)_0 & (m+r)_1 & . & . & . & (m+r)_{r-1} \end{vmatrix},$$

where k *is equal to one of the other suffixes, be multiplied by the corresponding element of the array*

$$\begin{array}{ccccccc} 1 & \alpha & \beta & . & . & . & \rho \\ 1 & \alpha+d & \beta+d & . & . & . & \rho+d \\ . & . & . & . & . & . & . \\ 1 & \alpha+rd & \beta+rd & . & . & . & \rho+rd, \end{array}$$

the resulting determinant is equal to

$$(k+1)(k+2) \ldots r \cdot d^{r-k} \cdot (md-\alpha)(\overline{m-1} \cdot d-\beta) \ldots,$$

where the number of factors following d^{r-k} *is* k. The other theorem of the pair is that, *If each element of the determinant*

$$\begin{vmatrix} m & (m)_1 & (m)_2 & . & . & . & (m)_r \\ m+1 & (m+1)_1 & (m+1)_2 & . & . & . & (m+1)_r \\ . & . & . & . & . & . & . \\ m+r & (m+r)_1 & (m+r)_2 & . & . & . & (m+r)_r \end{vmatrix}$$

be multiplied by the corresponding element of the same array as in the preceding, the resulting determinant is equal to

$$m(m+1) \ldots (m+r) \cdot d^r.$$

It will be noticed that the cases of these theorems where $d = 1$ are included in previous results; also that in each theorem the basic determinant has two columns alike. The case of the second theorem, where $d = 2$ and $\alpha = \beta = \gamma = \ldots = 1$, is readily transformable into

$$\begin{vmatrix} 1 & 1 & (m)_1 & \ldots & (m)_{r-1} \\ \frac{1}{3} & 1 & (m+1)_1 & \ldots & (m+1)_{r-1} \\ \frac{1}{5} & 1 & (m+2)_1 & \ldots & (m+2)_{r-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \frac{1}{2r+1} & 1 & (m+r)_1 & \ldots & (m+r)_{r-1} \end{vmatrix} = \frac{2 \cdot 4 \cdot 6 \ldots (2r)}{3 \cdot 5 \cdot 7 \ldots (2r+1)},$$

from which, with the help of Wallis' continued product for $\pi/4$, there is readily deduced the result that *as* r *approximates towards an infinitely great integer, the product of* 2r+1 *and the determinant approximates indefinitely near to* $\pi/2$.

It only remains to note that in the first six pages of the fourth Section cases are considered where the array of factors is in effect changed into its conjugate before the distributed multiplication is performed. The most interesting result reached is to the effect that, *If each element of the determinant*

$$\left| (\mathrm{m})_0 (\mathrm{m}+1)_1 \ldots . (\mathrm{m}+\mathrm{r})_\mathrm{r} \right|$$

be multiplied by the corresponding element of the array

$$\begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ \alpha & \alpha+1 & \alpha+2 & \ldots & \alpha+\mathrm{r} \\ \beta & \beta+1 & \beta+2 & \ldots & \beta+\mathrm{r} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \rho & \rho+1 & \rho+2 & \ldots & \rho+\mathrm{r} \end{vmatrix},$$

the resulting determinant is equal to

$$(\mathrm{m}+\alpha+1)(\mathrm{m}+\beta+2) \ldots . (\mathrm{m}+\rho+\mathrm{r}).$$

That $m+\alpha+1$ is a removable factor is established by performing the operation

$$\mathrm{row}_2 - \alpha \cdot \mathrm{row}_1,$$

and using the identity

$$(\alpha+s)(m+1)_s - \alpha(m)_s = (\alpha+m+1)(m)_{s-1}.$$

A similar procedure suffices to effect the removal of the factor $m+\beta+2$, and thereafter of the others in order, when it is found that the determinant as thus reduced is equal to 1.

TIRELLI, F. (1875).

[Quistione 36. *Giornale di Mat.*, xiii. pp. 167, 225 : solution by A. Landriani, xiii. pp. 356–358.]

Tirelli's results are somewhat obscured by his notation and by his adoption of an awkward order for the columns of his determinants. If instead of using one notation for $n(n+1)\ldots(n+k-1)/k!$ and another for $n(n-1)\ldots(n-k+1)/k!$, we use the same for expressing both, denoting them respectively by

$$(n+k-1)_k \quad \text{and} \quad (n)_k,$$

and if we reverse the order of the columns in each case, the identities submitted by him for proof are :

$$\begin{vmatrix} 1 & (p+1)_1 & (p+2)_2 & \ldots & (p+n-1)_{n-1} \\ 1 & (p+2)_1 & (p+3)_2 & \ldots & (p+n)_{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & (p+n)_1 & (p+n+1)_2 & \ldots & (p+2n-2)_{n-1} \end{vmatrix} = 1,$$

$$\begin{vmatrix} 3_1 & 4_1 \ldots & n_1 \\ 3_2 & 4_2 \ldots & n_2 \\ \ldots & \ldots & \\ \ldots & \ldots & n_{n-1} \end{vmatrix} = \begin{vmatrix} 2_1 & 3_1 & 4_1 & \ldots & n_1 \\ 3_2 & 4_2 & 5_2 & \ldots & (n+1)_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (n)_{n-1} & (n+1)_{n-1} & (n+2)_{n-1} & \ldots & (2n-2)_{n-1} \end{vmatrix} = n,$$

With the first determinant we are already familiar from Zeipel. The third, as we also know from Zeipel, is transformable into the second by repeatedly diminishing each row in backward order by the row in front of it, the number of rows affected being one fewer each time ; and the second, by the like operation performed on its columns, is changed into the persymmetric continuant

$$\begin{vmatrix} 2 & 1 & . & . & \ldots \\ 1 & 2 & 1 & . & \ldots \\ . & 1 & 2 & 1 & \ldots \\ . & . & 1 & 2 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_{n-1},$$

the value of which is n.

JANNI, V. (1876 ?).

[Nota sullo sviluppo di un determinante. 7 pp. ?.] *

The main result established here,

$$\left| m_0(m+1)_1(m+2)_2 \ldots (m+n)_n \right| = 1,$$

is Zeipel's. It closely resembles one previously given (Baltzer), and may be proved in exactly the same way, the effect of the first transformation being to show that the left-hand member is equal to the last of its own principal minors, namely,

$$\left| m_0(m+1)_1(m+2)_2 \ldots (m+n-1)_{n-1} \right|.$$

Attributed also to Janni is the simple recurrent † of which the five-line instance is

$$\begin{vmatrix} 1 & . & . & . & u_0 \\ 1 & 1_1 & . & . & u_1 \\ 1 & 2_1 & 2_2 & . & u_2 \\ 1 & 3_1 & 3_2 & 3_3 & u_3 \\ 1 & 4_1 & 4_2 & 4_3 & u_4 \end{vmatrix} = u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 \equiv \Delta^4 u_0.$$

It is at once established by performing the operation

$$\text{row}_5 - 4\,\text{row}_4 + 6\,\text{row}_3 - 4\,\text{row}_2 + \text{row}_1.$$

BONOLIS, A. (1876).

[Sviluppi di alcuni determinanti. *Giornale di Mat.*, xv. pp. 113–134.]

Bonolis' earlier results may be best viewed as slight generalizations of simple identities of Zeipel's, and be most easily evaluated by being made dependent on the latter. Thus the determinant

$$\begin{vmatrix} (m)_0 a^m & (m)_1 a^{m-t} & \ldots & (m)_r a^{m-rt} \\ (m+d)_0 a^{m+d} & (m+d)_1 a^{m+d-t} & \ldots & (m+d)_r a^{m+d-rt} \\ (m+2d)_0 a^{m+2d} & (m+2d)_1 a^{m+2d-t} & \ldots & (m+2d)_r a^{m+2d-rt} \\ \ldots & \ldots & \ldots & \ldots \\ (m+rd)_0 a^{m+rd} & (m+rd)_1 a^{m+rd-t} & \ldots & (m+rd)_r a^{m+rd-rt} \end{vmatrix}$$

* This pamphlet I have not seen. Garbieri in 1878 says in regard to it (*Bull. di Bibliogr. e di Storia delle sci. mat.*, xi. pp. 257–318), "*in* 8° di 7 pagine, senza nota d'anno, nè di luogo di stampa, nè di stamperia."

† It appears in an unnecessarily complicated form in Günther's text-book (p. 197), the reference being to "Janni, Lezioni di algebra complementare, Napoli, 1876, p. 23." It is taken over by Pascal (pp. 179–180) and by Leitzmann (p. 138). All of them state it incorrectly.

(p. 120) can be freed of all its powers of a by multiplying all the columns in order by

$$a^0, \quad a^t, \quad a^{2t}, \quad \ldots\ldots, \quad a^{rt},$$

and then removing from the rows in order the factors

$$a^m, \quad a^{m+d}, \quad a^{m+2d}, \quad \ldots\ldots, \quad a^{m+rd}.$$

In this way it is made clear that

$$a^{\{2m+r(d-t)\}\frac{1}{2}(r+1)}$$

is a factor, and that the cofactor is

$$\left|(m)_0(m+d)_1 \ldots\ldots (m+rd)_r\right|,$$

the value of the latter being known from Zeipel to be

$$d^{\frac{1}{2}r(r+1)}.$$

Of the other results the same cannot be said, and Bonolis' own procedure is lengthy. Thus, in addition to the space occupied in establishing auxiliary identities, six pages (§ 7, pp. 122–128) are taken up with proving that

$$\left|\begin{array}{llll}
(m)_0\alpha^0 & (m+d)_0\alpha^d & \ldots\ldots & (m+zd)_0\alpha^{zd} \\
(m)_1\alpha^0 & (m+d)_1\alpha^d & \ldots\ldots & (m+zd)_1\alpha^{zd} \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
(m)_{p-1}\alpha^0 & (m+d)_{p-1}\alpha^d & \ldots\ldots & (m+zd)_{p-1}\alpha^{zd} \\
(m)_0\beta^0 & (m+d)_0\beta^d & \ldots\ldots & (m+zd)_0\beta^{zd} \\
(m)_1\beta^0 & (m+d)_1\beta^d & \ldots\ldots & (m+zd)_1\beta^{zd} \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\
(m)_{q-1}\beta^0 & (m+d)_{q-1}\beta^d & \ldots\ldots & (m+zd)_{q-1}\beta^{zd}
\end{array}\right|,$$

where $z = p+q-1$, is equal to

$$d^{\frac{1}{2}p(p+1)+\frac{1}{2}q(q-1)} \cdot \alpha^{\frac{1}{2}p(p-1)d} \cdot \beta^{\frac{1}{2}q(q-1)d} \cdot (\beta^d-\alpha^d)^{pq};$$

and the remainder of the paper with the similar identities in which appear r additional rows involving γ as α and β are involved, and in which $z = p+q+r-1$.

KOSTKA, C. (1876).

(See under this heading in chap. v.)

GÜNTHER, S. (1879).

[Von der expliciten Darstellung der regulären Determinanten aus Binomialcoefficienten. *Zeitschrift f. Math. u. Phys.*, xxiv. pp. 96–103.]

This is an attempt to proceed beyond Zeipel by following Zeipel's own method, the quite general form

$$\left|(a)_\alpha(b)_\beta \ldots (r)_\rho\right|$$

being taken for investigation. In the first place a factor is removed from each row and a factor from each column, so as to make the elements of the first column of the reduced determinant all equal to 1 and the elements of all the other columns free of fractions; next, the multiplications indicated in the reduced elements are supposed to be performed; thirdly, we are directed to use Albeggiani's theorem in order to express the reduced determinant as an aggregate of determinants with monomial elements; and lastly, we are to trust to Naegelsbach for assistance in dealing with the evolved alternants.

The outcome is not very satisfying.

CHAPTER XXI.

ZERO-AXIAL DETERMINANTS, UP TO 1888.

WHAT was said in the introduction to the preceding chapter regarding classification applies also in part here. In addition, it has to be noted that as in the next twenty-year period there is only one paper concerned with zero-axial determinants, it is thought best to append the report of it to the present chapter.

BALTZER, R. (1870).

[Theorie und Anwendung der Determinanten, ... 3te verbesserte Aufl. . . . . viii+242 pp. Leipzig.]

In § 4. 2 (p. 29) Baltzer considers the question of the number of terms in the final development of an n-line determinant having all the elements of the diagonal equal to 0. Calling the number $\psi(n)$, he obtains the correct result

$$\psi(n) = n!\left\{1-1+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}\right\}, \qquad (\alpha)$$

although making two oversights in the reasoning, and thence deduces Stockwell's incidental result of 1860 (see p. 3 above),

$$\psi(n+1) = (n+1)\psi(n)+(-1)^{n+1}, \qquad (\beta)$$

the values of $\psi(1)$, $\psi(2)$, . . . being thus 0, 1, 2, 9, 44, 265, ... By using Cayley's development of 1847 he also obtains Stockwell's identity

$$n! = \psi(n) + n\psi(n-1) + \tfrac{1}{2}n(n-1)\psi(n-2) + \ldots + 1.$$

It is important to note that the problem here dealt with is identical with a much older one regarding arrangements, namely, *the finding*

of the number of permutations of n *letters subject to the condition that no letter is to be in its original place.* While studying this form of the problem, Euler in 1809 obtained the recurrence-formulae (β) and the formula

$$\psi(n+1) = n\{\psi(n)+\psi(n-1)\}.$$

Similarly (α) was obtained by Oettinger in 1837.*

WEYRAUCH, J. J. (1871).

[Zur Theorie der Determinanten. *Crelle's Journ.*, lxxiv. pp. 273–276.]

Weyrauch starts with the fact that the number of terms in $|a_{11} \ldots a_{nn}|$ which contain a_{11} is $(n-1)!$, and proves that the number containing a_{11} or a_{22} or both together is

$$(n-1)!$$
$$+(n-1)!-(n-2)!.$$

Having proceeded in this way, he then performs a summation, and finds that the number containing one or more elements of the diagonal is

$$n!\left\{1-\frac{1}{2!}+\frac{1}{3!}-\ldots+(-1)^{n-1}\frac{1}{n!}\right\}.$$

From this it at once follows that

$$\psi(n) = n!\left\{1-1+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}\right\};$$

that the number containing a particular set of m diagonal elements is $\psi(n-m)$; that the number containing m diagonal elements without restriction is $(n)_m\psi(n-m)$; that the number containing not more than m is †

$$\sum_{\mu=0}^{\mu=m}(n)_\mu\psi(n-\mu);$$

and that the number containing not less than m is

$$1+\sum_{\mu=m}^{\mu=n-2}(n)_\mu\psi(n-\mu).$$

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* Netto's *Lehrbuch der Combinatorik* (Leipzig, 1901).

† Putting $m = 1$ here we have the answer to question 445 of the *Nouv. Annales de Math.*, xvii. (1858), p. 262 (see *Hist.*, ii. p. 471).

MONRO, C. J. (1872).

[Baltzer on the number of terms in a determinant with a vanishing diagonal. *Messenger of Math.*, ii. pp. 38–39.]

Besides pointing out Baltzer's faulty reasoning, Monro provides a substitute. Expressing a zero-axial determinant of the $(n+1)^{th}$ order in terms of the elements of the first row and their cofactors, he obtains at once

$$\psi(n+1) = n\{\psi(n)+\psi(n-1)\}.$$

He then writes this in the form

$$\psi(n+1)-(n+1)\psi(n) = -[\psi(n)-n\psi(n-1)],$$

which leads finally to

$$\begin{aligned}\psi(n+1)-(n+1)\psi(n) &= (-1)^{n-1}[\psi(2)-2\psi(1)]\\ &= (-1)^{n-1}[1-0]\\ &= (-1)^{n-1}.\end{aligned}$$

WEIHRAUCH, K. (1874).

[Zur Determinantenlehre. *Zeitschrift f. Math. u. Phys.*, xix. pp. 354–360.]

After an introduction and a correction of Baltzer,—who had corrected himself the year before,*—Weihrauch gives two proofs of the result

$$\psi(n) = n!\left\{1-1+\frac{1}{2!}-\frac{1}{3!}+\ldots+(-1)^n\frac{1}{n!}\right\},$$

neither of them being short or attractive. Using the longer recurrence-formula, he obtains of course

$$\psi(n) = (-1)^n \begin{vmatrix} (n)_1 & 1 & . & \ldots & . & . \\ (n)_2 & (n-1)_1 & 1 & \ldots & . & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (n)_{n-1} & (n-1)_{n-2} & (n-2)_{n-3} & \ldots & 1 & . \\ (n)_n & (n-1)_{n-1} & (n-2)_{n-2} & \ldots & (1)_1 & 1 \\ (n)! & (n-1)! & (n-2)! & \ldots & 1! & 0! \end{vmatrix},$$

and this is his starting-point in both cases.

* *Berichte . . . Ges. d. Wiss.* (Leipzig), xxv. pp. 523–537.

CUNNINGHAM, A. (1874).

[An investigation of the number of constituents . . . *Quart. Journ. of Sci.*, (2), iv. pp. 212–228.]

In the sections (III.-VI.) which concern zero-axial determinants the fresh subject investigated is the number of terms in an n-line determinant having r zeros in the diagonal. One expression obtained for this is

$$\psi(n) + (n-r)_1\psi(n-1) + (n-r)_2\psi(n-2) + \ldots,$$

as we should expect; but by a process of 'symbolic inversion' he derives another therefrom, namely,

$$n! - (r)_1 \cdot (n-1)! + (r)_2 \cdot (n-2)! - \ldots,$$

which of course includes the familiar

$$\psi(n) = \frac{n!}{2!} - \frac{n!}{3!} + \ldots.$$

There is also obtained a recurrence-formula, which we may write in the form

$$u_{n,r} = u_{n,r+1} + u_{n-1,r}.$$

As an alternative source for the second expression for $u_{n,r}$ there is given a very interesting expansion for a determinant having r zeros in the diagonal, namely,

$$\begin{aligned}\Delta - \sum\left(a_{xx}\frac{\partial\Delta}{\partial a_{xx}}\right) + \sum\left(a_{xx}a_{yy}\frac{\partial^2\Delta}{\partial a_{xx}\partial a_{yy}}\right) \\ - \sum\left(a_{xx}a_{yy}a_{zz}\frac{\partial^3\Delta}{\partial a_{xx}\partial a_{yy}\partial a_{zz}}\right) \\ + \ldots\ldots\ldots\ldots\ldots\ldots \\ + (-1)^r \sum\left(a_{11}a_{22}\ldots a_{rr}\frac{\partial^r\Delta}{\partial a_{11}\partial a_{22}\ldots\partial a_{rr}}\right),\end{aligned}$$

where Δ is $|a_{11}a_{22}\ldots a_{nn}|$. Although the truth of this is said to be easily seen, it may be well to note that the zero elements are taken to be in the first r places of the diagonal; that $x, y, z, \ldots$ are not greater than r; that the Σ in the last term is not required; and that as an example we have

$$\begin{vmatrix} . & a_2 & a_3 & a_4 & a_5 \\ b_1 & . & b_3 & b_4 & b_5 \\ c_1 & c_2 & . & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix} = |a_1b_2c_3d_4e_5| - \sum a_1|b_2c_3d_4e_5| + \sum a_1b_2|c_3d_4e_5| - a_1b_2c_3|d_4e_5|.$$

The expansion in its limiting form,—that is, when $r = n$,—might have been called by Cunningham a 'symbolic inversion' of Cayley's expansion of 1847 (*Hist.*, ii. p. 42).

DICKSON, J. D. H. (1879).

[Discussion of two double series arising from the number of terms in determinants of certain forms. *Proceed. London Math. Soc.*, x. pp. 120–122.]

The first determinant dealt with is Cunningham's of 1874. The same recurrence-formula is obtained, and others less important. A table of the values of $u_{n,r}$ is given, although of course it is merely a table of the differences of $1 \cdot 2 \cdot 3 \ldots n$.

The other determinant is that which has zeros in the first r places of the primary diagonal and in the first $r-1$ places of the adjacent minor diagonal. The number of terms being $v_{n,r}$, it is stated that

$$v_{n,r} = v_{n,r+1} + 2v_{n-1,r} + v_{n-2,r-1};$$

also that

$$v_{n,n} = (n-1)(v_{n-1,n-1} + v_{n-2,n-2}) + v_{n-3,n-3};$$

and the values are tabulated as far as $v_{10,10}$.

We may note in passing that the Ψ of Muir's paper of 1877 is such that

$$v_{n,n} - v_{n-1,n-1} = \Psi(n).$$

HERTZSPRUNG, S. (1879).

[Losning og Udvidelse af Opgave 402. *Tidskrift for Math.*, (4) iii. pp. 134–140.]

After a short statement of the facts regarding the simpler problem Hertzsprung raises the question of *the number of terms in a determinant whose* two *diagonals contain nothing but zero elements.* From the outset, however, he views it as the problem of finding *the number of*

arrangements of the counters, 1, 2, . . . , n, *subject to the conditions that in every case the* k^{th} *counter shall occupy neither the* k^{th} *place from the beginning nor the* k^{th} *place from the end*, and determinants are not in any way referred to. His final result is the recurrence-formula

$$w_n = (n-1)w_{n-1} + \begin{cases} 2(n-2)w_{n-3} & \text{for } n \text{ even}, \\ 2(n-1)w_{n-2} & \text{for } n \text{ odd}; \end{cases}$$

so that, since $w_2 = 0$, $w_3 = 0$, $w_4 = 4$, he finds

$$w_5 = 16, \quad w_6 = 80, \quad w_7 = 672, \quad w_8 = 4752, \quad \ldots .$$

HANSTED, B. (1880).

[Trois théorèmes relatifs à la théorie des nombres. *Journ. de sci. math. e astron.*, ii. pp. 154–164.]

The second theorem established (pp. 156–158) is that the number of terms in an n-line zero-axial determinant is the nearest integer to $n!e^{-1}$.

SZÜTS, N. v. (1888).

[Zur Theorie der Determinanten. *Math. Annalen*, xxxiii. pp. 477–492.]

The chief object of the author here is to generalize Weyrauch's result of 1871, and this with a wealth of formulae he fully effects. He is unaware, however, of Cunningham's paper of 1874 and Dickson's of 1878. The way in which he formulates their and his principal result is: *The number of non-zero terms in an* n*-line determinant having* r *zeros in its main diagonal is the* $(n-r+1)^{th}$ *member of the* r^{th} *row of differences of*

$$1, \quad 1!, \quad 2!, \quad 3!, \quad 4! \ldots .$$

CHAPTER XXII.

THE LESS COMMON SPECIAL FORMS, FROM 1839 TO 1880.

WHAT remains now to be attended to are those special forms which are of much rarer occurrence during the period than any of those hitherto dealt with, the great majority of them, indeed, occurring only once. The chronological order of the papers will as hitherto be adhered to, save that, in the case of a form dealt with in more than one paper, the papers of the group will be brought together.

CATALAN, E. (1839).

[Sur la transformation des variables dans les intégrales multiples. *Mém. couronnés par l'Acad. . . . . de Bruxelles*, xiv. 2^{me} partie, 49 pp.]

The third and fourth sections (pp. 25–31, 32–47) of Catalan's memoir are occupied with applications of the main result of the second section (*Hist.*, i. pp. 356–358); and what we have now to note is that in the course of the work a fresh form of determinant makes its appearance, namely, that in which the elements are definite integrals.

The first example selected to illustrate the transformation is

$$\int \ldots \int \partial x_1 \partial x_2 \ldots \partial x_n,$$

in which the summation extends to all positive values of the x's subject to the condition

$$\frac{x_1^2}{a^2 - a_1^2} + \frac{x_2^2}{a^2 - a_2^2} + \ldots + \frac{x_n^2}{a^2 - a_n^2} \leqq 1,$$

where $a_1 > a_1 > a_2 \ldots > a_n$.

The value of the integral is known to be

$$\frac{(\frac{1}{2}\sqrt{\pi})^n}{\Gamma(1+\frac{1}{2}n)}\sqrt{(a^2-a_1^2)(a^2-a_2)\ldots(a^2-a_n^2)};$$

and the new result obtained is a consequence of the fact that when, by means of the equations

$$\left.\begin{aligned}
\frac{x_1^2}{u_1^2-a_1^2}+\frac{x_2^2}{u_1^2-a_2^2}+\ldots+\frac{x_n^2}{u_1^2-a_n^2}&=1\\
\frac{x_1^2}{u_2^2-a_1^2}+\frac{x_2^2}{u_2^2-a_2^2}+\ldots+\frac{x_n^2}{u_2^2-a_n^2}&=1\\
\ldots\ldots\ldots\ldots\ldots\ldots&\\
\frac{x_1^2}{u_n^2-a_1^2}+\frac{x_2^2}{u_n^2-a_2^2}+\ldots+\frac{x_n^2}{u_n^2-a_n^2}&=1
\end{aligned}\right\},$$

the variables of the integral are changed into $u_1, u_2, \ldots, u_n$ the integral itself becomes (p. 36) for the case where $a_n = 0$,

$$\begin{vmatrix}
\int_0^{a_{n-1}}\frac{\partial u_n}{R_n} & \int_{a_{n-1}}^{a_{n-2}}\frac{\partial u_{n-1}}{R_{n-1}} & \ldots & \int_{a_1}^{a}\frac{\partial u_1}{R_1}\\
\int_0^{a_{n-1}}\frac{u_n^2\partial u_n}{R_n} & \int_{a_{n-1}}^{a_{n-2}}\frac{u_{n-1}^2\partial u_{n-1}}{R_{n-1}} & \ldots & \int_{a_1}^{a}\frac{u_1^2\partial u_1}{R_1}\\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\\
\int_0^{a_{n-1}}\frac{u_n^{2n-2}\partial u_n}{R_n} & \int_{a_{n-1}}^{a_{n-2}}\frac{u_{n-1}^{2n-2}\partial u_{n-1}}{R_{n-1}} & \ldots & \int_{a_1}^{a}\frac{u_1^{2n-1}\partial u_1}{R_1}
\end{vmatrix},$$

where

$$\begin{aligned}
R_1 &= \sqrt{(u_1^2-a_1^2)(u_1^2-a_2^2)\ldots(u_1^2-a_{n-1}^2)},\\
R_2 &= \sqrt{(a_1^2-u_2^2)(u_2^2-a_2^2)\ldots(u_2^2-a_{n-1}^2)},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
R_p &= \sqrt{(a_1^2-u_p^2)\ldots(a_{p-1}^2-u_p^2)(u_p^2-a_p^2)\ldots(u_p^2-a_{n-1}^2)}.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
R_n &= \sqrt{(a_1^2-u_n^2)(a_2^2-u_n^2)\ldots(a_{n-1}^2-u_n^2)}.
\end{aligned}$$

If n be taken equal to 3 the result degenerates into the well-known theorem of Legendre's regarding the complete elliptic integrals of the first and second kinds.

Another result is similarly obtained (p. 40) from the transformation of the integral

$$\int \ldots \int \partial x_1 \partial x_2 \ldots \partial x_n \sqrt{1 + \left(\frac{\partial x}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial x}{\partial x_n}\right)^2},$$

where the summation extends as before, and x is given in terms of the other x's by the equation

$$\frac{x^2}{u^2 - a^2} + \frac{x_1^2}{u^2 - a_1^2} + \ldots + \frac{x_n^2}{u^2 - a_n^2} = 1.$$

ROBERTS, W. (1851).

[Sur quelques propriétés des intégrales définies, déduites de la méthode des coördonnées elliptiques. *Journ.* (*de Liouville*) *de Math.*, xvi. pp. 1–5.]

Roberts changes the variables of the integral

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \partial x \, \partial y,$$

whose value is $\frac{1}{4}\pi$, by means of the equations

$$\left.\begin{aligned} \frac{x^2}{u_1^2} + \frac{y^2}{u_1^2 - b^2} &= 1 \\ \frac{x^2}{u_2^2} - \frac{y^2}{b^2 - u_2^2} &= 1 \end{aligned}\right\},$$

and thence obtains

$$\tfrac{1}{4}\pi e^{-b} = \int_b^\infty \frac{e^{-u^2} u^2 \partial u}{\sqrt{u^2 - b^2}} \cdot \int_0^b \frac{e^{-u^2} \partial u}{\sqrt{b^2 - u^2}} - \int_b^\infty \frac{e^{-u^2} \partial u}{\sqrt{u^2 - b^2}} \cdot \int_0^b \frac{e^{-u^2} u^2 \partial u}{\sqrt{b^2 - u^2}},$$

the right-hand member of which would be suitably represented as a determinant of the second order. He then treats in similar fashion the integral

$$\int_0^\infty \int_0^\infty \int_0^\infty e^{-(x^2+y^2+z^2)} \partial x \, \partial y \, \partial z,$$

and obtains for $\frac{1}{8}\pi^{\frac{3}{2}} \cdot e^{-b^2-c^2}$ a six-termed expression representable as a determinant of the third order; and he suggests proceeding further.

Tissot's paper of the following year (*Hist.*, ii. pp. 461–462) seems to have had its origin in this note of Roberts'.

ENNEPER, A. (1861, 1865).

[Zur Theorie der bestimmten Integrale. *Zeitschrift f. Math. u. Phys.*, vi. pp. 289–310.]

[Ueber eine Determinante bestimmter Integrale. *Zeitschrift f. Math. u. Phys.*, xi. pp. 69–77.]

Of the five notes composing the first of these two papers the second (pp. 294–300) concerns Catalan's first determinant of 1839. This Enneper improves in neatness by reducing to the order $n-1$, the result now being

$$\begin{vmatrix} \int_0^{a_{n-1}} \frac{\partial u}{R_n} & \int_{a_{n-1}}^{a_{n-2}} \frac{\partial u}{R_{n-1}} & \cdots & \int_{a_2} \frac{\partial u}{R_2} \\ \int_0^{a_{n-1}} \frac{u^2 \partial u}{R_n} & \int_{a_{n-1}}^{a_{n-2}} \frac{u^2 \partial u}{R_{n-1}} & \cdots & \int_{a_2}^{a_1} \frac{u^2 \partial u}{R_2} \\ \cdots & \cdots & \cdots & \cdots \\ \int_0^{a_{n-1}} \frac{u^{2n-4} \partial u}{R_n} & \int_{a_{n-1}}^{a_{n-2}} \frac{u^{2n-3} \partial u}{R_{n-1}} & \cdots & \int_{a_2}^{a_1} \frac{u^{2n-4} \partial u}{R_2} \end{vmatrix} = \frac{(\sqrt{\pi})^n}{2^{n-1}\Gamma(\frac{1}{2}n)},$$

where

$$a_1 > a_2 > a_3 \ldots > a_{n-1},$$

$$R_p = \sqrt{(a_1^2-u^2)\ldots(a_{p-1}^2-u^2)\cdot(u^2-a_p^2)\ldots(u^2-a_{n-1}^2)}, \quad p<n,$$

and $R_n = \sqrt{(a_1^2-u^2)(a_2^2-u^2)\ldots(a_{n-1}^2-u^2)}$.

In the third note of the same paper he makes an equal improvement on Tissot's determinant of 1852; of Catalan's second determinant of 1839 he says nothing.

His paper of 1865, however, is devoted to showing that Catalan's two results are not essentially different, the first being deducible from the second,—a not unnatural conclusion since the two are viewable as generalizations of the expressions for the surface and volume of an ellipsoid.

HEINE, E. (1861).

[Die *Lame*'schen Functionen verschiedener Ordnungen. *Crelle's Journ.*, lx pp. 252–303.]

The transformation which Heine has occasion to use (pp. 292–293) is from the persymmetric determinant

$$\begin{vmatrix} \eta_0 & \eta_1 & \cdot\cdot\cdot & \eta_\sigma \\ \eta_1 & \eta_2 & \cdot\cdot\cdot & \eta_{\sigma+1} \\ \cdot & \cdot & \cdot\cdot\cdot & \cdot \\ \eta_\sigma & \eta_{\sigma+1} & \cdot\cdot\cdot & \eta_{2\sigma} \end{vmatrix}, \quad \text{where } \eta_m \equiv \int_\beta^\gamma \frac{x^m \partial x}{\sqrt{\psi(x)}},$$

to the multiple integral

$$\int_\beta^\gamma \frac{x_1 x_2 \ldots x_\sigma \cdot \xi^{\frac{1}{2}}(x_0, x_1, \ldots, x_\sigma) \cdot \partial x_0 \partial x_1 \ldots \partial x_\sigma}{\sqrt{\psi(x_0) \cdot \psi(x_1) \ldots \psi(x_\sigma)}},$$

x in the first row of the determinant having previously been replaced by x_0, x in the second row by x_1, x in the third row by x_2, and so on.

BRIOSCHI, F. (1854).

[Sulle funzioni simmetriche delle radici di una equazione. *Annali di sci. mat. e fis.*, v. pp. 422–428; or *Opere*, i. pp. 143–150.]

At the close (p. 427) Brioschi makes the noteworthy remark that 'on certain conditions the determinant form is also adapted to represent Waring's expressions for symmetric functions in terms of sums of like powers of the variables.' Thus, the expression for $\Sigma x_1^\alpha x_2^\beta x_3^\gamma$, namely,

$$s_\alpha s_\beta s_\gamma + 2s_{\alpha+\beta+\gamma} - s_{\alpha+\beta} s_\gamma - s_{\beta+\gamma} s_\alpha - s_{\gamma+\alpha} s_\beta,$$

is representable by the 3-line determinant $|a_{\alpha\alpha} a_{\beta\beta} a_{\gamma\gamma}|$ on condition that for

$$a_{\alpha\alpha}, \quad a_{\alpha\beta} a_{\beta\alpha}, \quad a_{\alpha\beta} a_{\beta\gamma} a_{\gamma\alpha}, \quad \ldots\ldots$$

in the ordinary expansion there be substituted

$$s_\alpha, \quad s_{\alpha+\beta}, \quad s_{\alpha+\beta+\gamma}, \quad \ldots\ldots$$

and similarly that in the case of $|a_{\alpha\alpha}a_{\beta\beta}a_{\gamma\gamma}a_{\delta\delta}|$ such substitutions as

$$s_{\alpha+\beta+\gamma\delta} \quad \text{for} \quad a_{\alpha\beta}a_{\beta\gamma}a_{\gamma\alpha}a_{\delta\delta},$$

$$s_{\alpha+\beta}s_{\gamma+\delta} \quad \text{for} \quad a_{\alpha\beta}a_{\beta\alpha}a_{\gamma\delta}a_{\delta\gamma},$$

be made. Nothing in the nature of proof is given.

This remark of Brioschi's has been sadly overlooked. In 1857 Bellavitis, apparently unaware of it, gave an incorrect view of his own: in 1876 Faà di Bruno ignored it and substituted a second incorrect view: it was not until 1908 that Muir, seeking to put Bellavitis and Bruno right, rediscovered the correct result in a different form.* (*Proceed. Edinburgh Math. Soc.*, xxvii. pp. 5–9.)

CLEBSCH, A. (1861): CHRISTOFFEL, E. B. (1864).

[Ueber eine Classe von Gleichungen, welche nur reelle Wurzeln besitzen. *Crelle's Journ.*, lxii. pp. 232–245.]

[Verallgemeinerung einiger Theoreme des Herrn Weierstrass. *Crelle's Journ.*, lxiii. pp. 255–272.]

These two papers deserve mention because they make use of determinants whose elements are complex numbers and follow up similar work previously referred to (*Hist.*, ii. pp. 449–452). Their main interest, however, is in connection with the subject of linear transformation and with the study of bilinear forms whose variables or coefficients or both are complex quantities.

SYLVESTER, J. J. (1863).

[On the centre of gravity of a truncated triangular pyramid, and . . . . *Philos. Magazine*, xxvi. pp. 167–183; or *Collected Math. Papers*, ii. pp. 342–357.]

In the course of his investigation Sylvester is led to the results

$$\begin{vmatrix} bc & c\alpha & \alpha\beta \\ \beta\gamma & ca & a\beta \\ b\gamma & \gamma a & ab \end{vmatrix} = (abc - \alpha\beta\gamma)^2, \qquad \begin{vmatrix} bcd & cd\alpha & d\alpha\beta & \alpha\beta\gamma \\ \beta\gamma\delta & cd\iota & da\beta & a\beta\gamma \\ b\gamma\delta & \gamma\delta a & da\iota & ab\gamma \\ bc\delta & c\delta a & \delta a\beta & abc \end{vmatrix} = (abcd - \alpha\beta\gamma\delta)^3,$$

* In our fourth volume it will be shown that the oversight here commented on was pointed out by E. D. Roe in 1898.

which he naturally views as the second and third cases of a series, of which the first case is

$$\begin{vmatrix} b & a \\ \beta & a \end{vmatrix} = (ab - a\beta)'.$$

The law of formation of the determinants is assumed to be evident, and no proof is given.* Two related identities, foreshadowing another series, are also given, namely,

$$\begin{vmatrix} ac(b+\beta) & ca & a\beta \\ \beta a(c+\gamma) & ca & a\beta \\ \gamma b(a+\alpha) & \gamma a & ab \end{vmatrix} = -\, a\alpha(bc - \beta\gamma)(abc - a\beta\gamma),$$

$$\begin{vmatrix} ad(bc+c\beta+\beta\gamma) & cda & da\beta & a\beta\gamma \\ \beta a(cd+d\gamma+\gamma\delta) & cda & da\beta & a\beta\gamma \\ \gamma b(da+a\delta+\delta\alpha) & \gamma\delta a & dab & ab\gamma \\ \delta c(ab+ba+a\beta) & c\delta a & \delta a\beta & abc \end{vmatrix} = a\alpha(bcd - \beta\gamma\delta)(abcd - a\beta\gamma\delta)^2,$$

where the determinants differ from the previous two in the first columns only.†

SARDI, C. (1864).

[Proprietà di un determinante. *Giornale di Mat.*, ii. pp. 376–380.]

The subject is in reality the construction of a magic square.

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* The operations

$$\begin{array}{llll} \mathrm{col}_1 \times a, & \mathrm{row}_2 \div a; & \mathrm{col}_2 \times ab, & \mathrm{row}_3 \div ab; \quad \mathrm{col}_3 \times abc, \quad \mathrm{row}_4 \div abc; \quad \ldots \\ \mathrm{row}_2 \times a, & \mathrm{col}_2 \div a; & \mathrm{row}_3 \times a\beta, & \mathrm{col}_3 \div a\beta; \quad \mathrm{row}_4 \times a\beta\gamma, \ \mathrm{col}_4 \div a\beta\gamma; \quad \ldots \end{array}$$

produce determinants involving only the product of all the Italic letters, say p, and the product of all the Greek letters, say π. For example, the four-line determinant becomes

$$\begin{vmatrix} p & p & p & 1 \\ \pi & p & p & 1 \\ \pi & \pi & p & 1 \\ \pi & \pi & \pi & 1 \end{vmatrix}, \quad i.e. \quad \begin{vmatrix} p-\pi & . & . & . \\ . & p-\pi & . & . \\ . & . & p-\pi & . \\ \pi & \pi & \pi & 1 \end{vmatrix}.$$

† In the case of the four-line determinant the operation

$$\mathrm{col}_1 - b\,\mathrm{col}_2 - c\,\mathrm{col}_3 - d\,\mathrm{col}_4$$

changes the first column into

$$0, \quad a(\beta\gamma\delta - bcd), \quad ab(\gamma\delta - cd), \quad abc(\delta - d),$$

and it is then found that the cofactors of the third and fourth of these elements vanish.

HORNER, J. (1865).

[Notes on determinants. *Quart. Journ. of Math.*, viii. pp. 157–162.]

As an example of his theorem regarding the reduction of the order of a determinant, Horner gives

$$\begin{vmatrix} 1 & a & b & c \\ \frac{1}{a} & 1 & d & e \\ \frac{1}{b} & \frac{1}{d} & 1 & f \\ \frac{1}{c} & \frac{1}{e} & \frac{1}{f} & 1 \end{vmatrix} = \begin{vmatrix} . & d-\frac{b}{a} & e-\frac{c}{a} \\ \frac{1}{d}-\frac{a}{b} & . & f-\frac{c}{b} \\ \frac{1}{e}-\frac{a}{c} & \frac{1}{f}-\frac{b}{c} & . \end{vmatrix}$$

$$= \begin{vmatrix} . & ad-b & ae-c \\ b-ad & . & (bf-c)d \\ (c-ae)f & (c-bf)e & . \end{vmatrix} \div a \cdot bd \cdot cef$$

$$= (b-ad)(c-bf)(ae-c)(e-df) \div abcdef.$$

HORNER, J. (1865).

[Notes on determinants. *Quart. Journ. of Math.*, viii. pp. 157–162.]

Instead of using { } in the symbolizing of a permanent, as Cayley did in 1857, Horner introduces $\overline{|\ |}$ and gives the identity

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \cdot \begin{vmatrix} d & e & f \\ d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \end{vmatrix}$$

$$= \overline{\begin{vmatrix} ad & be & cf \\ a_1d_1 & b_1e_1 & c_1f_1 \\ a_2d_2 & b_2e_2 & c_2f_2 \end{vmatrix}} - \overline{\begin{vmatrix} ad & be & cf \\ a_1d_2 & b_1e_2 & c_1f_2 \\ a_2d_1 & b_2e_1 & c_2f_1 \end{vmatrix}} + \overline{\begin{vmatrix} ad_1 & be_1 & cf_1 \\ a_1d_2 & b_1e_2 & c_1f_2 \\ a_2d & b_2e & c_2f \end{vmatrix}}$$

$$- \overline{\begin{vmatrix} ad_1 & be_1 & cf_1 \\ a_1d & b_1e & c_1f \\ a_2d_2 & b_2e_2 & c_2f_2 \end{vmatrix}} + \overline{\begin{vmatrix} ad_2 & be_2 & cf_2 \\ a_1d & b_1e & c_1f \\ a_2d_1 & b_2e_1 & c_2f_1 \end{vmatrix}} - \overline{\begin{vmatrix} ad_2 & be_2 & cf_2 \\ a_1d_1 & b_1e_1 & c_1f_1 \\ a_2d & b_2e & c_2f \end{vmatrix}}$$

as an instance of a general theorem which he establishes for expressing the product of two determinants in the form of an aggregate of what he calls 'conterminants.' His proof depends on the fact

that if each term of $|ab_1c_2|$ be multiplied by the corresponding term of $|de_1f_2|$, then similarly, by the corresponding term of

$$-|de_2f_1|, \quad |d_1e_2f|, \quad -|d_1ef_2|, \quad |d_2ef_1|, \quad -|d_2e_1f|,$$

the aggregate of the 36 terms thus obtained is the product desired. He also, without reference to Cayley (1851), gives the companion theorem

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \cdot \begin{vmatrix} d & e & f \\ d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \end{vmatrix} = |ad \cdot b_1e_1 \cdot c_2f_2| - |ad \cdot b_1c_2 \cdot c_2f_1| + \ldots,$$

and thus is able to indicate how the product of three or more determinants can be expressed.

In his illustrative examples Horner is unfortunate, the expression obtained for $|ab_1c_2|^2$ being quite incorrect, and the expression for $|a^0b^1c^2|^2$ being much more readily got from

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}^2 \quad \text{than from} \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}^2.$$

HAMMOND, J. (1879).

[Question 6001. *Educ. Times*, xxxii. p. 179; lii. p. 338: solution by T. Muir, lxv. p. 139, or *Math. from Educ. Times*, (2) xxii. pp. 49–50.]

The proposition of which proof is sought is that in the final expansion of the product

$$(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n)(a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n) \ldots \ldots (a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n)$$

the coefficients of $x_1^n, \quad x_1^{n-1}x_2, \ldots$

are all expressible as permanents, or, as Hammond rather unhappily calls them, 'alternate determinants.' He instances the first two coefficients, namely,

$$+\begin{vmatrix} a_{11} & \cdot & \cdot & \ldots & \cdot \\ \cdot & a_{21} & \cdot & \ldots & \cdot \\ \cdot & \cdot & a_{31} & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & a_{n1} \end{vmatrix}+, \qquad +\begin{vmatrix} a_{11} & \cdot & \cdot & \ldots & a_{12} \\ a_{11} & a_{21} & \cdot & \ldots & a_{22} \\ \cdot & a_{21} & a_{31} & \ldots & a_{32} \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & a_{n2} \end{vmatrix}+.$$

The proposition appears not to have been established. It was probably brought forward on seeing Scott's use of the functions in connection with alternants (see chap. v.). Muir's so-called solution is really the statement of a proposition to be substituted for Hammond's.

HUNYADY, E. v. (1866).

[Ueber ein Product zweier Determinanten. *Zeitschrift f. Math. u. Phys.*, xi. pp. 359–360.]

The interest of this is strictly geometrical.

CALDARERA, F. (1866).

[Dei determinanti a matrice magica. *Giornale di sci. nat. ed econ.* (Palermo), i. pp. 173–196.]

Evaluations such as

$$\begin{vmatrix} x & . & -1 & 1 & . \\ 1 & x & -1 & 1 & . \\ 1 & . & x-1 & . & 1 \\ . & 1 & -1 & x & 1 \\ . & 1 & -1 & . & x \end{vmatrix} = (x^3+1)(x^2-x+1)$$

are made with great fullness of detail, the centrosymmetry being unobserved.

SYLVESTER, J. J. (1867).

[Thoughts on inverse orthogonal matrices, simultaneous sign-successions, ... *Philos. Magazine*, xxxiv. pp. 461–475; or *Collected Math. Papers*, ii. pp. 615–628.]

A determinant matrix is said by Sylvester to be 'inversely orthogonal' when its elements are inversely proportional to the corresponding elements of the adjugate matrix. This is the same as to say that the product of any element by its cofactor in the determinant is constant, and therefore equal to $1/n$ of the value of the determinant when the order is n. The name properly implies contrast with an orthogonant, as the elements of the latter are *directly* proportional to the elements of its adjugate.

Taking the case of a three-line determinant, he finds that the number of independent equations to be satisfied in the construction of it is

$$(3^2-1)-(2\cdot 3-1)+1,$$

i.e.

$$(3-1)^2;$$

and he concludes that in the case of n lines the number of independent equations is $(n-1)^2$, and that therefore $2n-1$ of the elements are arbitrary. This warrants him for ease in construction to make each element of the first row and first column equal to 1. Doing so in the case of the third order, he consequently seeks to determine a, b, c, d so as to have

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{vmatrix}$$

inversely orthogonal,—that is to say, so as to have

$$ab-bc = d(a-1) = c(1-b) = b(1-c) = a(d-1).$$

These equations are equivalent to the two pairs

$$ad = c = b \quad \text{and} \quad bc = d = a,$$

from the first of which there is obtained

$$a^2d^2 = bc,$$

and thence, with the help of the second,

$$a^4 = a.$$

The roots 0 and 1 of this being inapplicable, and the others being denoted by γ, γ^2, the two sets of values found for a, b, c, d are

$$\begin{matrix} \gamma, & \gamma^2, & \gamma^2, & \gamma \\ \gamma^2, & \gamma, & \gamma & \gamma^2, \end{matrix}$$

which in effect result in one and the same determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & \gamma & \gamma^2 \\ 1 & \gamma^2 & \gamma \end{vmatrix}$$

with the constant product $\gamma^2-\gamma$ or $\sqrt{-3}$ and value

$$3(\gamma^2-\gamma) \quad \text{or} \quad (-3^3)^{\frac{1}{2}}.$$

Similarly, we are told, in the case of the fifth order there is obtained

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 & \epsilon^3 & \epsilon^4 \\ 1 & \epsilon^2 & \epsilon^4 & \epsilon^6 & \epsilon^8 \\ 1 & \epsilon^3 & \epsilon^6 & \epsilon^9 & \epsilon^{12} \\ 1 & \epsilon^4 & \epsilon^8 & \epsilon^{12} & \epsilon^{16} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \epsilon & \epsilon^2 & \epsilon^3 & \epsilon^4 \\ 1 & \epsilon^2 & \epsilon^4 & \epsilon & \epsilon^3 \\ 1 & \epsilon^3 & \epsilon & \epsilon^4 & \epsilon^2 \\ 1 & \epsilon^4 & \epsilon^3 & \epsilon^2 & \epsilon \end{vmatrix},$$

where ϵ is a primitive fifth root of 1, and the value of the determinant is $(5^5)^{\frac{1}{2}}$. It is added that this holds generally when the order-number n is a prime, the value being then

$$(\sqrt{-1})^{\frac{1}{2}(n-1)(n-2)} n^{\frac{1}{2}n}.$$

When the order is composite, the results are not so simple, the variety introduced being due to the different modes of resolving n into factors. The case where n is 4 is dealt with in detail.

VELTMANN, W. (1871).

[Beiträge zur Theorie der Determinanten. *Zeitschrift f. Math. u. Phys.*, xvi. pp. 516–525.]

The third of Veltmann's contributions is in effect the identity

$$\begin{vmatrix} x & a_1 & a_2 & \dots & a_{n-1} & 1 \\ a_1 & x & a_2 & \dots & a_{n-1} & 1 \\ a_1 & a_2 & x & \dots & a_{n-1} & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x & 1 \\ a_1 & a_2 & a_3 & \dots & a_n & 1 \end{vmatrix} = (x-a_1)(x-a_2) \dots (x-a_n).$$

He does not note that the identity obtained by deleting the last row and column in the left-hand member and substituting

$$x+a_1+a_2+\dots+a_{n-1}$$

for $x-a_n$ on the right is much more interesting, being naturally first in order of thought, and the parent of the other by 'bordering.' The case where the bordering lines cross in a zero is also worth noting.

ROSANES, J. (1872).

[Ueber Functionen, welche ein den Functionaldeterminanten analoges Verhalten zeigen. *Crelle's Journ.*, lxxv. pp. 166–171.]

What is here considered is a property of determinants of the form

$$\begin{vmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x\,\partial y} & \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial x\,\partial y} & \frac{\partial^2 v}{\partial y^2} \\ \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 w}{\partial x\,\partial y} & \frac{\partial^2 w}{\partial y^2} \end{vmatrix} \quad \text{or, say, } \mathrm{R}(u, v, w),$$

where u, v, w are binary quantics of one and the same degree. Such a determinant Rosanes writes in the form

$$\begin{vmatrix} f_1^{11} & f_1^{12} & f_1^{22} \\ f_2^{11} & f_2^{12} & f_2^{22} \\ f_3^{11} & f_3^{12} & f_3^{22} \end{vmatrix},$$

the functions undergoing differentiation being f_1, f_2, f_3, and the independent variables x_1, x_2. His theorem may be formulated thus: *If* f_1, f_2, f_3, f_4 *be binary* m-*thics*; ϕ_1, ϕ_2, ϕ_3, ϕ_4 *be the four* R*'s formable from the* f*'s, namely,* $-\mathrm{R}(\mathrm{f}_2, \mathrm{f}_3, \mathrm{f}_4)$, $\mathrm{R}(\mathrm{f}_1, \mathrm{f}_3, \mathrm{f}_4)$, $-\mathrm{R}(\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_4)$, $\mathrm{R}(\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3)$; ψ_1, ψ_2, ψ_3, ψ_4 *be formed from the* ϕ*'s as the* ϕ*'s from the* f*'s: then the* ψ*'s are proportional to the* f*'s, the common ratio being*

$$-3\mathrm{m}(\mathrm{m}-1)^2(\mathrm{m}-2)^{-5}(3\mathrm{m}-7)^2\left\{\mathrm{R}(\mathrm{f}_1, \mathrm{f}_2, \mathrm{f}_3, \mathrm{f}_4)\right\}^2.$$

The mode of proof, though effective, is neither as short nor as direct as one could wish. Two new magnitudes y_1, y_2, independent of the two x's, are introduced; and the consideration of the array

$$\left\| \begin{matrix} f_1^{11} & f_2^{11} & f_3^{11} & f_4^{11} \\ f_1^{12} & f_2^{12} & f_3^{12} & f_4^{12} \\ f_1^{22} & f_2^{22} & f_3^{22} & f_4^{22} \end{matrix} \right\|,$$

whose principal minors are the ϕ's, is replaced by the consideration of the array

$$\left\| \begin{matrix} f_1 & f_2 & f_3 & f_4 \\ \delta f_1 & \delta f_2 & \delta f_3 & \delta f_4 \\ \delta_2 f_1 & \delta_2 f_2 & \delta_2 f_3 & \delta_2 f_4 \end{matrix} \right\|,$$

where

$$\delta f \equiv y_1 \frac{\partial f}{\partial x_1} + y_2 \frac{\partial f}{\partial x_2} \quad \text{and} \quad \delta_2 f \equiv \delta(\delta f).$$

The replacement is justified by the fact that the first array when multiplied columnwise by $|x_1y_2|^3$ in the form

$$\begin{vmatrix} x_1^2 & x_1y_1 & y_1^2 \\ 2x_1x_2 & x_1y_2+x_2y_1 & 2y_1y_2 \\ x_2^2 & x_2y_2 & y_2^2 \end{vmatrix}$$

gives a multiple of the second array by $m(m-1)^2$. A consequence of this, however, is that there is necessitated the use of properties of the δ-operator, for example,

$$\delta(uv) = u\,\delta v + v\,\delta u,$$

$$\delta|x_1y_2| = 0,$$

$$\delta|u_1v_2w_3| = \begin{vmatrix} \delta u_1 & \delta u_2 & \delta u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ \delta v_1 & \delta v_2 & \delta v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \delta w_1 & \delta w_2 & \delta w_3 \end{vmatrix},$$

and the use of another transforming identity, namely,

$$\begin{vmatrix} f_1^{111} & f_2^{111} & f_3^{111} & f_4^{111} \\ f_1^{112} & f_2^{112} & f_3^{112} & f_4^{112} \\ f_1^{122} & f_2^{122} & f_3^{122} & f_4^{122} \\ f_1^{222} & f_2^{222} & f_3^{222} & f_4^{222} \end{vmatrix} \cdot |x_1y_2|^6 = \begin{vmatrix} f_1 & f_2 & f_3 & f_4 \\ \delta f_1 & \delta f_2 & \delta f_3 & \delta f_4 \\ \delta_2 f_1 & \delta_2 f_2 & \delta_2 f_3 & \delta_2 f_4 \\ \delta_3 f_1 & \delta_3 f_2 & \delta_3 f_3 & \delta_3 f_4 \end{vmatrix} m(m-1)^2(m-2)^3,$$

or, say,

$$\mathrm{R}(f_1, f_2, f_3, f_4) \cdot |x_1y_2|^6 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \cdot m(m-1)^2(m-2)^3,$$

which is established by taking the multiplier $|x_1x_2|^6$ in the form

$$\begin{vmatrix} x_1^3 & x_1^2y_1 & x_1y_1^2 & y_1^3 \\ 3x_1^2x_2 & x_1^2y_2+2x_1x_2y_1 & y_1^2x_2+2y_1y_2x_1 & 3y_1^2y_2 \\ 3x_1x_2^2 & x_2^2y_1+2x_1x_2y_2 & y_2^2x_1+2y_1y_2x_2 & 3y_1y_2^2 \\ x_2^3 & x_2^2y_2 & x_2y_2^2 & y_2^3 \end{vmatrix},$$

as was shown by Schläfli (1851) to be permissible (*Hist.*, ii. pp. 52–53).

With this preliminary explanation we shall employ Rosanes' method to find the ratio of ψ_1 to f_1.

The ψ's being derived from the ϕ's as the ϕ's from the f's, and the degree of the ϕ's being $3(m-2)$, we have by the initial transformation

$$|x_1y_2|^3 \cdot \psi_1 = -3(m-2)(3m-7)^2 \cdot \begin{vmatrix} \phi_2 & \phi_3 & \phi_4 \\ \delta\phi_2 & \delta\phi_3 & \delta\phi_4 \\ \delta_2\phi_2 & \delta_2\phi_3 & \delta_2\phi_4 \end{vmatrix}.$$

The next step, naturally, is to express each element of the determinant on the right in terms of the f's. For any element in the first row there is no difficulty, the transformation just mentioned giving us what we require ; and two operations with δ will give the substitutions for the corresponding elements in the other rows. For example,

$$|x_1y_2|^3 \cdot \phi_2 = m(m-1)^2 \cdot \begin{vmatrix} f_1 & f_3 & f_4 \\ \delta f_1 & \delta f_3 & \delta f_4 \\ \delta_2 f_1 & \delta_2 f_3 & \delta_2 f_4 \end{vmatrix} = m(m-1)^2 \binom{012}{134},$$

$$|x_1y_2|^3 \cdot \delta\phi_2 = m(m-1)^2 \binom{013}{134},$$

$$|x_1y_2|^3 \cdot \delta_2\phi_2 = m(m-1)^2 \left\{ \binom{023}{134} + \binom{014}{134} \right\}.$$

The result of the full substitution thus is

$$|x_1y_2|^{12}\psi_1 = -3(m-2)(3m-7)^2\{m(m-1)^2\}^3 \cdot \begin{vmatrix} \binom{012}{134} & -\binom{012}{124} & \binom{012}{123} \\ \binom{013}{134} & -\binom{013}{124} & \binom{013}{123} \\ \binom{023}{134}+\binom{014}{134} & -\binom{023}{124}-\binom{014}{124} & \binom{023}{123}+\binom{014}{123} \end{vmatrix}$$

where the determinant on the right is partitionable into two determinants, the first of which is equal to a principal minor of the adjugate of $\binom{0123}{1234}$, and thus will be found equal to

$$f_1 \cdot \binom{0123}{1234}^2,$$

and the second of which vanishes.* Thus far, therefore, we have

$$|x_1y_2|^{12} \cdot \psi_1 = -3m^3(m-1)^6(m-2)(3m-7)^2 \cdot f_1 \cdot \binom{0123}{1234}^2.$$

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* Because, if we multiply

$$\begin{vmatrix} f_2 & f_3 & f_4 \\ \delta f_2 & \delta f_3 & \delta f_4 \\ \lambda & \mu & \nu \end{vmatrix}$$

rowwise by it, we shall obtain a determinant with two rows proportional. More generally,

$$\left| |a_1b_3c_4| \cdot |a_1b_2d_4| \cdot |a_1b_2e_3| \right| = 0,$$

where the left-hand member is our extensional of $\left| |b_3c_4| \cdot |b_2d_4| \cdot |b_2e_3| \right|$, and this last is such that $b_2\,\mathrm{col}_1 - b_3\,\mathrm{col}_2 + b_4\,\mathrm{col}_3 = 0$.

But from the second transforming identity we obtain by squaring

$$|x_1y_2|^{12} \cdot \left\{\mathrm{R}(f_1, f_2, f_3, f_4)\right\}^2 = m^2(m-1)^4(m-2)^6 \cdot \begin{pmatrix}0123\\1234\end{pmatrix}^2.$$

Consequently, by division

$$\frac{\psi_1}{\left\{\mathrm{R}(f_1, f_2, f_3, f_4)\right\}^2} = -3m(m-1)^2(m-2)^{-5}(3m-7)^2 \cdot f_1$$

as desired.

It should be noted that the determinant here which Rosanes would associate with the Jacobian is shown two years later by Pasch to be in a certain case closely related to the Wronskian. (See chap. viii.)

DOSTOR, G. (1874).

[Propriété des déterminants. *Archiv d. Math. u. Phys.*, lvi. pp. 238–240.]

The two pages used here are occupied with the formal establishment and illustration of the so-called theorem of which

$$2\begin{vmatrix} a & a & a & a \\ -a & a & b & c \\ -a & -a & a & d \\ -a & -a & -a & a \end{vmatrix} = (2a)^4$$

is an example. It is manifestly a simple case of the equality

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ -a_1 & b_2 & x_1 & x_2 \\ -a_1 & -a_2 & c_3 & x_3 \\ -a_1 & -a_2 & -a_3 & d_4 \end{vmatrix} = a_1(a_2+b_2)(a_3+c_3)(a_4+d_4).$$

In republishing it the same year in his text-book (pp. 45–46) he follows it up with another theorem of which

$$\begin{vmatrix} . & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \frac{1}{a_2a_3a_4 \cdot b_1c_1d_1} \begin{vmatrix} . & 1 & 1 & 1 \\ 1 & a_3a_4 \cdot b_2 \cdot c_1d_1 & a_2a_4 \cdot b_3 \cdot c_1d_1 & a_2a_3 \cdot b_4 \cdot c_1d_1 \\ 1 & a_3a_4 \cdot c_2 \cdot b_1d_1 & a_2a_4 \cdot c_3 \cdot b_1d_1 & a_2a_3 \cdot c_4 \cdot b_1d_1 \\ 1 & a_3a_4 \cdot d_2 \cdot b_1c_1 & a_2a_4 \cdot d_3 \cdot b_1c_1 & a_2a_3 \cdot d_4 \cdot b_1c_1 \end{vmatrix}$$

is an example (cf. *Hist.*, ii. p. 147).

CAYLEY, A. (1874).

[Note sur une formule d'intégration indéfinie. *Comptes Rendus . . . . Acad. des Sci.* (Paris), lxxviii. pp. 1624–1629; or *Collected Math. Papers*, ix. pp. 500–503.]

The peculiar structure of the interesting determinant which turns up here is best made known by giving the examples

$$\begin{vmatrix} 1 & mp-nq \\ q & m(p^2+pq) \end{vmatrix}, \qquad \begin{vmatrix} 1 & (m-1)p-nq & 1 \\ 2q & (m-1)(p^2+pq) & (m+1)p-(n-1)q \\ q^2 & . & m(p^2+pq) \end{vmatrix}$$

$$\begin{vmatrix} 1 & (m-2)p-nq & 1 & . \\ 3q & (m-2)(p^2+pq) & mp-(n-1)q & 2 \\ 3q^2 & . & (m-1)(p^2+pq) & (m+2)p-(n-2)q \\ q^3 & . & . & m(p^2+pq) \end{vmatrix},$$

$$\begin{vmatrix} 1 & (m-3)p-nq & 1 & . & . \\ 4q & (m-3)(p^2+pq) & (m-1)p-(n-1)q & 2 & . \\ 6q^2 & . & (m-2)(p^2+pq) & (m+1)p-(n-2)q & 3 \\ 4q^3 & . & . & (m-1)(p^2+pq) & (m+3)p-(n-3)q \\ q^4 & . & . & . & m(p^2+pq) \end{vmatrix}.$$

These are stated to be equal to the expressions

$$mp^2+nq^2, \qquad m(m-1)p^4 + 2mnp^2q^2 + n(n-1)q^4,$$

$$m(m-1)(m-2)p^6 + 3m(m-1)np^4q^2 + 3m(n-1)np^2q^4 + n(n-1)(n-2)q^6,$$

$$m(m-1)(m-2)(m-3)p^8 + 4m(m-1)(m-2)np^6q^2 + \ldots + n(n-1)(n-2)(n-3)q^8,$$

which again are denoted by

$$([m]p^2+[n]q^2)^1, \quad ([m]p^2+[n]q^2)^2, \quad ([m]p^2+[n]q^2)^3, \quad \ldots\ldots,$$

on the understanding that

$$[m]^r = m(m-1)(m-2)\ldots.(m-r+1).$$

Although in each determinant the complementary minors of the elements of the first column are calculated, it is not clear that this was how the results were obtained.

GÜNTHER, S. (1874): GLAISHER, J. W. L. (1874): SERDOBINSKY, V. E. (1877).

[Zur mathematischen Theorie des Schachbretts. *Archiv d. Math. u. Phys.*, lvi. pp. 281–292.]

[On the problem of the eight queens. *Philos. Magazine*, (4) xlviii. pp. 457–467.]

[Note on determinants: a chessboard problem (In Russian). *Mat. Sbornik* . . . . (Moscow), x. (1), pp. 74–86.]

The problem here dealt with is to find the number of ways in which eight queens can be disposed on an ordinary chessboard, so that no one of them may be seizable by any other. Günther, after giving an interesting historical sketch of the problem—a sketch in which Gauss is the prominent figure—points out that the problem may be generalized, the queens becoming n in number with a corresponding change in the board, and that there is an exactly equivalent problem in determinants, namely, to find the number of terms in

$$\begin{vmatrix} a_1 & c_2 & e_3 & g_4 & k_5 \\ b_2 & a_3 & c_4 & e_5 & g_6 \\ d_3 & b_4 & a_5 & c_6 & e_7 \\ f_4 & d_5 & b_6 & a_7 & c_8 \\ h_5 & f_6 & d_7 & b_8 & a_9 \end{vmatrix},$$

subject to the condition that no term shall contain the same letter or the same suffix more than once. The reason for this is that if a queen be in the $(r, s)^{\text{th}}$ place, her 'castle'-like moves require that no other queen shall be in the r^{th} row or in the s^{th} column, and her 'bishop'-like moves require that no other queen shall be in the same diagonal with her. It is the last requirement that suggests the marking of each diagonal of the one set with a special letter and each diagonal of the other by a special suffix. By actual expansion of the determinants Günther finds that in the case where n is 4 the number of dispositions is 2, namely, those corresponding to the terms

$$c_2e_5d_3b_6, \quad b_2d_5e_3c_6,$$

and that in the case where n is 5 the number is 10.

Glaisher, profiting by Günther's initiative and skilfully economizing labour, pushes the investigation forward and finds the numbers

4, 40, 92

for the cases where n is 6, 7, 8, his result in the last of these cases agreeing with that which Gauss had finally reached after two or three trials.*

Serdobinsky in his procedure does not really use determinants.

GÜNTHER, S. (1875).

[Auflösung eines besonderen Systemes linearer Gleichungen. *Archiv d. Math. u. Phys.*, lvii. pp. 240–254.]

The system in question is that in which the determinant of the $2n$ unknowns is of the form

$$\begin{vmatrix} a & b & b & a \\ c & d & -d & -c \\ e & f & f & e \\ g & h & -h & -g \end{vmatrix},$$

—that is to say, has $a_{r,s} = (-1)^{r-1} a_{r,2n-s+1}$. So far as our subject is concerned, all that is effected is the resolution of the determinant into two of the n^{th} order: for example, the four-line determinant just given is equal to

$$2^2 \cdot \begin{vmatrix} a & b \\ e & f \end{vmatrix} \cdot \begin{vmatrix} c & d \\ g & h \end{vmatrix}.$$

In the application to the axisymmetric case where

$$a_{r,s} = \sin \frac{r \cdot s \cdot \pi}{2n+1},$$

the stage reached falls short of Hunyady's of 1872 (p. 104 above).

SPOTTISWOODE, W. (1872).

[On determinants of alternate numbers. *Proceed. London Math. Soc.*, vii. pp. 100–112.]

What is here meant by a determinant of alternate or alternating

* See Bellavitis in *Atti del R. Istituto veneto*, (5) iii. (1876–77), pp. 186–187.

numbers is a determinant in which the elements are such that the product of any one by itself is zero and the product of any one by another changes sign when a change is made in the order of the factors. It is also implied—and some such implication is readily seen to be necessary—that in the formation of the terms of the determinant the elements are to be taken from the rows in order. There is no longer any idea of the numbers in question forming a *set*: and as this idea is involved in the use made of such numbers to resolve an ordinary general determinant into linear factors (*Hist.*, ii. pp. 27–30, 43–46, . . .) it is as well to keep the two usages apart,—a precaution which Spottiswoode does not observe.

The first proposition noted is that *if in the formation of the terms the elements be taken from the columns in order, the signs are all positive*; for example,

$$\begin{aligned} |\lambda_1\mu_2\nu_3| &\equiv \lambda_1\mu_2\nu_3 - \lambda_1\mu_3\nu_2 + \ldots \\ &= \lambda_1\mu_2\nu_3 + \lambda_1\nu_2\mu_3 + \ldots . \end{aligned}$$

The next is that *the interchange of two consecutive rows makes no change in the value of the determinant, but that the interchange of two consecutive columns alters its sign*, and that therefore *if two columns be identical the determinant vanishes.* The validity of the so-called addition-law of ordinary determinants is also noted.

On the other hand, nothing is said about the effect of multiplying a row, for example, the obtaining of

$$\begin{vmatrix} . & \lambda_2\lambda_1 & \lambda_3\lambda_1 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix} \quad \text{from} \quad |\lambda_1\mu_2\nu_3|$$

by multiplying by λ_1; nor about the increase of a row by a multiple of another row. Laplace's expansion-theorem is also passed by, although it could not be for the same reason.

Considerable space (pp. 103–107) is devoted to the subject of multiplication. He first expands

$$|\lambda_1\mu_2| \cdot |\rho_1\sigma_2| \quad \text{and} \quad \begin{vmatrix} \lambda_1\rho_1+\lambda_2\rho_2 & \lambda_1\sigma_1+\lambda_2\sigma_2 \\ \mu_1\rho_1+\mu_2\rho_2 & \mu_1\sigma_1+\mu_2\sigma_2 \end{vmatrix},$$

and finds that, if he takes both terms of the latter to be positive,

the one expansion differs only in sign from the other. Using the later notation of permanents, we may thus say that his result is

$$|\lambda_1\mu_2| \cdot |\rho_1\sigma_2| = -\begin{vmatrix} \lambda_1\rho_1+\lambda_2\rho_2 & \lambda_1\sigma_1+\lambda_2\sigma_2 \\ \mu_1\rho_1+\mu_2\rho_2 & \mu_1\sigma_1+\mu_2\sigma_2 \end{vmatrix}^{+}.$$

He then proceeds to examine farther, and affirms, but really without sufficient proof, that the result holds generally. When he advances to the consideration of *powers*, many simplifications of course occur ; in fact, it is only then that one of the halves of the property of alternating numbers comes into play. Here his first assertion is that the second power is skew and zero-axial ; for example,

$$\begin{aligned} |\lambda_1\mu_2|^2 &= -\begin{vmatrix} . & \Sigma\lambda\mu \\ \Sigma\mu\lambda & . \end{vmatrix}^{+} \\ &= (\lambda_1\mu_1+\lambda_2\mu_2)^2 \\ &= -2\lambda_1\lambda_2\mu_1\mu_2 ; \\ |\lambda_1\mu_2\nu_3|^2 &= -\begin{vmatrix} . & \Sigma\lambda\mu & \Sigma\lambda\nu \\ -\Sigma\lambda\mu & . & \Sigma\mu\nu \\ -\Sigma\lambda\nu & -\Sigma\mu\nu & . \end{vmatrix}^{+} \\ &= 0 ; \end{aligned}$$

and so on. The third example, however, is of doubtful accuracy.

At this stage the convention is made that the elements of each row of the given determinant form a *set*, the alteration being that henceforward we have

$$\lambda_1\lambda_2 \ldots \lambda_n = 1, \quad \mu_1\mu_2 \ldots \mu_n = 1, \quad \ldots ,$$

but it being still held that $\lambda_r\mu_s = -\mu_s\lambda_r$, just as if the λ's and μ's belonged to the same set.

Compound determinants are next dealt with (pp. 107–110), and finally certain simplifiable products such as

$$|\lambda_1\mu_2| \cdot |\mu_1\nu_2| \cdot |\nu_1\lambda_2|.$$

The paper, which embodies at intervals results taken from notes by Clifford, is lacking in thoroughness. It is supplemented in the concluding paragraph of a paper of Scott's on cubic determinants in *Proceed. London Math. Soc.*, xi. pp. 17–29.

HUGEL, T. (1876): STUDNIČKA, F. J. (1876).

[Das Problem der magischen Systeme. 48 pp. Neustadt.]

[Ueber das Verhältniss der magischen Quadrate zur Determinanten-Theorie. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), pp. 269–271.]

The main fact noted by Studnička is the vanishing of all determinants of the order $16n^2$, whose elements are identical with those of magic squares constructed in accordance with the so-called second rule of Moschopoulos. For example, the operations

$$\text{col}_4 - \text{col}_1, \quad \text{col}_3 - \text{col}_2$$

show that

$$\begin{vmatrix} \alpha+1 & \beta-1 & \beta-2 & \alpha+4 \\ \beta-4 & \alpha+6 & \alpha+7 & \beta-7 \\ \alpha+8 & \beta-6 & \beta-5 & \alpha+5 \\ \beta-3 & \alpha+3 & \alpha+2 & \beta \end{vmatrix} = 0,$$

where the elements of the determinant are those of a magic square with the common sum $2(\alpha+\beta+1)$.

As for Hugel it suffices to note that in the course of his investigations he makes effective enough use of determinants, so-called and actual.

MUIR, T. (1877).

[Note on determinant expressions for the sum of a harmonical progression. *Proceed. R. Soc. Edinburgh*, ix. pp. 361–363.]

The expression obtained for

$$\frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \ldots + \frac{1}{a+(n-1)b}$$

is

$$\begin{vmatrix} 1 & -b & -b & -b & -b & \ldots & -b \\ 1 & na & -2b & -3b & -4b & \ldots & -(n-1)b \\ 1 & (n-1)a & a & -3b & -4b & \ldots & -(n-1)b \\ 1 & (n-2)a & a & a & -4b & \ldots & -(n-1)b \\ 1 & (n-3)a & a & a & a & \ldots & -(n-1)b \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 2a & a & a & a & \ldots & a \end{vmatrix} \div a(a+b)\ldots(a+\overline{n-1}b$$

MUIR, T. (1877, 1881).

[On Professor Tait's problem of arrangement. *Proceed. R. Soc. Edinburgh,* ix. pp. 382–387.]

[Additional note on a problem of arrangement. *Proceed. R. Soc. Edinburgh,* xi. pp. 187–190.]

The problem, which arose in the investigation of the theory of knots, is to find the number of possible arrangements of n things, subject to the condition that the first is not to be in the last or first place, the second not in the first or second place, the third not in the second or third place, and so on. Muir shows that it is the same as to find the number U_n of terms in the final expansion of an n-line determinant having zeros in the last and first places of the first row, the first and second places of the second row, the second and third places of the third row, and so on : and viewing it thus he succeeds in proving that

$$\begin{aligned} U_n = {} & (n-2)U_{n-1} + (2n-4)U_{n-2} + (3n-6)U_{n-3} \\ & + (4n-10)U_{n-4} + (5n-14)U_{n-5} \\ & + (6n-20)U_{n-6} + (7n-26)U_{n-7} \\ & + \ldots\ldots\ldots\ldots \\ & + \frac{1-(-1)^n}{2}, \end{aligned}$$

and consequently that

$$1, \quad 2, \quad 13, \quad 80, \quad 579, \quad 4738, \quad \ldots\ldots$$

are the values of

$$U_3, \quad U_4, \quad U_5, \quad U_6, \quad U_7, \quad U_8, \quad \ldots\ldots$$

In his second paper he reaches the much simpler recurrence-formula

$$U_n = n\,U_{n-1} + \frac{n}{n-2}U_{n-2} + (-1)^{n-1}\frac{4}{n-2},$$

which is also interesting as implying that

$$2U_n + (-1)^{n+1}4$$

is a multiple of n.

STUDNIČKA, F. J. (1878).

[Poznámka k nauce o determinantech. *Časopis pro pěstováni math. a fys.*, vii. pp. 31–33.]

This paper is not quite in keeping with its title, the subject being the expression for double the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3). (There are similar papers in *Casopis*, ii. pp. 69–82, etc.)

PAIGE, C. LE (1878).

[Sur une transformation de déterminants. *Nouv. Corresp. Math.*, iv. pp. 79–82.]

This is the transformation of Ψ into $(\alpha'-\alpha)\Upsilon$ already familiar from Cayley (1858) when dealing with the geometric theories of involution and homography. (*Hist.*, ii. pp. 453–455.)*

PRATT, O. (1878).

[Problem 231. *Analyst*, v. p. 190; vi. p. 25.]

The assertion here made is in effect that if

$$\phi(x) = \tfrac{1}{2}x(x-1)m - x(x-2),$$

the determinant

$$\begin{vmatrix} \phi(1) & \dots & \phi(n) \\ \phi(n+1) & \dots & \phi(2n) \\ \dots & \dots & \dots \\ \phi(n^2-n+1) & \dots & \phi(n^2) \end{vmatrix}$$

vanishes for all values of n above 3. The reason is that

$$\phi(x) - 2\phi(x-1) + \phi(x-2) = m-2.$$

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* We might have noted somewhat earlier a distantly related identity dealt with by J. J. Walker in *Proceed. London Math. Soc.*, iv. p. 413. Cayley's result is

$$\left\{\Psi\right\}_{\substack{\delta=\alpha' \\ \delta'=\alpha}} = (\alpha'-\alpha)\begin{vmatrix} 1 & \alpha+\alpha' & \alpha\alpha' \\ 1 & \beta+\beta' & \beta\beta' \\ 1 & \gamma+\gamma' & \gamma\gamma' \end{vmatrix}.$$

SCOTT, R. F. (1879): (ANON.) (1879).

[On some symmetrical forms of determinants. *Messenger of Math.*, viii. pp. 131–138.]

In § 5 Scott gives the result

$$\begin{vmatrix} . & a_2 & a_3 & a_4 & \dots \\ b_1 & . & a_3 & a_4 & \dots \\ b_1 & b_2 & . & a_4 & \dots \\ b_1 & b_2 & b_3 & . & \dots \\ . & . & . & . & . \end{vmatrix} = (-1)^{n-1}\{b_1a_2a_3a_4\dots a_n + b_1b_2a_3a_4\dots a_n + b_1b_2b_3a_4\dots a_n + \dots + b_1b_2b_3b_4\dots b_{n-1}a_n\},$$

where the determinant is zero-axial in form and unisignant in development. He also notes

$$\begin{vmatrix} c_1 & a & \dots & a & 1 \\ b & c_2 & \dots & a & 1 \\ . & . & . & . & . \\ b & b & \dots & c_n & 1 \\ 1 & 1 & \dots & 1 & . \end{vmatrix} = \frac{f(a)-f(b)}{a-b},$$

where $f(x) \equiv (c_1-x)(c_2-x)\dots(c_n-x)$.

In the *Cambridge Univ. Exam. Papers*, viii. p. 228, there appears the determinant which has $a, b, c, d, \dots$ for its first row, $a, a, a, a, \dots$ for its first column, and any one of its other elements, (r, s) say, such that

$$(r, s) = (r-1, s)+(r-1, s-1)+\dots+(r-1, 1).$$

Its value is a^n. It is essentially the same as Caldarera's of 1871.

SCOTT, R. F. (1879).

[Notes on determinants. *Messenger of Math.*, viii. pp. 182–187.]

In § 8 Scott affirms that *if there are two* n*-row arrays*

$$\begin{matrix} a_1 & a_2 & \dots \\ b_1 & b_2 & \dots \\ . & . & . \\ l_1 & l_2 & \dots \end{matrix} \qquad \begin{matrix} \alpha_1 & \alpha_2 & \dots \\ \beta_1 & \beta_2 & \dots \\ . & . & . \\ \lambda_1 & \lambda_2 & \dots \end{matrix}$$

and

$$d^r_{hk} \equiv (a_h-\alpha_k)^r+(b_h-\beta_k)^r+\dots+(l_h-\lambda_k)^r,$$

then

$$\begin{vmatrix} d^r_{11} & d^r_{12} & \dots & d^r_{1s} \\ \dots & \dots & \dots & \dots \\ d^r_{s1} & d^r_{s2} & \dots & d^r_{ss} \end{vmatrix} = 0, \quad \textit{when } s = n(r-1)+3,$$

and

$$\begin{vmatrix} d^r_{11} & \dots & d^r_{1s} & 1 \\ \dots & \dots & \dots & \dots \\ d^r_{s1} & \dots & d^r_{ss} & 1 \\ 1 & \dots & 1 & . \end{vmatrix} = 0, \quad \textit{when } s = n(r-1)+2.$$

No proof is given, but doubtless the results were reached by a natural extension of Cayley's procedure of 1841 and Sylvester's of 1852, a procedure which shows that in the one case s only needs to be *greater than* $n(r-1)+2$ and in the other $> n(r-1)+1$. For example, when $n = 2$ and $r = 3$ the typical element

$$(a-\alpha)^3+(b-\beta)^3$$

would be expressed as the product of the two rows

$$a^2 \quad a \quad b^2 \quad b \quad 1 \quad a^3+b^3,$$

$$-3\alpha \quad 3\alpha^2 \quad -3\beta \quad 3\beta^2 \quad \alpha^3+\beta^3 \quad 1 \quad ;$$

and the determinant of the first result would thus be seen to be the product of the two arrays

$$\left\| \begin{matrix} a_1^2 & a_1 & b_1^2 & b_1 & 1 & a_1^3+b_1^3 \\ a_2^2 & a_2 & b_2^2 & b_2 & 1 & a_2^3+b_2^3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_s^2 & a_s & b_s^2 & b_s & 1 & a_s^3+b_s^3 \end{matrix} \right\|, \qquad \left\| \begin{matrix} -3a_1 & 3a_1^2 & -3\beta_1 & 3\beta_1^2 & a_1^3+\beta_1^3 & 1 \\ -3a_2 & 3a_2^2 & -3\beta_2 & 3\beta_2^2 & a_2^3+\beta_2^3 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -3a_s & 3a_s^2 & -3\beta_s & 3\beta_s^2 & a_s^3+\beta_s^3 & 1 \end{matrix} \right\|,$$

and therefore to be equal to 0, if s were greater than the number of columns, namely, $2(3-1)+1+1$. By affixing to the first array the additional row

$$0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1$$

and to the second array the additional row

$$0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0,$$

the second result would be quite similarly obtained.

The author's remark, that "these two theorems include most of the determinant relations in geometry," surpasses Gundelfinger's of 1873. It would be less of an overstatement if all the lower values of s had not been excluded and if a slightly more general form of d^r_{hk} had been taken. Readers interested in such applications would do well to consult the chapter on the subject in Scott's text-book published a year later (chap. xiv. pp. 180–212), and especially the last seventeen pages.

In the said text-book Scott notes (p. 216) three special determinants with elements constructed from $|a_{1n}|$, namely,

$$\beta_{rs} = (a_{r1}+a_{r2}+ \ldots +a_{rn})-a_{rs},$$
$$\gamma_{rs} = (a_{r1}+a_{r2}+ \ldots +a_{rn})-2a_{rs},$$
$$\delta_{rs} = |a_{1n}| \text{ with } a_{rs} \text{ in it increased by } x,$$

the evaluations being when amended

$$|\beta_{1n}| = (-1)^{n-1}\cdot(n-1)\cdot|a_{1n}|,$$
$$|\gamma_{1n}| = (-2)^{n-1}\cdot(n-2)\cdot|a_{1n}|,$$
$$|\delta_{1n}| = x^{n-1}\cdot|A_{1n}|\cdot(x+\Sigma a_{rs}).$$

The second is Dostor's of 1879. (See above p. 77.)

WALKER, J. J. (1879).

[Question 6025. *Educ. Times*, xxxii. pp. 205, 268; or *Math. from Educ. Times*, xxxii. p. 49.]

The question concerns the establishment of the identity

$$\begin{vmatrix} 1 & 1 & a-b \\ x & ay & a(1-b)z \\ b(1-x) & 1-y & (a-1)bz \end{vmatrix} = \begin{vmatrix} 2x-1 & 2y-1 & (a-b)(2z-1 \\ x & ay & a(1-b)z \\ b(x-1) & y-1 & (a-1)bz \end{vmatrix},$$

a matter which is of more interest than might at first sight appear. We may note that the common value is

$$-xy(a-b)(1-ab) + yz(ab-2b+1)a + zx(ab-2a+1)b + x(a-b) - y(a-b)ab - z(1-b)^2a.$$

STUDNIČKA, F. J. (1880).

[Ueber eine neue Determinanteneigenschaft. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), pp. 50–54.]

The 'new property' here is a suitable companion for Dostor's of 1874, the difference merely being that the evaluation is now accomplished by *subtracting* the first row from each of the others instead of adding. It is the simple case of the equality

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & a_2 & a_3 & a_4 & a_5 \\ x_1 & b_2 & a_3 & a_4 & a_5 \\ x_2 & x_3 & b_3 & a_4 & a_5 \\ x_4 & x_5 & x_6 & b_4 & a_5 \end{vmatrix} = (a_1-b_1)(a_2-b_2)(a_3-b_3)(a_4-b_4)a_5,$$

where the b's are each -1 and all the other elements $+1$.

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THE THEORY OF DETERMINANTS in the HISTORICAL ORDER OF DEVELOPMENT

VOLUME FOUR

THE PERIOD 1880 TO 1900.

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PREFACE.

With the issue of this volume, dealing with the period 1880–1900, my effort to present a historical account of the Theory of Determinants up to the close of the nineteenth century comes to an end. May it give to present and future students all the help which some such book would certainly have given to me half a century ago.

The work connected with the preparation of the volume has been more than ordinarily onerous. Considerably over 800 writings of one kind and another have had to be dealt with, that is to say, about a half more than in the case of the third volume. In view of the fact that each volume deals with a twenty-year period, the increase thus indicated in the attention given to the study of determinants is most striking, even when one makes allowance for the considerably increased facilities for publication during the latter period.

On first thoughts the bulk of the fourth volume ought, therefore, to be about a half more than that of the third ; but then we have to reflect that the increased number of students and the increased facilities referred to are not accompanied by the like increase of original matter. It is only that workers rush into print a little more hastily than formerly, and that scientific societies and editors do not always scrupulously exert themselves to withstand the rush,— a state of matters that after all may be well-ordered in the interest of instruction and the spread of knowledge. Be this as it may, it is an undoubted fact that the number of writings which the

chronicler of scientific progress can without unkindness dismiss in a sentence grows apace. All that can be fairly expected of him is to try his hardest to be just in his judgments, and, in the interest of his readers, to guard against hurtful condensation.

The bibliographical basis of the work, as is probably now well known among students of the subject, is the six Lists of Writings published in the *Quarterly Journal of Mathematics* at intervals from 1881 to 1916. A seventh List was added in 1920.* Since in the compiling of every one of these lists after 1900 writings belonging to the preceding century continued to be carefully sought for, it is hoped that little matter of any serious importance has been passed over that was needed for the History.

Although the number of chapters is the same as in the third volume, the distribution of subjects is somewhat different. 'Text-books' have been separated from other writings on general determinants and given a chapter to themselves. The presence of a second new chapter is due to the inclusion of a new subject 'Determinants having Invariant-factors.' On the other hand, there is no longer a chapter on 'Zero-axial determinants' nor on 'Determinants of Combinatory Numbers.' The writings on these subjects having greatly fallen off in number and importance, the few of them calling for some slight attention have been included in the chapter on 'Less common special forms.'

Of the contents of the volume only an exceedingly small fraction has seen the light before, namely, a portion of the chapter (xvi.) on Circulants.†

RONDEBOSCH, SOUTH AFRICA,
6th April, 1922.

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* *Quart. Journ. of Math.*, xlix. pp. 51–73.

† *Proc. R. Soc. Edinburgh*, xxxvi. (1915) pp. 151–173.

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CHAPTER I.

DETERMINANTS IN GENERAL, FROM 1880 TO 1900.

THE number of writings dealt with in this chapter differs very little from the number in the corresponding chapter of the preceding volume; but, as all writings of a merely text-book character are no longer retained in the chapter, having been set aside to form a chapter by themselves (Chap. II.), it will at once be understood what a notable increase of activity marks the period now under consideration. About four-fifths of the writings are in English, French or German, the predominance of English being to a considerable extent due to American contributions; but fourteen different languages are represented, Greek making its appearance for the first time.

Having spoken of the period of Vol. II. (1841–1860) as being in a sense 'the period of Cayley and Sylvester,' we feel it due to their memory to recall their continued work throughout the period of Vol. III. (1860–1880), and to note that only now in the course of Vol. IV. their last contributions fall to be chronicled, Cayley having died in 1895 (see p. 52) and Sylvester in 1897 (see p. 131).

PICQUET, (1880).

[Question 215. *Journ. de Math. Élém.*, iv. pp. 46, 325, 326.]

The theorem here proved is that already known regarding the differential-coefficient of a determinant.

SYLVESTER, J. J. (1880): HAMMOND, J. (1881).

[Question 6188. *Educ. Times*, xxxiii. p. 32: *Math. from Educ. Times*, xxxiv. pp. 55–56.]

[Question 6815. *Educ. Times*, xxxiv. p. 218.]

Hammond's theorem, properly substituted for a narrow one of Sylvester's, is in the case of the fourth order to the effect that *if* Δ_r *be the determinant formed from* $|a_1b_2c_3d_4|$ *by deleting the* r^{th} *column*

$$a_r,\ b_r,\ c_r,\ d_r$$

and substituting for it

$$\lambda_1 d_r,\ \lambda_2 a_r,\ \lambda_3 b_r,\ \lambda_4 c_r,$$

then

$$\Delta_1+\Delta_2+\Delta_3+\Delta_4 = 0.$$

No proof is given; but we may note that the cofactors of the λ's are easily shown to vanish, for example,

$$\text{cof}\,\lambda_1 = a_4A_1+b_4B_1+c_4C_1+d_4D_1 = 0.$$

LINDELÖF, L. (1880).

[Bidrag till läran om determinanter. *Öfversigt af Finska Vet. Soc. Förh.*, xxii. pp. 123–154.]

This long and carefully written paper is divided into two sections, of which the first is somewhat the longer. From our point of view the first is also considerably the more important, as it deals with properties of general determinants, whereas the second consists of easy deductions from it regarding axisymmetric determinants and of applications to binary quadrics. The ultimate subject is the solution of linear equations: but the two theorems dealing therewith are naturally and effectively led up to by a series of nine theorems on the vanishing of square and oblong arrays. Among them all the really fresh result is a generalization of the old theorem regarding the resolvability of the determinant obtainable by bordering a null determinant (*Hist.*, ii. p. 153; iii. pp. 28–29, 453). Put shortly this generalization is that *if an* n*-line determinant of nullity* p *have affixed to it a* p*-line border, the product of the resulting determinant by any* (n–p)*-line minor of the original is expressible as the product of two* n*-line determinants.* Thus, when n is 4 and p is 3, we have at once

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & x_1 & y_1 & z_1 \\ b_1 & b_2 & b_3 & b_4 & x_2 & y_2 & z_2 \\ c_1 & c_2 & c_3 & c_4 & x_3 & y_3 & z_3 \\ d_1 & d_2 & d_3 & d_4 & x_4 & y_4 & z_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \alpha_1 & \alpha_2 & \alpha_3 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & \beta_1 & \beta_2 & \beta_3 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & x_1 & y_1 & z_1 \\ b_1 & b_2 & b_3 & b_4 & x_2 & y_2 & z_2 \\ c_1 & c_2 & c_3 & c_4 & x_3 & y_3 & z_3 \\ d_1 & d_2 & d_3 & d_4 & x_4 & y_4 & z_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & . & . & . \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 & . & . & . \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 & . & . & . \end{vmatrix}$$

because of the vanishing of $|\, a_1 b_2 c_3 d_4 \,|$ and its primary and secondary minors. In the next place the latter is, by Laplace's expansion theorem,

$$\begin{aligned} = & -|\, \xi_2\eta_3\xi_4 \,| \cdot \{a_1 |\, x_2y_3z_4 \,| - b_1 |\, x_1y_3z_4 \,| + \ldots.\} \\ & +|\, \xi_1\eta_3\xi_4 \,| \cdot \{a_2 |\, x_2y_3z_4 \,| - b_2 |\, x_1y_3z_4 \,| + \ldots.\} \\ & -|\, \xi_1\eta_2\xi_4 \,| \cdot \{a_3 |\, x_2y_3z_4 \,| - b_3 |\, x_1y_3z_4 \,| + \ldots.\} \\ & +|\, \xi_1\eta_2\xi_3 \,| \cdot \{a_4 |\, x_2y_3z_4 \,| - b_4 |\, x_1y_3z_4 \,| + \ldots.\}; \end{aligned}$$

and since, by reason again of p being 3, we have

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = \ldots., \quad \frac{a_3}{a_1} = \frac{b_3}{b_1} = \ldots., \quad \frac{a_4}{a_1} = \frac{b_4}{b_1} = \ldots.,$$

it follows that this expansion must contain the factor

$$a_1 |\, x_2y_3z_4 \,| - b_1 |\, x_1y_3z_4 \,| + \ldots.$$

with the cofactor

$$-|\, \xi_2\eta_3\xi_4 \,| + \frac{a_2}{a_1} |\, \xi_1\eta_3\xi_4 \,| - \frac{a_3}{a_1} |\, \xi_1\eta_2\xi_4 \,| + \frac{a_4}{a_1} |\, \xi_1\eta_2\xi_3 \,|.$$

Thus our given bordered determinant multiplied by a_1 equals

$$\{a_1 |\, x_2y_3z_4 \,| - b_1 |\, x_1y_3z_4 \,| + \ldots\} \{a_1 |\, \xi_2\eta_3\xi_4 \,| - a_2 |\, \xi_1\eta_3\xi_4 \,| + \ldots.\},$$

i.e.

$$\begin{vmatrix} a_1 & x_1 & y_1 & z_1 \\ b_1 & x_2 & y_2 & z_2 \\ c_1 & x_3 & y_3 & z_3 \\ d_1 & x_4 & y_4 & z_4 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{vmatrix},$$

as was to be proved.

We have only to remark for ourselves that the form of the result suggests an entirely different kind of proof, namely, a deduction from Muir's theorem of 1879 regarding the product of a determinant by one of its own minors (*Hist.*, iii. pp. 79–81).

PAIGE, C. LE (1880).

[Sur quelques points de la théorie des formes algébriques. *Mém. . . . Soc. . . . des Sci.* (Liège), (2) ix. no. 4, 23 pp.]

[Sur quelques propriétés des déterminants. *Nouv. Corresp. Math.*, vi. pp. 490–496.]

Incidental to and consequent on his main subject, Le Paige deduces the theorem that if A, Y, Z *be the determinants got by leaving out from an* n*-by-*(n+1) *array the* 1st, *the* nth, *and the* (n+1)th *columns respectively, and if* Δ_r *be the determinant whose* rth *row agrees with the* rth *row of* A *and whose other rows agree with those of* Z, *then*

$$\Sigma\Delta_r = Y.$$

For example, the array being

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4, \end{matrix}$$

we have

$$\begin{vmatrix} a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_2 & c_3 & c_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}.$$

Some deductions are also made (pp. 19–23) depending on the vanishing of a determinant and its coaxial minors down to that of the 2nd order. A proof of the multiplication-theorem which he gives we refer to later.

LANDRÉ, C. L. (1880).

[Een stelling omtrent determinanten. *Nieuw Archief v. Wisk.*, vi. pp. 208–211.]

In correcting an error in Dostor's *Éléments* (p. 62) Landré is led to point out, as Dostor himself had by this time done, that *if from any given* n*-line determinant* A *another* B *be formed such that each row of* B *is the corresponding row of* A *diminished by all the other rows of* A, *then* $B = 2^{n-1}(2-n)A$; the reason being that B is the product of A by the circulant

$$C(1, -1, -1, -1, \ldots).$$

TEIXEIRA, F. G. (1880).

[Note sur la dérivation des déterminants. *Proceed. London Math. Soc.*, xii. pp. 14, 212.]

Inquiry here is made as to the result of repeatedly applying the ordinary rule for expressing the derivate of a determinant as a sum of determinants. The simplest statement of it given is that *if the determinant* Δ *be of the* n^{th} *order, its* r^{th} *derivate is*

$$(d_1+d_2+\ldots+d_n)^r\Delta,$$

where d_h *denotes differentiation of the* h^{th} *column.* This and a variant of it by Cayley are given in the absence of the actual wording of the paper itself.

WEYR, ED. (1880): PAIGE, C. LE (1881): TEIXEIRA, F. G. (1881).

[Verification der Multiplicationsformel für Determinanten. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), 1880, pp. 55–56.]

[Sur la règle de multiplication des déterminants. *Bull. Soc. Math. de France*, ix. pp. 67–69.]

[Sobre a multiplição dos determinantes. *Jorn. de Sci. math. e astron.*, iii. pp. 185–186.]

The verification referred to turns on the partitioning of the product-determinant.

In effect Le Paige proves that *If* A, B *be two* n*-line determinants*, C *the determinant formed from* A, B *by row-by-row multiplication*, α *the first of the primary minors of* A, β *the first of the primary minors of* B, *and* γ *the determinant formed from* α, β *as* C *from* A, B, *then* C *can be transformed into*

$$\frac{AB}{\alpha\beta}\cdot\gamma:$$

thus showing that if the multiplication-theorem hold for the $(n-1)^{th}$ order,—that is to say, if $\gamma=\alpha\beta$,—it must also hold for the n^{th}.

Teixeira's note I have not seen: it is reported to have no really fresh point of interest.

IGEL, B. (1880).

[Zur Theorie der Determinanten. *Sitzungsb. d. k. Acad. d. Wiss.* (Wien), lxxxii. pp. 1288–1294.]

Without reference to Jacobi (*Hist.*, i. p. 263), Igel obtains such a set of equations as

$$\begin{array}{rrrrrrrrrrrrrrrrrrrr}
. & +a_2|b_3c_4d_5| & -a_3|b_2c_4d_5| & +a_4|b_2c_3d_5| & -a_5|b_2c_3d_4| & = & |a_2b_3c_4d_5| \\
-a_1|b_3c_4d_5| & . & +a_3|b_1c_4d_5| & -a_4|b_1c_3d_5| & +a_5|b_1c_3d_4| & = & -|a_1b_3c_4d_5| \\
a_1|b_2c_4d_5| & -a_2|b_1c_4d_5| & . & +a_4|b_1c_2d_5| & -a_5|b_1c_2d_4| & = & |a_1b_2c_4d_5| \\
-a_1|b_2c_3d_5| & +a_2|b_1c_3d_5| & -a_3|b_1c_2d_5| & . & +a_5|b_1c_2d_3| & = & -|a_1b_2c_3d_5| \\
a_1|b_2c_3d_4| & -a_2|b_1c_3d_4| & +a_3|b_1c_2d_4| & -a_4|b_1c_2d_3| & . & = & |a_1b_2c_3d_4|
\end{array}$$

and notes that the determinant of the set is zero-axial skew. He then proceeds on his own account to affirm that as a_1, a_2, a_3, a_4, a_5 are four-fold indeterminate, the determinant of their coefficients in the set of equations must vanish as well as all its minors of the fourth and third orders. In particular, by restricting himself to the coaxial minors, he deduces a proposition regarding vanishing Pfaffians, namely, that *if* $\alpha\beta$ *denote a minor got from an* (n−2)-by-n *array by leaving out the* α^{th} *and* β^{th} *columns, then*

$$\begin{vmatrix} \alpha\beta & \alpha\gamma & \alpha\delta \\ & \beta\gamma & \beta\delta \\ & & \gamma\delta \end{vmatrix} = 0 \quad \text{or} \quad [\alpha\beta\gamma\delta] = 0,$$

$$[\alpha\beta\gamma\delta\epsilon\xi] = 0,$$

and so on. He does not note, however, that the first of these is already well known, being Desnanot's extensional of 1819 (*Hist.*, i. p. 140).

HENRY, C. (1881).

[Sur le calcul des dérangements. *Nouv. Annales de Math.*, (2) xx. pp. 5–9.]

The subject here is a problem taken from Bertrand's *Traité d'Algèbre* (1850), p. 146, the result established being that *the total number of inverted pairs in the permutations of* 1, 2, 3, . . . , n *is* $n! \frac{1}{4}n(n-1)$.

MUIR, T. (1881).

[The law of extensible minors in determinants. *Transac. R. Soc. Edinburgh*, xxx. pp. 1–4.]

Here there is first the enunciation of a rediscovery,—the Law of Complementaries—reference being made to the previous mention of it by Cayley in speaking of Tanner's paper of 1878. The wording is: *To every general theorem which takes the form of an identical relation between a number of the minors of a determinant or between the determinant itself and a number of its minors there corresponds another theorem derivable from the former by merely substituting for every minor its cofactor in the determinant, and then multiplying any term by such a power of the determinant as will make the terms of the same degree.* As a simple example of its application the well-known identity employed by Hermite

$$\begin{vmatrix} |a_1b_2| & |a_2b_3| & |a_3b_4| \\ |a_1c_2| & |a_2c_3| & |a_3c_4| \\ |a_1d_2| & |a_2d_3| & |a_3d_4| \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} a_2a_3$$

is taken, and the identity

$$\begin{vmatrix} |c_3d_4| & |c_1d_4| & |c_1d_2| \\ |b_3d_4| & |b_1d_4| & |b_1d_2| \\ |b_3c_4| & |b_1c_4| & |b_1c_2| \end{vmatrix} = |b_1c_3d_4| \cdot |b_1c_2d_4|$$

at once derived therefrom. Without a knowledge of the law, these identities, it is pointed out, had to be discovered separately.

The Law of Extensible Minors is: *If any identical relation be established between a number of the minors of a determinant or between the determinant itself and a number of its minors, the determinants being denoted by means of their principal diagonals, then a new theorem is always obtainable by merely choosing a line of new letters with new suffixes and annexing it to the end of the diagonal of every determinant, including those of order* 0, *occurring in the identity.* By way of proof it is pointed out that, the established identity concerning $|a_1b_2c_3 \ldots l_n|$

being (A), we have only to take the Complementary identity (B) and then the Complementary of (B) with respect to

$$|\, a_1 b_2 c_3 \ldots l_n r_\alpha s_\beta \ \ldots z_\omega \,|.$$

Thus, the least possible extension being used, the simple identity

$$|\, a_1 b_2 c_3 \,| \ = \ a_1 \,|\, b_2 c_3 \,| - a_2 \,|\, b_1 c_3 \,| + a_3 \,|\, b_1 c_2 \,|$$

gives rise to

$$d_4 \,|\, a_1 b_2 c_3 d_4 \,| \ = \ |\, a_1 d_4 \,|\, |\, b_2 c_3 d_4 \,| - |\, a_2 d_4 \,|\, |\, b_1 c_3 d_4 \,| + |\, a_3 d_4 \,|\, |\, b_1 c_2 d_4 \,|,$$

and the first of the two identities above gives rise to

$$\begin{vmatrix} |\, a_1 b_2 e_5 \,| & |\, a_2 b_3 e_5 \,| & |\, a_3 b_4 e_5 \,| \\ |\, a_1 c_2 e_5 \,| & |\, a_2 c_3 e_5 \,| & |\, a_3 c_4 e_5 \,| \\ |\, a_1 d_2 e_5 \,| & |\, a_2 d_3 e_5 \,| & |\, a_3 d_4 e_5 \,| \end{vmatrix} = |\, a_1 b_2 c_3 d_4 e_5 \,| \ |a_2 e_5 \,| \ |\, a_3 e_5 \,|$$

and

$$e_5{}^3 \begin{vmatrix} |\, a_1 b_2 e_5 \,| & |\, a_2 b_3 e_5 \,| & |\, a_3 b_4 e_5 \,| \\ |\, a_1 c_2 e_5 \,| & |\, a_2 c_3 e_5 \,| & |\, a_3 c_4 e_5 \,| \\ |\, a_1 d_2 e_5 \,| & |\, a_2 d_3 e_5 \,| & |\, a_3 d_4 e_5 \,| \end{vmatrix} = \begin{vmatrix} |\, a_1 e_5 \,| & |\, a_2 e_5 \,| & |\, a_3 e_5 \,| & |\, a_4 e_5 \,| \\ |\, b_1 e_5 \,| & |\, b_2 e_5 \,| & |\, b_3 e_5 \,| & |\, b_4 e_5 \,| \\ |\, c_1 e_5 \,| & |\, c_2 e_5 \,| & |\, c_3 e_5 \,| & |\, c_4 e_5 \,| \\ |\, d_1 e_5 \,| & |\, d_2 e_5 \,| & |\, d_3 e_5 \,| & |\, d_4 e_5 \,| \end{vmatrix} |a_2 e_5 \,|\, |a_3 e_5 \,|.$$

Attention is also drawn to the case where a theorem is its own Complementary.

Instances of separately discovered theorems that were merely 'extensionals' began to appear early in the history of the subject and have been duly noted by us, the earliest being due to Monge (1809) and Desnanot (1819).

PERRY, H. M. (1881).

[A rule of signs in determinants. *Johns Hopkins Univ. Circ.*, i. p. 151.]

The rule is that when by transposition of rows and of columns any minor of a determinant is shifted to occupy the upper left-hand corner, the sign-factor to be prefixed to the new form is $(-1)^{s+s'}$, where s and s' are the sums of the original numbers of the rows and columns respectively.

KRONECKER, L. (1881).

[Zur Theorie der Elimination einer Variabeln aus zwei algebraischen Gleichungen. *Monatsb. . . . Akad. d. Wiss.* (Berlin), pp. 535–600: or *Werke*, ii. pp. 115–192. Abstract in *Bull. des Sci. Math.*, (2) x. (2) pp. 47–58.]

The main subject of this important memoir is the problem of finding two multipliers $A(x)$, $B(x)$ so related to two given integral functions $f(x)$, $g(x)$ of the degrees n, $n-n_1$ respectively that

$$(Ag - Bf) \cdot A$$

may be of a lower degree than the n^{th}. Included in the investigation, however, and forming an essential part of it (pp. 554–571) is a set of results belonging to the pure theory of determinants, some on determinants in general (pp. 554–558) and some on determinants that are persymmetric.

The first of the former is to the effect that *if an* n-*by*-(n+1) *array be such that all its* n–*line determinants vanish except one, then the column common to those that vanish must consist of zeros.* We have only got to denote the non-zero determinant by $|a_{1n}|$ and the extra column by

$$a_{10}, \quad a_{20}, \quad \ldots, \quad a_{n0}$$

in order to see that from the data we have the n equations

$$\left.\begin{aligned} A_{11} \cdot a_{10} + A_{21} \cdot a_{20} + \ldots + A_{n1} \cdot a_{n0} &= 0 \\ A_{12} \cdot a_{10} + A_{22} \cdot a_{20} + \ldots + A_{n2} \cdot a_{n0} &= 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \\ A_{1n} \cdot a_{10} + A_{2n} \cdot a_{20} + \ldots + A_{nn} \cdot a_{n0} &= 0 \end{aligned}\right\};$$

and, as the determinant of these is equal to $|a_{1n}|^{n-1}$, and therefore by the remaining datum does not vanish, it follows that

$$a_{10}, \quad a_{20}, \quad \ldots, \quad a_{n0}$$

are all equal to 0. A generalization is to the effect that *if in an* n-*by*-(n+1) *array* n–m *of the* n-*line minors vanish and one does not vanish, the array common to the said* n-m *minors is evanescent*: for example, if in

$$\begin{matrix} b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{matrix}$$

we have given

$$|b_1c_2d_3e_4| = 0 = |b_1c_2d_3e_5| \quad \text{and} \quad |b_1c_2d_4e_5| \gtrless 0,$$

then

$$\left\| \begin{matrix} b_1 & c_1 & d_1 & e_1 \\ b_2 & c_2 & d_2 & e_2 \\ b_3 & c_3 & d_3 & e_3 \end{matrix} \right\| = 0.$$

The proof is exactly similar to that of the simpler theorem (which is the case of this where m is 0), the basis of it being the obtaining of four equations having for quasi-unknowns

$$|c_1d_2e_3|, \quad |b_1d_2e_3|, \quad |b_1c_2e_3|, \quad |b_1c_2d_3|$$

—the minors, that is to say, which are to be proved equal to 0,—and for coefficients the elements of the given non-vanishing minor $|b_1c_2d_4e_5|$. Kronecker prefers, however, to enunciate this more general theorem differently, saying that *if* $|a_{0n}|$ *be a linear function of the first* $m+1$ *indeterminates*

$$a_{00}, \quad a_{01}, \quad \ldots, \quad a_{0m}$$

only, and the cofactor of a_{00} *in it be not equal to* 0, *then all the* $(m+1)$*-line determinants of the array*

$$\begin{matrix} a_{10} & a_{11} & \ldots & a_{1m} \\ a_{20} & a_{21} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n0} & a_{n1} & \ldots & a_{nm} \end{matrix}$$

must vanish.

The next theorem is at most only fresh in form, being that *the solution of the equations*

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = 0 \end{matrix}\right\},$$

in whose determinant $|a_{1m}|$ *is the non-vanishing minor of highest order, is equivalent to finding the values of the* x*'s which will satisfy the equality*

$$\left| \begin{matrix} a_{01} & a_{02} & \ldots & a_{0m} & a_{0m+1}x_{m+1} + a_{0m+2}x_{m+2} + \ldots + a_{0n}x_n \\ a_{11} & a_{12} & \ldots & a_{1m} & a_{1m+1}x_{m+1} + a_{1m+2}x_{m+2} + \ldots + a_{1n}x_n \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} & a_{m2} & \ldots & a_{mm} & a_{mm+1}x_{m+1} + a_{mm+2}x_{m+2} + \ldots + a_{mn}x_n \end{matrix} \right|$$
$$= (-1)^m \cdot |a_{1m}| \cdot (a_{01}x_1 + a_{02}x_2 + \ldots + a_{0n}x_n) \qquad \text{(T)}$$

on the understanding that $a_{01}, a_{02}, \ldots, a_{0n}$ *are indeterminates*—that is to say, *is equivalent to equating coefficients of* $a_{01}, a_{02}, \ldots, a_{0n}$ *in the said equality.* As a preparation for proof the last column of the left-hand member of (T) is increased by

$$x_m \mathrm{col}_m + x_{m-1}\mathrm{col}_{m-1} + \ldots + x_1 \mathrm{col}_1,$$

and then the two members equally diminished, the result being

$$\begin{vmatrix} a_{01} & a_{02} & \ldots & a_{0m} & \cdot \\ a_{11} & a_{12} & \ldots & a_{1m} & f_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \ldots & a_{mm} & f_m \end{vmatrix} = 0, \qquad (T')$$

where $f_1, f_2, \ldots$ stand for the left-hand members of the given set of equations. It is then clear that the first half of the proof is made good, namely, that the existence of (S) necessitates the existence of (T′): and that part of the converse half follows from the preceding theorem, namely, that the existence of (T′) necessitates the existence of the first m equations in (S). All that is then necessary is to show that the first m equations entail the existence of the others, and this has already been more than once done.*

The last theorem is to the effect that *if in the array*

$$\begin{array}{cccccccc} c_{00} & c_{01} & \ldots & c_{0,n-1} & f_{0,n} & f_{0,n+1} & \ldots \\ c_{10} & c_{11} & \ldots & c_{1,n-1} & f_{1,n} & f_{1,n+1} & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n0} & c_{n1} & \ldots & c_{n,n-1} & f_{n,n} & f_{n,n+1} & \ldots, \end{array}$$

where each column after the n^{th} *is an aggregate of multiples of the first* n *columns, there be at least one* m*-line minor of the* c*-array that does not vanish, while the minors of higher order all vanish, then each row of the whole array is a linear function of the rows from which the said non-vanishing minor is formed: further, if there be in the* c*-array an* m*-line minor that vanishes, its rows in their full extent are connected by a linear relation.*

The remaining theorems concern persymmetric determinants.

* It is not clear what grounds Kronecker had for preferring (T) to (T′). Further, so far as ordinary usefulness is concerned, all that is got from either of them by the process of equating coefficients of $a_{01}, \ldots, a_{0m}$ is obtained directly by solving the first m equations for $x_1, \ldots, x_m$.

LAQUIÈRE, E.: PELLET, E. (1881).

[Démonstration rationnelle des premiers principes des déterminants. *Assoc. française* . . . (Congrès d'Alger), pp. 226-230.]

[Sur les déterminants. *Assoc. française* . . . (Congrès d'Alger), p. 231.]

The first of these is a helpful introduction to the subject, keeping its origin well in view. Of the second the title only is given.

MUIR, T. (1881).

[On new and recently discovered properties of certain symmetric determinants. *Quart. Journ. of Math.*, xviii. pp. 166–177.]

The paper draws attention to quite a number of special forms of determinants that can be viewed as centrosymmetric and therefore resolvable into two determinant factors. Ultimately (§ 11) the general result

$$|\alpha_1\beta_2\gamma_3| \cdot |x_1y_2z_3| = \frac{1}{2^6} \begin{vmatrix} \alpha_1+x_1 & \alpha_2+x_2 & \alpha_3+x_3 & \alpha_3-x_3 & \alpha_2-x_2 & \alpha_1-x_1 \\ \beta_1+y_1 & \beta_2+y_2 & \beta_3+y_3 & \beta_3-y_3 & \beta_2-y_2 & \beta_1-y_1 \\ \gamma_1+z_1 & \gamma_2+z_2 & \gamma_3+z_3 & \gamma_3-z_3 & \gamma_2-z_2 & \gamma_1-z_1 \\ \gamma_1-z_1 & \gamma_2-z_2 & \gamma_3-z_3 & \gamma_3+z_3 & \gamma_2+z_2 & \gamma_1+z_1 \\ \beta_1-y_1 & \beta_2-y_2 & \beta_3-y_3 & \beta_3+y_3 & \beta_2+y_2 & \beta_1+y_1 \\ \alpha_1-x_1 & \alpha_2-x_2 & \alpha_3-x_3 & \alpha_3+x_3 & \alpha_2+x_2 & \alpha_1+x_1 \end{vmatrix}$$

is reached, and note taken that when the determinants to be multiplied are both axisymmetric the product determinant is doubly axisymmetric.

MUIR, T. (1881).

[A list of writings on determinants. *Quart. Journ. of Math.*, xviii. pp. 110–149.]

The compiler explains that the writings meant to be included are those which concern the *theory* of determinants and the *history* of the theory, and which have appeared before the end of 1880, writings dealing with *applications* being only inserted when found to contain incidentally matter touching on the theory. The list extends to 40 pages, and contains 589 titles of books, memoirs, etc., arranged in chronological order beginning with the year 1693.

DERUYTS, J. (1881).

[Note sur quelques propriétés des déterminants multiples. *Mém. . . . Soc. roy.* (Liège), (2) x. no. 7, 11 pp.]

Two purely determinantal proofs are here given of Le Paige's theorem of 1880, and a generalization is made in which the number of determinants constructed from the same array as before is $(n)_p$ instead of $(n)_1$.

KRONECKER, L. (1882).

[Die Subdeterminanten symmetrischer Systeme. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), pp. 821–824 : or *Bull. des Sci. Math.*, (2) xi. (2) pp. 102–107 : or *Werke*, ii. pp. 391–396.]

This beginning of an unfinished record is based on the conception of what are called ' reciprocal systems,' the term ' reciprocal ' being used in the sense * employed by Cayley in his memoir on Matrices (1857). Indeed, ' matrix ' in the less specialized sense, or ' square array,' might fitly be employed throughout in translating ' system ' in Kronecker's paper.

Only one fresh theorem regarding general determinants is enunciated, but it is important : unfortunately, no proof is given.† It is to the effect that *if two determinants be reciprocals, their* k^{th} *compounds are also reciprocals.* Having stated this, the author then recalls the fact that if each of the elements of the adjugate be divided by the original determinant, a determinant is obtained that

* ' Reciprocal ' in this sense may be applied with advantage to *determinants* also, now that the improper use of it for ' adjugate ' is becoming rare. Two determinants of the same order would then be styled *reciprocal* if their row-by-column product were

$$\begin{vmatrix} 1 & . & . & \dots \\ . & 1 & . & \dots \\ . & . & 1 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

† Probably the best mode of procedure is to use the extended multiplication-theorem. Thus, $|\,a_1b_2c_3d_4\,|$ and $|\,\alpha_1\beta_2\gamma_3\delta_4\,|$ being reciprocals, the equality of the two forms of

$$\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix}\right\| \cdot \left\|\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{matrix}\right\|$$

suffices to establish the reciprocity of the second compounds.

is the reciprocal of the said original: and using with this the new theorem, he concludes that *the complementary of any minor divided by the determinant of the complete system is equal to the corresponding minor of the reciprocal system.** He then adds, rather unfairly, that Jacobi's theorem thus enunciated includes Francke's 'apparent' generalization of 1862.

The remainder of the paper concerns axisymmetric systems.

SYLVESTER, J. J. (1882).

[Question 7121. *Educ. Times*, xxxv. p. 235.]

Sylvester's proposition, when stated without using any new terminology, is that *if each* r-*line minor of any* n-*line determinant* A *be multiplied by the conjugate of the corresponding minor of any other* n-*line determinant* B, *the sum of the products so obtained* (which is manifestly not altered when A and B change places) "*depends only on the value of* AB *or* BA." No proof is given, and as a matter of fact the sum in question can be shown to be *equal to the sum of the* r-*line coaxial minors of* AB *or* BA.

EDDY, H. T.; VAN VELZER, C. A.; MUIR, T. (1882).

[Query. *Analyst*, ix. p. 64. Solutions, etc., pp. 94, 116–118, 127, 184–185.]

[Note on the transformation of a determinant into any other equivalent determinant. *Analyst*, x. pp. 8–9.]

An inquiry regarding the equality

$$\begin{vmatrix} . & b & c \\ b & 1 & a \\ c & a & 1 \end{vmatrix} = \begin{vmatrix} 2abc & b & c \\ b & 1 & . \\ c & . & 1 \end{vmatrix}$$

* The following variant would probably be oftener useful: *If two* n-*line determinants be such that their product equals*

$$\begin{vmatrix} s & . & . & . & \dots \\ . & s & . & . & \dots \\ . & . & s & . & \dots \\ . & . & . & s & \dots \\ . & . & . & . & \dots \end{vmatrix}_n,$$

then any m-*line minor of either is equal to* s^m *multiplied by the cofactor of the corresponding minor of the other and divided by that other.*

leads Van Velzer to raise wider issues, and to show (1) that *any determinant may in general be transformed into any other equivalent determinant of the same order,* and (2) that *any vanishing determinant may be transformed into one having two rows identical and at the same time two columns identical.*

In supporting Van Velzer, Muir supplies a basis for both propositions by proving that *the first three elements of any column, the last say, of* $|\,a_1b_2c_3d_4\,|$ *may be replaced by any three magnitudes whatever,* α, β, γ, *provided the fourth element be changed into*

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4-\alpha \\ b_1 & b_2 & b_3 & b_4-\beta \\ c_1 & c_2 & c_3 & c_4-\gamma \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} \div |\,a_1b_2c_3\,|.$$

At the same time, as a contrast to the simplicity of the originating example, he gives from the theory of circulants * the equality

$$\begin{vmatrix} 1 & a_5+b_2 & a_4+b_3 & a_3+b_4 & a_2+b_5 \\ 1 & a_1+b_3 & a_5+b_4 & a_4+b_5 & a_3+b_1 \\ 1 & a_2+b_4 & a_1+b_5 & a_5+b_1 & a_4+b_2 \\ 1 & a_3+b_5 & a_2+b_1 & a_1+b_2 & a_5+b_3 \\ 1 & a_4+b_1 & a_3+b_2 & a_2+b_3 & a_1+b_4 \end{vmatrix} = \begin{vmatrix} 1 & a_5-b_2 & a_4-b_3 & a_3-b_4 & a_2-b_5 \\ 1 & a_1-b_3 & a_5-b_4 & a_4-b_5 & a_3-b_1 \\ 1 & a_2-b_4 & a_1-b_5 & a_5-b_1 & a_4-b_2 \\ 1 & a_3-b_5 & a_2-b_1 & a_1-b_2 & a_5-b_3 \\ 1 & a_4-b_1 & a_3-b_2 & a_2-b_3 & a_1-b_4 \end{vmatrix}$$

DERUYTS, J. (1882).

[Sur certaines sommes de déterminants. *Mém. . . . Soc. . . . de Liège,* (2) x. no. 4, 11 pp.]

The author's main object is, as in the preceding year, to generalize Le Paige's theorem of 1880. The lemma, however, which he now uses for this purpose is more important than the said theorem or its generalization. It is that A *and* B *being* n*-line determinants, the sum of the* $(n)_p$ *determinants each consisting of* n − p *rows from* A *and* p *rows from* B *is equal to the sum obtained in performing the same operation with the columns.* The enunciation, we may remark, would have been improved by adding that the rows and columns are to occupy the same places in the new determinants as in the originals. By way of proof it is pointed out that the coefficient of $x^{n-p}y^p$ in

* See below under year 1881 in Chap. XVI.

$$\begin{vmatrix} a_{11}x + b_{11}y & a_{12}x + b_{12}y & \dots & a_{1n}x + b_{1n}y \\ a_{21}x + b_{21}y & a_{22}x + b_{22}y & \dots & a_{2n}x + b_{2n}y \\ \dots & \dots & \dots & \dots \\ a_{n1}x + b_{n1}y & a_{n2}x + b_{n2}y & \dots & a_{nn}x + b_{nn}y \end{vmatrix}$$

is the same as in the conjugate determinant.

The deduction which includes Le Paige's theorem is of course at once given, but need not be repeated here in view of a further generalization to which considerable space is devoted (pp. 4–8). Unfortunately, as before, the enunciation of this wider generalization is not so satisfactory as could be wished. It is that "le signe sommatoire se rapportant à tous les arrangemens p à p des indices 1, 2, 3, . . . , n, analogues à g, h, . . . , t,

$$\sum \begin{vmatrix} a_{1,1} & a_{1.2} & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots \\ a_{g,k_1} & a_{g,k_1+1} & \dots & a_{g,k_1-2} \\ \dots & \dots & \dots & \dots \\ a_{h,k_2} & a_{h,k_2+1} & \dots & a_{h,k_2-2} \\ \dots & \dots & \dots & \dots \\ a_{t,k_p} & a_{t,k_p+1} & \dots & a_{t,k_p-2} \\ \dots & \dots & \dots & \dots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{vmatrix} = (-1)^{\Sigma\kappa} \cdot p!\, \mathrm{D}_{n+1+p-\Sigma\kappa},"$$

where D_r denotes the determinant got by deleting the r^{th} column of the array

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1,n+1} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} & a_{n,n+1}. \end{matrix}$$

The mode of proof is gradational, the equality being known to hold when p is 1, and the case for any other value of p being made to depend on that for a lower value.

The remainder of the paper is occupied with other instances of a sum of determinants being equal to a multiple of one determinant.

We note for ourselves that when in the above quoted lemma A and B are the first and last n-line determinants of an n-by-$(n+1)$ array, the lemma degenerates into Le Paige's theorem of 1880 : and that another interesting case of the same lemma arises when B is the conjugate of A.

SYLVESTER, J. J. (1882, 1883).

[On the properties of a split matrix. *Johns Hopkins Univ. Circ.*, i. pp. 210–211 : or *Collected Math. Papers*, iii. pp. 645–646.]

[Question 7480. *Educ. Times*, xxxvi. p. 293.]

The fundamental property given here is a rediscovery, having been enunciated in its first form by Brill in 1871. As viewed by Sylvester it may be put thus : *If an* n*-line determinant be such that the product of each of its first* r *rows by each of the last* n-r *rows be* 0, *then the ratio which any* r*-line minor belonging to the first* r *rows bears to its complementary minor is constant.* Another property is that *if the sum of the squares of all the said* r*-line minors be multiplied by the sum of the squares of all the complementaries, the product is equal to the square of the determinant.* This is easily established by multiplying the determinant by itself.

There is also given a theorem of a different type, namely, *the nullity of the product of any number of* n*-line determinants cannot be less than the nullity of any one of them nor greater than the sum of the nullities of all*, it being explained that a determinant which vanishes but whose primary minors do not all vanish is a ' simple ' null or ' of nullity 1,' that a determinant whose primary minors vanish but whose secondary minors do not all vanish is ' of nullity 2,' and so on.

In regard to this idea of *degrees of nullity* note must be taken that doubtless the originating train of thought is the same as that which suggested Rouché's ' critical minor ' of 1875 and Frobenius' ' rank ' of 1877. In fact if the order of the determinant be n and the nullity be k, the rank is evidently $n-k$: the one number being an index of the degree of nullity, the other is an index of the degree of non-nullity, and the sum of the two naturally n.

BONOLIS, A. (1882).

[Di un nuovo e semplice modo di sviluppare i determinanti di grado qualunque, e sua applicazione all ricerca della resultante di due equazioni qualsivogliano. *Giornale di Mat.*, xxi. pp. 336–342.]

The method referred to consists in forming every term from the elements in a diagonal or in two mutually parallel minor diagonals,

being an extension of that used by Muir the previous year for three-line determinants. If the determinant be n-line, the number of terms thus to be got at once is $2n$, so that what Bonolis seeks to give is $n!/2n$ forms of the determinant, each furnishing a different set of $2n$ terms, and this he obtains by transposition of columns. For example, the final expansion of $|\ a_1b_2c_3d_4\ |$ may be got from

$$|\ a_1b_2c_3d_4\ |,\quad -|\ a_1b_2c_4d_3\ |,\quad |\ a_1b_4c_2d_3\ |,$$

the order being

$$\begin{array}{llll} 1234 - 2341 + 3412 - 4123 & & + 4321 - 3214 + 2143 - 1432 \\ - 1243 + 2431 - 4312 + 3124 & & - 3421 + 4213 - 2134 + 1342 \\ + 1423 - 4231 + 2314 - 3142 & & + 3241 - 2413 + 4132 - 1324 \end{array}$$

where, for shortness' sake, the letters a, b, c, d are omitted from each term. The method is considered to be specially useful in the case of bigradients, and to these the second half of the paper is devoted.

When the determinants are taken in the form $|\ a_1b_2c_3\ldots|$ or $|\ a_{11}a_{22}a_{33}\ldots|$ the method is essentially the same as that considered by Cayley in 1871 (*Hist.*, iii. p. 31), namely, where the terms are got by $(n-1)!$ cyclical permutations. Probably it is best viewed in this latter light, as the points requiring demonstration are less likely to be overlooked.

DAVIS, E. W. (1882).

[The maximum value of a certain determinant. *Johns Hopkins Univ. Circulars*, ii. no. 20, p. 22 ; or *Bull. American Math. Soc.*, (2) xiv. pp. 17–18.]

The result here stated is that *if the elements of an* n*-line determinant* ($n > 2$) *be restricted to variation between the limits* $-a$ *and* $+a$, *a numerical maximum is* $C(-a, a, a, \ldots, a)_n$. The value of the circulant, we may note, is known from Sylvester (*Hist.*, ii. pp. 406–407) to be $(-1)^{n-1}(n-2)\,2^{n-1}a^n$.

ANTONELLI, G. B. (1883).

[Nota sulle relazioni indipendenti tra le coordinate di una forma fondamentale in uno spazio di qualsivoglia dimensioni, e .... *Annali di Scuola Normale Sup.* (Pisa), iii. pp. 71–77.]

What calls for notice here is the old subject of *vanishing aggregates*

of products of pairs of determinants. The case mainly dealt with is in the umbral notation

$$\sum_{t=1}^{t=n+1}(-1)^t \begin{vmatrix} 1 & 2 & \dots & n \\ h_1 & h_2 & \dots & h_{n+1} \end{vmatrix} \begin{vmatrix} 1 & 2 & 3 & \dots & m \\ h & k_2 & k_3 & \dots & k_p \end{vmatrix} = 0.$$

It dates from 1825, being found substantially in Schweins (*Hist.*, i. p. 171; xxiii. 8, 1). When $m = n$ it is identical with Sylvester's of 1839, and when $m < n$ either may be viewed as including the other.*

KAPTEYN, W. (1883).

[Over een paar stellingen uit de leer der determinanten. *Nieuw Archief v. Wisk.*, x. pp. 180–185.]

A careful statement and verification of the expansions of Laplace and Cauchy (*Hist.*, i. pp. 104–105) with a wealth of differentiation-symbols.

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* Thus, having proved, say in Cayley's manner (*Hist.*, iii. p. 65), that

$$|a_1b_2c_3|\,|a_4b_5c_6| - |a_1b_2c_4|\,|a_3b_5c_6| + |a_1b_2c_5|\,|a_3b_4c_6| - |a_1b_2c_6|\,|a_3b_4c_5| = 0,$$

we can put $a_1, b_1, c_1 = 1, 0, 0$ and obtain

$$|b_2c_3|\,|a_4b_5c_6| - |b_2c_4|\,|a_3b_5c_6| + |b_2c_5|\,|a_3b_4c_6| - |b_2c_6|\,|a_3b_4c_5| = 0;$$

or, having proceeded again in Cayley's way, thus,

$$0 = \begin{vmatrix} a_1 & a_2 & . & . & . & . & . \\ b_1 & b_2 & . & . & . & . & . \\ c_1 & c_2 & . & . & . & . & . \\ . & . & a_3 & a_4 & a_5 & a_6 & a_7 \\ . & . & b_3 & b_4 & b_5 & b_6 & b_7 \\ . & . & c_3 & c_4 & c_5 & c_6 & c_7 \\ . & . & d_3 & d_4 & d_5 & d_6 & d_7 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ . & . & a_3 & a_4 & a_5 & a_6 & a_7 \\ . & . & b_3 & b_4 & b_5 & b_6 & b_7 \\ . & . & c_3 & c_4 & c_5 & c_6 & c_7 \\ . & . & d_3 & d_4 & d_5 & d_6 & d_7 \end{vmatrix}$$

$$= |a_1b_2c_3|\,|a_4b_5c_6d_7| - |a_1b_2c_4|\,|a_3b_5c_6d_7| + \dots + |a_1b_2c_7|\,|a_3b_4c_5d_6|,$$

we can put $a_7, b_7, c_7, d_7 = 0, 0, 0, 1$ and obtain the equality with which we started.

As we have pointed out elsewhere (Textbook, § 87), however, the theorem here formulated by Antonelli does not exhaust the case where the orders of the determinants are different. Thus, the equality

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ . & . & a_3 & a_4 & a_5 \\ . & . & b_3 & b_4 & b_5 \end{vmatrix} = \begin{vmatrix} a^1 & a^2 & & . & . \\ b_1 & b_2 & . & . & . \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ . & . & a_3 & a_4 & a_5 \\ . & . & b_3 & b_4 & b_5 \end{vmatrix}$$

gives us

$$|a_1b_2c_3|\,|a_4b_5| - |a_1b_2c_4|\,|a_3b_5| + |a_1b_2c_5|\,|a_3b_4| - |a_3b_4c_5|\,|a_1b_2| = 0,$$

which is not included in the said theorem.

SYLVESTER, J. J. (1883).

[On the equation to the secular inequalities in the planetary theory. *Philos. Magazine*, (5) xvi. pp. 267–269 : or *Collected Math. Papers*, iv. pp. 110–111.]

Notwithstanding the title here used Sylvester is now dealing with a much more general equation, namely, that which has for its non-zero member any determinant whatever with x subtracted from each of its diagonal elements. The roots of such an equation, called in the preceding year *lambdaic* roots,* he now calls the *latent roots* of the determinant—or matrix, rather—and the non-zero member the *latent function* ; and with this preparation formulates the theorem of the paper, namely, *The sum of the* k-*ary products of the latent roots of the product of two determinants is equal to the sum of the products got by multiplying each* k-*line minor of the first by the corresponding minor of the second.* A proof is promised. At present, however, it is sufficient to point out in regard to the latent roots of

$$|a_1b_2c_3| \cdot |a_1\beta_2\gamma_3|,$$

that is to say, the roots of

$$\begin{vmatrix} \Sigma aa - x & \Sigma a\beta & \Sigma a\gamma \\ \Sigma ba & \Sigma b\beta - x & \Sigma b\gamma \\ \Sigma ca & \Sigma c\beta & \Sigma c\gamma - x \end{vmatrix} = 0,$$

that we manifestly have

$$x_1 + x_2 + x_3 = \Sigma aa + \Sigma b\beta + \Sigma c\gamma,$$

$$x_1x_2 + x_1x_3 + x_2x_3 = \begin{vmatrix} \Sigma aa & \Sigma a\beta \\ \Sigma ba & \Sigma b\beta \end{vmatrix} + \begin{vmatrix} \Sigma aa & \Sigma a\gamma \\ \Sigma ca & \Sigma c\gamma \end{vmatrix} + \begin{vmatrix} \Sigma b\beta & \Sigma b\gamma \\ \Sigma c\beta & \Sigma c\gamma \end{vmatrix},$$

$$x_1x_2x_3 = \begin{vmatrix} \Sigma aa & \Sigma a\beta & \Sigma a\gamma \\ \Sigma ba & \Sigma b\beta & \Sigma b\gamma \\ \Sigma ca & \Sigma c\beta & \Sigma c\gamma \end{vmatrix},$$

and thence see that the first of these expressions on the right is the sum of the products got by multiplying together each pair of corresponding elements ; that the second

* v. *Comptes Rendus . . . Acad. des Sci.* (Paris), xciv. pp. 55–59 ; 396–399 : or *Collected Math. Papers*, iii. pp. 562–567, ...

$$= \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix} \right\| \cdot \left\| \begin{matrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \end{matrix} \right\| + \ldots .$$

$$= |\, a_1 b_2 \,|\, |\, a_1 \beta_2 \,| + |\, a_1 b_3 \,|\, |\, a_1 \beta_3 \,| + |\, a_2 b_3 \,|\, |\, a_2 \beta_3 \,| + \ldots .;$$

and the third $|\, a_1 b_2 c_3 \,| \cdot |\, a_1 \beta_2 \gamma_3 \,|$. It must be noted, however, that Sylvester always speaks of *matrix* unless when compelled to say *determinant*; and, as the multiplication of matrices is row-by-column, the wording of the theorem is to the necessary extent different.

In giving his definition at the outset he announces that *the latent roots of any function of a matrix are respectively the same functions of the latent roots of the matrix itself*,—an interesting generalization of a special case which he had used thirty years before (*Hist.*, ii. p. 123).

It is strictly necessary, however, to draw attention to the fact that much of what Sylvester says regarding latent roots in these papers is novel only in form. As a preliminary we note that so long ago as 1840 the general determinantal equation had made its appearance in a paper on linear differential equations by Cauchy,* who styled it the "équation caractéristique"; and Frobenius in 1877, when dealing with bilinear forms,† took over Cauchy's term, calling the equation

$$\begin{vmatrix} a_1 - x & a_2 & a_3 & \ldots \\ b_1 & b_2 - x & b_3 & \ldots \\ c_1 & c_3 & c_3 - x & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0$$

the *characteristic equation*, and the left-hand member of it the *characteristic determinant*,‡ of the form

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & \ldots & \\ \hline a_1 & a_2 & a_3 & \ldots & \xi_1 \\ b_1 & b_2 & b_3 & \ldots & \xi_2 \\ c_1 & c_2 & c_3 & \ldots & \xi_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

* *Exercices d'analyse et de phys. math.*, i. p. 53: or *Œuvres complètes*, 2e sér. xi. p. 79.

† *Crelle's Journ.*, lxxxiv. p. 10.

‡ While still, be it remarked, calling $|\, a_1 b_2 c_3 \ldots \,|$ *the* determinant of the form.

If we bear this in mind and make allowance for difference in terminology a single proposition from Frobenius will establish our point. He says (p. 11): *If* $r_1, r_2, \ldots, r_n$ *be the roots of the characteristic equation of* A, *then* $f(r_1), f(r_2), \ldots, f(r_n)$ *are those of the characteristic equation of* f(A). The corresponding proposition of Sylvester is that which we have spoken of above as a generalization.

MUIR, T. (1883).

[On the development of determinants with polynomial elements. *Messenger of Math.*, xiii. pp. 135–138.]

Here attention is first recalled to Albeggiani's theorem of 1874, a restatement of it being made with a view to supplying a clearer conception of the nature of the development. The given determinant, $\Delta_{n,p}$, being of the n^{th} order with p-termed elements, and D_1 being the partial determinant whose elements are all the first terms of the polynomials, D_2 the determinant whose elements are all the second terms, and so on, then Albeggiani's theorem gives $\Delta_{n,p}$ in terms of $D_1, D_2, \ldots, D_p$ and their minors of every order. Thus the development of $\Delta_{3,4}$ consists of 4 terms which are three-line determinants, 108 terms involving two-line determinants, and 144 terms involving one-line determinants.

The following theorem is next stated and proved: *If* $D_{n,p}$ *be a determinant of the* n^{th} *order each of whose elements consists of* p *terms*, p *being* $> n$; *if* $\Sigma D_{n,p-1}$ *denote the sum of the* p *determinants formed from* $D_{n,p}$ *by omitting firstly all the first terms of the elements, secondly all the second terms, and so on*; *if* $\Sigma D_{n,p-2}$ *denote the sum of the* $\frac{1}{2}p(p-1)$ *determinants formed by omitting firstly all the first and all the second terms of the elements of* $D_{n,p}$, *secondly all the first and all the third terms, and so on, and if* $\Sigma D_{n,p-3}$, $\Sigma D_{n,p-4}, \ldots$ *bear similar interpretations*: *then*

$$D_{n,p} - \Sigma D_{n,p-1} + \Sigma D_{n,p-2} - \ldots = 0.$$

From this is readily derived a property of row-multiplication, namely, *If* $\pi_{n,p}$ *denote the product of* n p*-termed expressions*, p *being* $> n$; *if* $\Sigma\pi_{n,p-1}$ *denote the sum of the* p *products formed from* $\pi_{n,p}$ *by omitting firstly all the first terms of the expressions, secondly all the*

second terms, and so on; *if* $\Sigma\pi_{n,p-2}$, $\Sigma\pi_{n,p-3}$, . . . *bear similar interpretations*: *then*

$$\pi_{n,p} - \Sigma\pi_{n,p-1} + \Sigma\pi_{n,p-2} - \ldots = 0.$$

The illustrations are taken from $D_{2,4}$, and by specialization end with the expression of a sum of seven squares as a sum of eight squares.

WALECKI, . (1884): FALK, M. (1885).

[La multiplication des déterminants. *Journ. de Math. Spéc.*, iii. pp. 8–9.]

[Beweis der Multiplications-Theorems für die Determinanten. *Nachrichten . . . Ges. d. Wiss.* (Göttingen), pp. 181–183.]

The first proof is one depending again on the partitioning of the product-determinant, and the second is another of the gradational type. (See above, p. 5.)

MÉRAY, CH. (1884).

[Exposition nouvelle de la théorie des formes linéaires et des déterminants. *Journ.* (*de Liouville*) *de Math.*, (3) x. pp. 181–280.]

Taking an exaggerated view of the extent to which in teaching determinants the subject is separated from its origin and artificially developed this author sets out ' to recast the whole affair, putting each thing in its own place, linear forms on the first plane and determinants on the second.' The result is a long and carefully constructed memoir of 100 pages, in which the subject of determinants ultimately emerges in a sufficiently subordinate position. The first chapter (pp. 182–201) deals with ' Systems of linear forms in general,' and the second (pp. 201–222) with ' Simultaneous linear equations.' In the third (pp. 222–248), which bears the heading ' Determinants,' ten pages are required to prepare for the definition, from which it appears that a determinant is a particular case of an entity called a ' covanescent.' The fourth (pp. 249–268) is described as ' Developments on the composition of systems of linear forms ' ; and the fifth (pp. 269–280) as ' Relations between the major and minor determinants of a system of linear forms.' Unfortunately the memoir is not easy reading, the writer, oddly enough, having faults of style in

common with the extreme determinantalist whom he chides, and every subject touched on being more or less covered over with fresh technicalities. As an example we take a brief proposition (p. 250) connected with the multiplication-theorem, namely, "Si l'un des abaques inducteurs est vanescent par ses files inductorielles, l'abaque induit l'est aussi par les files de son sens d'aggrégation à l'autre inducteur." It also serves as an illustration of the fact that anything new on determinants which the memoir may contain is applicational.

MUIR, T. (1884).

[An overlooked discoverer in the theory of determinants. *Philos. Magazine* (5), xviii. pp. 416–427.]

The discoverer referred to is Schweins, whose *Producte mit Versetzungen* extending to 113 quarto pages had been overlooked, even in Germany, from its publication in 1825 until this paper appeared—a period of sixty years. The loss involved in this neglect will be understood on reading the pages devoted to Schweins in our first volume (*Hist.*, i. pp. 159–175, 311–322, 479–485). In the paper now under consideration Muir, after a short introduction regarding Schweins and his treatise, gives an account of the first and most important section (pp. 317–368).

Taking Schweins' fundamental theorem regarding products of pairs of determinants, and writing it in the umbral notation in the easily remembered form

$$\sum \pm \begin{vmatrix} a_1 & \dots & a_{n-q}\, b_1 & \dots & b_q \\ a_1 & \dots & \dots & \dots & a_n \end{vmatrix} \begin{vmatrix} b_{q+1} & \dots & \dots & b_m \\ \beta_1 & \dots & \dots & \beta_{m-q} \end{vmatrix}$$

$$= \sum \pm \begin{vmatrix} a_1 & \dots & a_{n-q} \\ a_1 & \dots & a_{n-q} \end{vmatrix} \begin{vmatrix} b_1 & \dots & b_q & b_{q+1} & \dots & b_m \\ a_{n-q+1} & \dots & a_n & \beta_1 & \dots & \beta_{m-q} \end{vmatrix}$$

he shows, by putting $m=q+n$ and $\beta_1, \dots, \beta_n=a_1, \dots, a_n$, that it includes Sylvester's theorem of 1851, namely,

$$\sum \pm \begin{vmatrix} a_1 & \dots & a_{n-q}\, b_1 & \dots & b_q \\ a_1 & \dots & \dots & \dots & a_n \end{vmatrix} \begin{vmatrix} b_{q+1} & \dots & b_{q+n} \\ a_1 & \dots & a_n \end{vmatrix} = 0,$$

where the meaning of the horizontal lines will be got from Schweins' mode of writing the first of these equalities (*Hist.*, i. p. 169) or from our verbal enunciation of the second (*Hist.*, iv. p. 81).

KRONECKER, L. (1884).

[Näherungsweise ganzzahlige Auflösung linearer Gleichungen. *Sitzungsb. . . . Akad. d. Wiss.*, (Berlin) Jahrg. 1884, pp. 1179–1193, 1271–1299 ; or *Werke*, iii. (1), pp. 47–109.]

In this (p. 1192, § 5) and an immediately preceding memoir Kronecker defines his use of ' Rang ' or ' Stufenzahl ' in connection with a rectangular array, without making any mention of Frobenius (1877).

BRUNEL, G. (1884).

[Note sur l'analyse indéterminée et la géometrie à n dimensions. *Mém. . . . Soc. des Sci. . . .* (Bordeaux), (3) ii. pp. 129–143.]

In the closing pages (pp. 137, . . .) the writer deals with the relations between the 3-line determinants of any 3-by-5 array, showing that not more than three of them are independent (see below, pp. 55-56). The determinants being 1, 2, . . . , 10, the three relations obtained are

$$\left.\begin{array}{l} 1\cdot 6 \;-2\cdot 5+3\cdot 4=0 \\ 1\cdot 9 \;-2\cdot 8+3\cdot 7=0 \\ 1\cdot 10-4\cdot 8+5\cdot 7=0 \end{array}\right\}.$$

In this connection it is desirable again to recall d'Ovidio's overlooked work of 1877 (*Hist.* iii. p. 68).

DERUYTS, J. (1884).

[Sur l'analyse combinatoire des déterminants. *Mém. . . . Soc. des Sci.* (Liège), (2) xi. 11 pp.]

Notwithstanding the difference in the title, the subject here is quite similar to that of the author's papers of 1881 and 1882. The determinants, however, that now go to form his aggregates are no longer constructed from the elements of an n-by-$(n+1)$ array, but, like Hammond's of 1881, from the elements of a single determinant. His fundamental theorem—changed in form for simplicity's sake—is that *if from* $|\ a_{1n}\ |$ *a new determinant be formed by altering the order of the elements in the first row, a second determinant by making the same alteration in the elements of the second row, and so on, the sum of the* n *determinants thus derived is* p *times the original, where* p *is the*

number of elements of a row that after the change of order occupy the same positions as before. Hammond's is merely the case of this where p is 0; on the other hand, however, by the introduction of coefficients with the elements in their changed order, Hammond's is in a way more general. Verification of the theorem is readily accomplished by expanding the n determinants in terms of the elements in the altered rows and their cofactors, and then making a different summation of the n^2 terms thus obtained.

By taking the n determinants of this fundamental theorem and applying the said theorem to each of them while holding one row exempt from further change, there are obtained $n(n-1)$ determinants whose sum can be readily specified as before. In this way the author obtains his second theorem, and then in like manner a third theorem. He also is fertile in examples and analogous results.

Strangely enough note is not taken by him of the further equalities resulting from the fact that to each theorem obtained as above by operating on rows there corresponds a theorem obtained by similarly operating on columns.

CAUCHY, A. L. (1885).

[Algebraische Analysis. Deutsch herausg. von C. Itzigsohn, xii+398 pp. Berlin.]

A German translation of the *Analyse Algébrique* of 1821 (*Hist.*, i. pp. 148–150.)

HUMBERT, E. (1885): HOPPE, R. (1885).

[Note sur le développement d'un déterminant. *Nouv. Annales de Math.*, (3) iv. pp. 289–294.]

[Ein Satz über Determinanten. *Archiv d. Math. u. Phys.*, (2) ii. pp. 106–107.]

The first of these is a simple exposition of Laplace's expansion-theorem: and the theorem referred to in the second is the extension of

$$|a_1b_2|\cdot|c_1d_2| - |a_1c_2|\cdot|b_1d_2| + |a_1d_2|\cdot|b_1c_2| = 0$$

made by Desnanot in 1819.

MUIR, T. (1885.)

[On bipartite functions. *Transac. R. Soc. Edinburgh*, xxxii. pp. 461–482.]

The functions here introduced and discussed have points of resemblance with determinants, differing mainly from them in being non-alternating. Besides this we have an interest in them from the fact that the one theory is of substantial value in the study of the other. The matrix representation of any such function consists of m arrays, of which the first and last are single rows of n elements each, and the others are arrays of n-by-n elements ; and the ordinary algebraical representation is the aggregate of the terms that contain one and only one element from each array, subject to the condition that the element to be taken from any one array must be in the same row or column with the element taken from the preceding array, and in the same column or row with the element taken from the following array : for example,

$$\begin{array}{cccc} a & b & c & d \\ \alpha & \beta & \gamma & \delta \end{array} \equiv a\alpha + b\beta + c\gamma + d\delta,$$

$$\begin{array}{ccc|c} x & y & z & \\ \hline a_1 & a_2 & a_3 & \xi \\ b_1 & b_2 & b_3 & \eta \\ c_1 & c_2 & c_3 & \zeta \end{array} \equiv \begin{array}{r} a_1x\xi + b_1x\eta + c_1x\zeta \\ + a_2y\xi + b_2y\eta + c_2y\zeta \\ + a_3z\xi + b_3z\eta + c_3z\zeta, \end{array}$$

$$\begin{array}{cc|cc} a & b & & \\ c & d & g & h \\ e & f & i & j \\ \hline & & k & l \end{array} \equiv \begin{array}{r} acgk + achl + aeik + aejl \\ + bdgk + bdhl + bfik + bfjl. \end{array}$$

After definitions, illustrations and elementary properties (§§ 1–11), the author gives a long series of results analogous to results in determinants, namely, expansions of the type of Laplace's (§§ 12–19), theorems regarding 'condensation' and 'compounds' (§§ 20–27), and theorems on the 'adjugate,' including the Law of Complementaries. A few miscellaneous matters are next dealt with, including the resolution of a bordered adjugate determinant (*Hist.*, ii. p. 118) ; and then the paper concludes with the 'applications.' Of these by far the most important for us consists in the fact that

the elements of the product of any number of determinants are for the first time made representable in a uniform way: for example,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \begin{vmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{vmatrix} \underline{a_1\ a_2\ a_3} & \underline{a_1\ a_2\ a_3} & \underline{a_1\ a_2\ a_3} \\ a_1\ \beta_1\ \gamma_1 & a_2\ \beta_2\ \gamma_2 & a_3\ \beta_3\ \gamma_3 \\ \underline{b_1\ b_2\ b_3} & \underline{b_1\ b_2\ b_3} & \underline{b_1\ b_2\ b_3} \\ a_1\ \beta_1\ \gamma_1 & a_2\ \beta_2\ \gamma_2 & a_3\ \beta_3\ \gamma_3 \\ \underline{c_1\ c_2\ c_3} & \underline{c_1\ c_2\ c_3} & \underline{c_1\ c_2\ c_3} \\ a_1\ \beta_1\ \gamma_1 & a_2\ \beta_2\ \gamma_2 & a_3\ \beta_3\ \gamma_3 \end{vmatrix},$$

$$|\, a_1 b_2 c_3 \,| \cdot |\, a_1 \beta_2 \gamma_3 \,| \cdot |\, x_1 y_2 z_3 \,| = \begin{vmatrix} \underline{a_1\ a_2\ a_3} & & \underline{a_1\ a_2\ a_3} & & \underline{a_1\ a_2\ a_3} & \\ a_1\ \beta_1\ \gamma_1 & x_1 & a_1\ \beta_1\ \gamma_1 & x_2 & a_1\ \beta_1\ \gamma_1 & x_3 \\ a_2\ \beta_2\ \gamma_2 & y_1 & a_2\ \beta_2\ \gamma_2 & y_2 & a_2\ \beta_2\ \gamma_2 & y_3 \\ a_3\ \beta_3\ \gamma_3 & z_1 & a_3\ \beta_3\ \gamma_3 & z_2 & a_3\ \beta_3\ \gamma_3 & z_3 \\ \underline{b_1\ b_2\ b_3} & & \underline{b_1\ b_2\ b_3} & & \underline{b_1\ b_2\ b_3} & \\ a_1\ \beta_1\ \gamma_1 & x_1 & a_1\ \beta_1\ \gamma_1 & x_2 & a_1\ \beta_1\ \gamma_1 & x_3 \\ a_2\ \beta_2\ \gamma_2 & y_1 & a_2\ \beta_2\ \gamma_2 & y_2 & a_2\ \beta_2\ \gamma_2 & y_3 \\ a_3\ \beta_3\ \gamma_3 & z_1 & a_3\ \beta_3\ \gamma_3 & z_2 & a_3\ \beta_3\ \gamma_3 & z_3 \\ \underline{c_1\ c_2\ c_3} & & \underline{c_1\ c_2\ c_3} & & \underline{c_1\ c_2\ c_3} & \\ a_1\ \beta_1\ \gamma_1 & x_1 & a_1\ \beta_1\ \gamma_1 & x_2 & a_1\ \beta_1\ \gamma_1 & x_3 \\ a_2\ \beta_2\ \gamma_2 & y_1 & a_2\ \beta_2\ \gamma_2 & y_2 & a_2\ \beta_2\ \gamma_2 & y_3 \\ a_3\ \beta_3\ \gamma_3 & z_1 & a_3\ \beta_3\ \gamma_3 & z_2 & a_3\ \beta_3\ \gamma_3 & z_3 \end{vmatrix}$$

and the determinant which is the equivalent of

$$|\, a_1 b_2 c_3 \,| \cdot |\, a_1 \beta_2 \gamma_3 \,| \cdot |\, x_1 y_2 z_3 \,| \cdot |\, h_1 k_2 l_3 \,|$$

has for its first element

$$\begin{array}{ccc|ccc} \underline{a_1} & \underline{a_2} & \underline{a_3} & h_1 & k_1 & l_1 \\ a_1 & \beta_1 & \gamma_1 & x_1 & x_2 & x_3 \\ a_2 & \beta_2 & \gamma_2 & y_1 & y_2 & y_3 \\ a_3 & \beta_3 & \gamma_3 & z_1 & z_2 & z_3 \end{array}.$$

As a consequence the investigation of the laws of determinantal multiplication are much facilitated. This application, however, is not entered on, the author promising a separate communication on the subject.*

* This was not published, as a number of the results were found to be already known to Cayley, and indeed were afterwards come across in his writings on Matrices: for example, the validity of the Associative Law, and the effect of sub-

MUIR, T. (1885).

[New relations between bipartite functions and determinants, with a proof of Cayley's theorem in matrices. *Proceed. London Math. Soc.*, xvi. pp. 276–286.]

The first theorem is, for the case of the 4th degree,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \cdot \begin{vmatrix} a_r & b_r & c_r & d_r \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \cdot \Sigma a_1 \begin{matrix} \\ a_r \\ b_r \\ c_r \\ d_r \end{matrix}$$

$$+ \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_r & b_r & c_r & d_r \end{vmatrix} \cdot \Sigma \mid a_1 b_2 \mid - a_r \Sigma \mid a_1 b_2 c_3 \mid + \mid a_r b_2 c_3 d_4 \mid = 0,$$

where Σa_1 stands for $a_1+b_2+c_3+d_4$, $\Sigma \mid a_1 b_2 \mid$ for the sum of the two-line coaxial minors, and so forth. For higher degrees there is used a more compact notation, easily understood with the help of the case just written in full, and warranted by a property of the functions: for example,

$$\frac{a_1\ a_2\ \ldots\ a_6 \mid a_r\ b_r\ \ldots\ f_r}{\Delta^4}$$

$$- \frac{a_1\ a_2\ \ldots\ a_6 \mid a_r\ b_r\ \ldots\ f_r}{\Delta^3} \cdot \Sigma a_1$$

$$+ \frac{a_1\ a_2\ \ldots\ a_6 \mid a_r\ b_r\ \ldots\ f_r}{\Delta^2} \cdot \Sigma \mid a_1 b_2 \mid$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- a_r \cdot \Sigma \mid a_1 b_2 c_3 d_4 e_5 \mid$$

$$+ \mid a_1 b_2 c_3 d_4 e_5 f_6 \mid = 0.$$

---

stituting for a determinant the conjugate determinant. There, however, they are uniformly used without proof,—a matter of little moment perhaps, save in the case of the Associative Law. Of course his multiplication is row-by-column: if it be taken row-by-row the results are less shapely: for example, in the former case we have

$$ABC = A(BC), \quad ABCD = A(BC)D = AB \cdot CD = \ldots$$
$$(AB)' = B'A', \quad (ABC)' = C'B'A', \ \ldots,$$

and in the latter

$$ABC = AC'B$$
$$(AB)' = BA, \quad (ABC)' = C(AB) = CB'A, \ \ldots,$$

where A' is very conveniently used for *conjugate of* A.

The second theorem is, for the case of the 4th degree,

$$\begin{vmatrix} b_2-x & b_3 & b_4 \\ c_2 & c_3-x & c_4 \\ d_2 & d_3 & d_4-x \end{vmatrix} \div \begin{vmatrix} a_1-x & a_2 & a_3 & a_4 \\ b_1 & b_2-x & b_3 & b_4 \\ c_1 & c_2 & c_3-x & c_4 \\ d_1 & d_2 & d_3 & d_4-x \end{vmatrix}$$

$$= -x^{-1} - a_1x^{-2} - \frac{a_1\,a_2\,a_3\,a_4}{a_1\,b_1\,c_1\,d_1}x^{-3} - \frac{a_1\,a_2\,a_3\,a_4 \mid a_1\,b_1\,c_1\,d_1}{\Delta}x^{-4}$$

$$- \frac{a_1\,a_2\,a_3\,a_4 \mid a_1\,b_1\,c_1\,d_1}{\Delta^2}x^{-5} - \ldots,$$

the quotient of the two determinants being thus viewable as the generating function of expressions of the form

$$\frac{a_1\,a_2\,a_3\,a_4 \mid a_1\,b_1\,c_1\,d_1}{\Delta^m}.$$

The third theorem is

$$\begin{vmatrix} a_1-x & a_2 & a_3 & a_4 \\ b_1 & b_2-x & b_3 & b_4 \\ c_1 & c_2 & c_3-x & c_4 \\ d_1 & d_2 & d_3 & d_4-x \end{vmatrix} \div \begin{vmatrix} b_2-x & b_3 & b_4 \\ c_2 & c_3-x & c_4 \\ d_2 & d_3 & d_4-x \end{vmatrix}$$

$$= -x + a_1 + \frac{a_2\,a_3\,a_4}{b_1\,c_1\,d_1}x^{-1} + \frac{a_2\,a_3\,a_4 \mid b_1\,c_1\,d_1}{\mathrm{D}}x^{-2}$$

$$+ \frac{a_2\,a_3\,a_4 \mid b_1\,c_1\,d_1}{\mathrm{D}^2}x^{-3} + \ldots,$$

where D stands for the square array of $\mid b_2c_3d_4 \mid$. Advantage is of course taken of the fact that the expressions dealt with in the last two theorems are reciprocals, the resulting deduction being exemplified by

$$\begin{array}{ccc|ccc} a_1 & a_2 & a_3 & h_1 & k_1 & l_1 \\ b_1 & c_1 & d_1 & e_1 & e_2 & e_3 \\ b_2 & c_2 & d_2 & f_1 & f_2 & f_3 \\ b_3 & c_3 & d_3 & g_1 & g_2 & g_3 \end{array}$$

$$= a_1 \cdot \frac{\begin{matrix} h_1 & k_1 & l_1 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{matrix}}{\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}} + \frac{\begin{matrix} a_2 & a_3 \\ c_1 & d_1 \end{matrix}}{\begin{matrix} h_1 & k_1 & l_1 \\ e_1 & e_2 & e_3 \end{matrix}} + \frac{\begin{matrix} a_2 & a_3 \\ c_2 & d_2 \\ c_3 & d_3 \end{matrix}}{\begin{matrix} f_1 \\ g_1 \end{matrix}} \cdot h_1 + \frac{\begin{matrix} a_2 & a_3 \\ c_2 & d_2 \\ c_3 & d_3 \end{matrix}}{\begin{matrix} k_1 & l_1 \\ f_2 & f_3 \\ g_2 & g_3 \end{matrix}}.$$

The theorem of Cayley's to which the title refers is that concerning the matrix which is a latent root of a determinant. For the 4th order it is that *

$$\text{if } \mathrm{M} \equiv \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}, \quad \text{then} \quad \begin{vmatrix} a_1-\mathrm{M} & a_2 & a_3 & a_4 \\ b_1 & b_2-\mathrm{M} & b_3 & b_4 \\ c_1 & c_2 & c_3-\mathrm{M} & c_4 \\ d_1 & d_2 & d_3 & d_4-\mathrm{M} \end{vmatrix} = 0,$$

and as the expansion of the determinant here is

$$\mathrm{M}^4 - \mathrm{M}^3 \cdot \Sigma a_1 + \mathrm{M}^2 \cdot \Sigma |\, a_1 b_2 \,| - \mathrm{M} \cdot \Sigma |\, a_1 b_2 c_3 \,| + \mathrm{M}^0 \cdot |\, a_1 b_2 c_3 d_4 \,|,$$

the connection with the foregoing is sufficiently indicated.

HAMBURGER, M. (1885).

(See under this heading in Chap. VII.)

MERTENS, F. (1885, 1890).

[Ueber eine Formel der Determinantentheorie. *Sitzungsb. d. k. Akad. d. Wiss.* (Wien), xci. ii A, pp. 622–636.]

[O funkcyjach całkowitych ukladu *mn* zmiennych, tworzących *m* wierszy i *n* kolumn. *Pamiętnik Akad. . . . Wydz. mat.-przyr.*, xvii. pp. 143–165.]

The title of the first paper here is quite misleading: there is nowhere in it any formula of the Theory of Determinants. Its subject throughout is a definitely specified integral function of the elements of an *m*-by-*n* array, various expressions for the function being arrived at, in which determinants not unnaturally appear.

The title of the second paper is appropriate to both, the subject being now carried a little further, and being now more closely connected with determinants, in that conditions are investigated for the divisibility of the function by a power of a determinant. The case where $m=n$ the author had dealt with in 1880 (*Sitzungsb. . . .*, lxxxi. ii A, pp. 260–270.)

* Cayley, A. A memoir on the theory of Matrices. *Philos. Transac. R. Soc. London*, cxlviii. pp. 17–37; or *Collected Math. Papers*, ii. pp. 475–496.

MUIR, T. (1885).

[Question 14792. *Educ. Times*, liv. p. 83 : *Math. from Educ. Times*, (2) i. pp. 52–53.]

The theorem here proved by Muir is that *if in the case of each row of a determinant the square root of the sum of the squares of the elements be taken, the product of the said square roots is greater than the determinant.* It is stated to be due to Sir William Thomson, afterwards Lord Kelvin, who suspected it to be true and desired a proof.* Taking the determinant $|\, a_1 b_2 c_3 d_4 \,|$ and putting

$$a_r^2 + b_r^2 + c_r^2 + d_r^2 \equiv s_r^2, \quad \mathrm{A}_r^2 + \mathrm{B}_r^2 + \mathrm{C}_r^2 + \mathrm{D}_r^2 \equiv \mathrm{S}_r^2,$$

we have the positive quantity

$$\left(\frac{a_r}{s_r} - \frac{\mathrm{A}_r}{\mathrm{S}_r}\right)^2 + \left(\frac{b_r}{s_r} - \frac{\mathrm{B}_r}{\mathrm{S}_r}\right)^2 + \left(\frac{c_r}{s_r} - \frac{\mathrm{C}_r}{\mathrm{S}_r}\right)^2 + \left(\frac{d_r}{s_r} - \frac{\mathrm{D}_r}{\mathrm{S}_r}\right)^2$$

$$= \frac{a_r^2 + b_r^2 + c_r^2 + d_r^2}{s_r^2} - 2\,\frac{a_r \mathrm{A}_r + b_r \mathrm{B}_r + \dots}{s_r \mathrm{S}_r} + \frac{\mathrm{A}_r^2 + \mathrm{B}_r^2 + \dots}{\mathrm{S}_r^2}$$

$$= 2 - 2\,\frac{|\, a_1 b_2 c_3 d_4 \,|}{s_r \mathrm{S}_r},$$

from which it follows that

$$s_r \mathrm{S}_r > |\, a_1 b_2 c_3 d_4 \,|.$$

This being used four times, it follows that

$$s_1 s_2 s_3 s_4 \cdot \mathrm{S}_1 \mathrm{S}_2 \mathrm{S}_3 \mathrm{S}_4 > |\, a_1 b_2 c_3 d_4 \,|^4,$$

and $\therefore$ $$> |\, a_1 b_2 c_3 d_4 \,| \cdot |\, \mathrm{A}_1 \mathrm{B}_2 \mathrm{C}_3 \mathrm{D}_4 \,|,$$

whence the deduction

$$s_1 s_2 s_3 s_4 > |\, a_1 b_2 c_3 d_4 \,|,$$

since to suppose otherwise would lead to a contradiction.

VANEČEK, M. N. (1886).

[O souvislosti subdeterminantů. *Věstník kral. České Společnosti Náuk.* Třída math., pp. 21–28.]

The title here is somewhat indefinite, the theorems actually brought forward—without reference to their original authors—being Jacobi's regarding a minor of the adjugate and Sylvester's 'extensional' of 1851.

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* See *Transac. R. Soc. S. Africa*, i. p. 323, footnote.

WEST, É. (1886).

[Exposé des méthodes générales en mathématiques . . . d'après Hoëné Wronski. x+314 pp. Paris.]

In this valuable guide to the study of Wronski's mathematical work, the portion directly connected with our subject is that section of the *Complément* which deals with the ' loi suprême ' (pp. 235–251).* It is helpful towards a clearer understanding of the law, but does not add anything to what we have already noted regarding ' schin ' functions.

MUIR, T. (1886-1889).

[The theory of determinants in the historical order of development. *Proceed. R. Soc. Edinburgh,* xiii. pp. 547–590 ; xiv. pp. 452–518 ; xv. pp. 481–544 ; xvi. pp. 207–234, 389–448, 748–772.]

The original draft of Part I. of the first volume of the present book (see *Hist.*, i. p. 3).

WELTZIEN, C. (1886).

[Zur Theorie der homogenen linearen Substitutionen. Sch.-Progr. 20 pp. Berlin.]

In reality the subject of this simply and clearly written paper is the relations between the elements of a set of four determinants, namely, any 3-line determinant, its adjugate, its square by row-multiplication, and its square by column-multiplication. The most important result in it, however, is a rediscovery, namely, Brioschi's or Beltrami's theorem regarding the latent roots of the last two determinants of the four (*Hist.*, iii. pp. 96–97, 294).

BAGNERA, G. (1886).

[Sopra i determinanti che si possono formare cogli stessi n^2 elementi. *Giornale di Mat.*, xxv. pp. 228–231.]

All that is here said on the general subject is that the number of different determinants formable from n^2 elements is $(n^2)\,!\,(n\,!)^2$,

* The portion preceding this *Complément* had appeared previously as a series of articles in the *Journ. (de Liouville) de Math.*

difference only in sign being considered an essential difference. The matter really dealt with is the determinants formable by permutation of the elements of a single row, and the object is to discover relations connecting such determinants. The first result reached is that *If the elements of the* r^{th} *row of* $|\ a_{1n}\ |$ *be permuted in every possible way, and any one, the* μ^{th} *say, of the* n ! *determinants so formable be singled out and be subtracted from every other one that differs from it by a single interchange in the row in question, then the product of these remainders is independent of* μ, *being equal to*

$$\pm\ \xi^{\frac{1}{2}}(a_{r1},\ a_{r2},\ \ldots,\ a_{rn}) \cdot \xi^{\frac{1}{2}}(A_{r1},\ A_{r2},\ \ldots,\ A_{rn}).$$

This is obtained by expressing in terms of a_{r1}, a_{r2}, ... and A_{r1}, A_{r2}, ... the two determinants concerned in any of the subtractions referred to, and noting that the remainder after subtraction can be put into the form of the product of the difference of two a's by the difference of two A's. The other result concerns the vanishing of a certain determinant whose elements are $(n+1)^2$ of the n ! determinants spoken of.

LORIA, G. (1886).

[Nota sulla moltiplicazione di due determinanti. *Jornal de Sci. math. e astron.*, vii. pp. 101–106.]

Loria's result differs from Sylvester's of 1852 (*Hist.*, ii. pp. 75–77) in that the product-determinant obtained involves a square array of elements unconnected with the factors. If, for example, the factors be $|\ a_1b_2c_3d_4\ |$, $|\ a_1\beta_2\gamma_3\delta_4\ |$, their product is evidently representable by an 8-line determinant whose first four columns are the columns of $|\ a_1b_2c_3d_4\ |$ followed by zeros, and whose last four columns are the columns of

$$\begin{vmatrix} \lambda_1 & \lambda_2 & . & . \\ \mu_1 & \mu_2 & . & . \\ . & . & -1 & . \\ . & . & . & -1 \end{vmatrix}$$

followed by the columns of $|\ a_1\beta_2\gamma_3\delta_4\ |$: and equally evidently the order of this is reducible to the 6^{th} by taking advantage of the

presence of the elements -1, -1,—an operation which does not oust the intruding λ's and μ's. The result thus reached by Loria is, however, obtained even more readily by Sylvester's process, namely, by multiplying

$$\begin{vmatrix} . & . & . & . & . & 1 \\ . & . & . & . & 1 & . \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & . & . \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & . & . \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & . & . \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 & . & . \end{vmatrix} \text{ by } \begin{vmatrix} . & . & c_1 & d_1 & b_1 & a_1 \\ . & . & c_2 & d_2 & b_2 & a_2 \\ . & . & c_3 & d_3 & b_3 & a_3 \\ . & . & c_4 & d_4 & b_4 & a_4 \\ 1 & . & . & . & \mu_1 & \lambda_1 \\ . & 1 & . & . & \mu_2 & \lambda_2 \end{vmatrix}.$$

(See *Hist.*, ii. pp. 75-77).

RADOS, G. (1886).

[A theorem in determinants (in Magyar). *Math. és természett. Értesitö*, iv. pp. 268–271 : German translation in *Math. u. naturw. Berichte aus Ungarn*, viii. pp. 60–64.]

The theorem in question is Zehfuss' of 1858 (*Hist.*, ii. pp. 102–104) ; but the proof is not purely determinantal, being dependent on Grassmann's methods.

MUIR, T. (1886).

[A supplementary list of writings on determinants. *Quart. Journ. of Math.*, xxi. pp. 299–320.]

This list contains 84 titles belonging to the period of the previous list (see above, p. 12) and 176 belonging to the five-year period 1881–1885.

NEKRASSOFF, P. (1886) : HAEBLER, T. (1888).

[The importance and the historical development of the theory of determinants. (In Russian.) *Fizmat. nauki* . . . . (Moskva), ii. pp. 169–178.]

[Betrachtungen über Determinanten. (Sch.-Prog.) Grimma.]

These I have not seen. They are reported to contain no really fresh matter.

GIBBS, J. W. (1886).

[Multiple algebra. *Proceed. American Assoc. Adv. of Sci.*, xxxv. 32 pp.]

Besides much that is indirectly interesting to us there is, on Sylvester's lines of 1884, a two-page criticism on the ordinary mode of expounding the multiplication of determinants, and, in particular, on row-by-row multiplication. Probably something less one-sided on the part of both writers would have been more effective. As a matter of fact, Sylvester up to the last used with determinants any kind of multiplication that came handy : and, of course, papers of his like that of 1852 (*Hist.*, ii. pp. 75–77) are not to be cast aside as useless.

FOURET, G. (1886, 1887).

[Sur un mode de transformation des déterminants. *Bull. Soc. Math. de France*, xiv. pp. 146–151.]

[Remarques sur certains déterminants numériques. *Bull. Soc. Math. de France*, xv. pp. 146–147.]

In the so-called transformation there is nothing really new (*Hist.*, iii. p. 77), nor, as the author himself afterwards points out, in the evaluation of the circulant (*Hist.*, ii. p. 107) which he uses to effect the transformation.

PRESLE, A. DE (1886).

[Multiplication de deux déterminants de même degré. *Bull. Soc. Math. de France*, xiv. pp. 157–158.]

Shows no advance in the mode of proof.

POUJADE, . (1887).

[Une propriété des déterminants. *Journ. de Math. Spéc.*, (3) i. pp. 10–11.]

The property is the Sylvester-Kronecker condition for the evanescence of all the k-line minors of an n-line determinant (*Hist.*, iii. p. 14). The proof is dependent on the theory of homogeneous linear equations.

DESPLANQUES, . (1887).

[Théorème d'algèbre. *Journ. de Math. Spéc.*, xi. pp. 12–13.]

The test for non-evanescence given by Lévy in 1881 is here improved on, namely, *If each diagonal element of a determinant exceeds in absolute value the sum of the other éléments of the row to which it belongs, the determinant cannot vanish.* The proof depends on the theory of homogeneous linear equations.

STUDNIČKA, F. J. (1887).

[Novi način kojim možemo izvesti treći temeljni poučak determinanta. *Rad yugoslavenske akad. . . . u Zagrebu,* lxxxiii. pp. 146–152.]

This is concerned with a detailed proof that the familiar theorem

$$|A_{1n}| = |a_{1n}|^{n-1}$$

can be established if Jacobi's theorem regarding minors of $|A_{1n}|$ holds for the two-line minors.

MUIR, T. (1887).

[An incorrect footnote and its consequences. *Nature,* xxxvii. pp. 246–247, (343, 344), 438–439, (445).]

The footnote in question attributes the *Demonstratio eliminationis Crameriana*e of 1811 not to its author Prasse, but to Mollweide. Strange to say, the error occurs in all the five editions of Baltzer's text-book. Muir gives interesting details in regard to the matter, and these are augmented in the correspondence called forth by his letter.

MUIR, T. (1888).

[On vanishing aggregates of determinants. *Proceed. R. Soc. Edinburgh,* xv. pp. 96–105.]

The first theorem here enunciated concerns an aggregate of $n+1$ determinants of the n^{th} order, the elements involved being those of a rectangular array of $n-1$ rows and $n+1$ columns together with

those of a Pfaffian array of $n+1$ frame-lines. If in the case of $n=5$ the arrays referred to be

$$\begin{array}{cccccc} & & & & & x\ \ y\ \ z\ \ u\ \ v \\ a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \quad\ \ a\ \ \beta\ \ \gamma\ \ \delta \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \quad\ \ \mathrm{X}\ \ \mathrm{Y}\ \ \mathrm{Z} \\ c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \quad\ \ p\ \ q \\ d_0 & d_1 & d_2 & d_3 & d_4 & d_5, \quad\ \ \Omega, \end{array}$$

the identity is

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ x & y & z & u & v \end{vmatrix} - \begin{vmatrix} a_0 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_2 & b_3 & b_4 & b_5 \\ c_0 & c_2 & c_3 & c_4 & c_5 \\ d_0 & d_2 & d_3 & d_4 & d_5 \\ x & a & \beta & \gamma & \delta \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_3 & b_4 & b_5 \\ c_0 & c_1 & c_3 & c_4 & c_5 \\ d_0 & d_1 & d_3 & d_4 & d_5 \\ y & a & \mathrm{X} & \mathrm{Y} & \mathrm{Z} \end{vmatrix}$$

$$- \begin{vmatrix} a_0 & a_1 & a_2 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_4 & b_5 \\ c_0 & c_1 & c_2 & c_4 & c_5 \\ d_0 & d_1 & d_2 & d_4 & d_5 \\ z & \beta & \mathrm{X} & p & q \end{vmatrix} + \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_5 \\ c_0 & c_1 & c_2 & c_3 & c_5 \\ d_0 & d_1 & d_2 & d_3 & d_5 \\ u & \gamma & \mathrm{Y} & p & \Omega \end{vmatrix} - \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \\ c_0 & c_1 & c_2 & c_3 & c_4 \\ d_0 & d_1 & d_2 & d_3 & d_4 \\ v & \delta & \mathrm{Z} & q & \Omega \end{vmatrix} = 0.$$

A verification is readily effected by observing the cofactors of

$$x,\ y,\ z,\ u,\ v,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \mathrm{X},\ \mathrm{Y},\ \mathrm{Z},\ p,\ q,\ \Omega\ ;$$

thus, the cofactor of γ in its first position is $| a_0 b_2 c_3 d_5 |$, and in its second position the same determinant with the opposite sign.* Note is taken that the number of elements involved is $\frac{1}{2}(n+1)(3n-2)$: and the theorem is used to establish Kronecker's relation between m-line minors of a $2m$-line axisymmetric determinant.

The second theorem is: *If any two determinants* A *and* B *of the* nth *order be taken, and from them two sets of determinants be formed, namely, first, a set of* n *determinants each of which is in one row identical with* A *and in the remaining rows with* B, *and, secondly, a set of* n *determinants each of which is in one column identical with* A *and*

* The theorem may be simply formulated thus: *If there be a rectangular array of* n − 1 *rows and* n + 1 *columns and a Pfaffian array of* n + 1 *frame-lines, and* Δ_r *be the determinant formed by taking all the columns of the former array except the* rth *and as a last row the* rth *frame-line of the Pfaffian array, then*

$$\Delta_1 - \Delta_2 + \Delta_3 - \ldots + (-1)^n \Delta_{n+1} = 0.$$

in the remaining columns with B, *then the sum of the first set of determinants is equal to the second set.* Thus, with the originating determinants

$$|\,a_1b_2c_3\,|\quad|\,a_4b_5c_6\,|,$$

we have the identity

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 \\ c_4 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_5 & a_6 \\ b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_5 & a_6 \\ b_1 & b_5 & b_6 \\ c_1 & c_5 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_2 & a_6 \\ b_4 & b_2 & b_6 \\ c_4 & c_2 & c_6 \end{vmatrix} + \begin{vmatrix} a_4 & a_5 & a_3 \\ b_4 & b_5 & b_3 \\ c_4 & c_5 & c_3 \end{vmatrix}.$$

In verification we have only got to observe the cofactors of the elements of either of the original determinants. Also, passing mentally from cofactors of elements to cofactors of *minors* a generalization is suggested which is coextensive with the expansion-theorem of Laplace, but which had already been obtained otherwise by Deruyts (1882). This is: *If any two determinants* A *and* B *of the* n^{th} *order be taken, and from them two sets of determinants be formed, namely, first, a set of* $(n)_r$ *determinants, each of which is in* r *rows identical with* A *and in the remaining rows with* B, *and, secondly, a set of the same number of determinants, each of which is in* r *columns identical with* A *and in the remaining columns with* B, *then the sum of the first set of determinants is equal to the sum of the second set.* For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_5 & c_6 & c_7 & c_8 \\ d_5 & d_6 & d_7 & d_8 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_5 & b_6 & b_7 & b_8 \\ c_1 & c_2 & c_3 & c_4 \\ d_5 & d_6 & d_7 & d_8 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_5 & b_6 & b_7 & b_8 \\ c_5 & c_6 & c_7 & c_8 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

$$+ \begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_5 & d_6 & d_7 & d_8 \end{vmatrix} + \begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ b_1 & b_2 & b_3 & b_4 \\ c_5 & c_6 & c_7 & c_8 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} + \begin{vmatrix} a_5 & a_6 & a_7 & a_8 \\ b_5 & b_6 & b_7 & b_8 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

$$= |\,a_1b_2c_7d_8\,| + |\,a_1b_6c_3d_8\,| + |\,a_1b_6c_7d_4\,|$$
$$+ |\,a_5b_2c_3d_8\,| + |\,a_5b_2c_7d_4\,| + |\,a_5b_6c_3d_4\,|.$$

The simpler form of the second theorem (Deruyts') is applied in the same manner as the first theorem, namely, to establish a linear relation between n-line minors of a $2n$-line *centro*-symmetric determinant, thus producing an analogue to Kronecker's theorem above referred to.

VANDERMONDE, N. (1888).

[Abhandlungen aus der reinen Mathematik von N. Vandermonde. In deutscher Sprache von C. Itzigsohn. 104 pp. Berlin.]

The important *Mémoire sur l'élimination* (*Hist.*, i. pp. 17–24) is the fourth here translated.

MOUCHEL, J. (1888).

[Correspondance. *Nouv. Annales de Math.*, (3) vii. p. 400.]

Statement of two theorems, one being Dostor's of 1879 and the other still less important.

RAHUSEN, A. E. (1888).

[Sur quelques proprietés des déterminants, appliquées à une question de géométrie à n dimensions. *Annales de l'École Polyt.* (Delft), iv. pp. 104–139.]

The properties in question (pp. 104–109) are almost entirely connected with the evanescence of a rectangular array, the other matter being notational and not of any permanent importance. The case is first taken where $|\, a_{1n} \,|$ and one of its primary minors vanish, and then in a step-by-step manner a general theorem is reached, namely, that *If in an oblong* n-*by*-m *array all the* n-*line minors vanish that include a fixed group of less than* n *columns, then either the array is wholly evanescent or the array composed of the said group is evanescent.* For example, if in the array

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{array}$$

we have given

$$|\, 2345 \,| \;=\; |\, 1345 \,| \;=\; |\, 1245 \,| \;=\; 0,$$

then either $|1235| = |1234| = 0,$

or $$\left\| \begin{matrix} a_4 & b_4 & c_4 & d_4 \\ a_5 & b_5 & c_5 & d_5 \end{matrix} \right\| = 0.$$

By way of proof the author forms four vanishing determinants by annexing to the given array as a fifth row a repetition of one of the other rows : from these by expansion he of course obtains

$$\begin{aligned} a_1|2345| - a_2|1345| + a_3|1245| - a_4|1235| + a_5|1234| &= 0 \\ b_1|2345| - b_2|1345| + b_3|1245| - b_4|1235| + b_5|1234| &= 0 \\ c_1|2345| - c_2|1345| + c_3|1245| - c_4|1235| + c_5|1234| &= 0 \\ d_1|2345| - d_2|1345| + d_3|1245| - d_4|1235| + d_5|1234| &= 0, \end{aligned}$$

i.e.

$$\left.\begin{aligned} - a_4|1235| + a_5|1234| &= 0 \\ - b_4|1235| + b_5|1234| &= 0 \\ - c_4|1235| + c_5|1234| &= 0 \\ - d_4|1235| + d_5|1234| &= 0 \end{aligned}\right\},$$

which is viewed as equivalent to the desired result.

STUDNIČKA, F. J. (1888).

[Nové odvození třetí základní poučky determinantní. *Časopis pro pěstování math. a fys.*, xvii. pp. 193–199.]

Holding the opinion that Jacobi's theorem regarding a minor of the adjugate is the third fundamental theorem of determinants, the first being Laplace's expansion-theorem and the second the multiplication-theorem, Studnička nevertheless is at pains to show that the third can be established without the help of the second by using along with the first the condensation-theorem of 1879.

MUIR, T. (1888).

[Nomenclature of determinants. *Nature*, xxxviii. p. 589.]

The subject, raised in a review (pp. 537–538), is the impropriety and harmfulness of multiplying synonyms, the main illustrative examples being proposed substitutes for *adjugate*, *persymmetric*, and *skew*. "*Adjugate* as applied to a determinant was a generation old before *reciprocal* was proposed, and carries with it the sanction of the highly-honoured names of Gauss and Cauchy. To outweigh these

claims there is very little to be said for the rival word. It is not more appropriate: indeed, the kind of connection to be indicated does not convey the idea of reciprocity at all . . . ." And so on, the other words being similarly dealt with.

HERMITE, C. (1889).

[Question 9930. *Educ. Times*, xlii. p. 37 : *Math. from Educ. Times*, li. pp. 49–50.]

What Hermite sets for proof is a case of Cayley's theorem of 1843 (*Hist.*, ii. p. 15), being to the effect that

if $|a_1b_2c_3| = |a_1b_2c_4| = 0$, then $|a_1b_3c_4| = |a_2b_3c_4| = 0$.

The proof given by Muir consists in taking the additional equality $|a_1b_2c_1| = 0$ and eliminating $|a_1b_2|$, $|a_1c_2|$, $|b_1c_2|$. It is also pointed out that the identity

$$|a_1b_2| \cdot |a_1b_3c_4| - |a_1b_3| \cdot |a_1b_2c_4| + |a_1b_4| \cdot |a_1b_2c_3| = 0$$

only establishes the vanishing of $|a_1b_3c_4|$ when it is known that $|a_1b_2|$ is not zero; whereas, if $|a_1b_2|$ be 0 the three equations of the other procedure become

$$\frac{a_3}{b_3} = \frac{a_4}{b_4} = \frac{a_1}{b_1} = \frac{|a_1c_2|}{|b_1c_2|}$$

and give $|a_1b_3c_4| = 0$ more easily than before.

HENSEL, K. (1889).

[Ueber die Darstellung der Determinante eines Systems welches aus zwei anderen componirt ist. *Acta Math.*, xiv. pp. 317–319.]

The theorem here dealt with is Zehfuss' of 1858 (*Hist.*, ii. pp. 102–104), although not so attributed by Hensel. Associating the composite determinant in question, $D_{4,2}$ say as before, with a set of linear homogeneous equations, Hensel shows that D_4 and D_2 are the only possible factors,—in other words, that the equivalent of $D_{4,2}$ must be of the form

$$cD_4^{\alpha}D_2^{\beta},$$

—and then by taking a case with very special elements he finds c, α, β to be 1, 2, 4.

MUIR, T. (1889).

[On self-conjugate permutations. *Proceed. R. Soc. Edinburgh*, xvii. pp. 7–13.]

After recalling previous work on the subject, the author fills a long vacant niche by giving a proof of Rothe's recurrence-formula

$$u_n = u_{n-1} + (n-1)\,u_{n-2}.$$

He then establishes the new formula

$$u_n = 1 + (n)_2 + 1\cdot 3\,(n)_4 - 1\cdot 3\cdot 5\,(n)_6 + \ldots,$$

the first eight values thus being

1, 2, 4, 10, 26, 76, 232, 764, ...

Passing then from permutations to terms of a determinant he proves independently the same two results, using in the case of the second the lemma that *in a zero-axial* n*-line determinant*

$$\begin{aligned} \mathrm{u_n} &= 0 \quad \textit{when } \mathrm{n} \textit{ is odd,} \\ &= 1\cdot 3\cdot 5\ldots(\mathrm{n}-1) \quad \textit{when } \mathrm{n} \textit{ is even.} \end{aligned}$$

KRONECKER, L. (1889).

(See under this heading in Chap. IV.)

CASPARY, F. (1889).

[Sur une méthode générale de la géometrie. . . . *Bull. des sci. math.*, (2) xiii. pp. 202–240.]

The method referred to is Grassmann's, and there naturally appears a short account of the relation between the ' outer product ' and determinants (§§ 24–26, pp. 228–231).

FISKE, T. S. (1889).

[Notes on modern higher algebra. 1. Converse of the determinant multiplication-theorem. *Messenger of Math.*, (2) xix. pp. 89–90.]

The converse in question is that *if a rational integral function of the elements of a square array be subject to the same law of multiplication*

as a determinant, it must be either a determinant or a power of a determinant. Denoting the function by ψ, we have by hypothesis

$$\psi\begin{Vmatrix} l_1 & l_2 & l_3 & \dots \\ m_1 & m_2 & m_3 & \dots \\ n_1 & n_2 & n_3 & \dots \\ \dots & \dots & \dots & \dots \end{Vmatrix} \cdot \psi\begin{Vmatrix} \lambda_1 & \cdot & \cdot & \dots \\ \cdot & \mu_2 & \cdot & \dots \\ \cdot & \cdot & \nu_3 & \dots \\ \dots & \dots & \dots & \dots \end{Vmatrix} = \psi\begin{Vmatrix} l_1\lambda_1 & l_2\mu_2 & l_3\nu_3 & \\ m_1\lambda_1 & m_2\mu_2 & m_3\nu_3 & \\ n_1\lambda_1 & n_2\mu_2 & n_3\nu_3 & \\ \dots & \dots & \dots & \dots \end{Vmatrix},$$

from which it is concluded that the function is homogeneous in the elements of a column, and therefore must vanish if there be a column of zeros. We thus know that if

$$\left.\begin{aligned} l_1\lambda_1 + l_2\lambda_2 + l_3\lambda_3 + \dots &= 0 \\ m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + \dots &= 0 \\ n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 + \dots &= 0 \\ \dots\dots\dots\dots\dots\dots\dots &\end{aligned}\right\},$$

or, what is the same thing, if

$$| l_1m_2n_3 \dots . | = 0$$

(for of course the λ's are understood to be non-zero), then the product

$$\psi\begin{Vmatrix} l_1 & l_2 & l_3 & \dots \\ m_1 & m_2 & m_3 & \dots \\ n_1 & n_2 & n_3 & \dots \\ \dots & \dots & \dots & \dots \end{Vmatrix} \cdot \psi\begin{Vmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \dots \\ \mu_1 & \mu_2 & \mu_3 & \dots \\ \nu_1 & \nu_2 & \nu_3 & \dots \\ \dots & \dots & \dots & \dots \end{Vmatrix}$$

must vanish,—a result which is seen to imply that its first factor is a multiple of $| l_1m_2n_3 \dots . |$. The step from 'multiple' to 'power' is thereupon readily taken.

POMEY, E. (1890): TISSOT, A. (1890).

[Sur un théorème de déterminants (théorème de multiplication). *Journ. de Math. Spéc.*, xiv. pp. 3–4.]

[Sur la multiplication des déterminants. *Journ. de Math. Spéc.*, (3) iv. pp. 193–194.]

Nothing noteworthy in either.

MUIR, T. (1890).

[The Theory of Determinants in the Historical Order of its Development. Part I. Determinants in general; Leibnitz (1693) to Cayley (1841). xii+278 pp. London.]

A reprint, separately published, of six papers communicated to the *Royal Society of Edinburgh* during the years 1886-1889 (see above, p. 33). It takes up to, but does not include, Cayley.

KRONECKER, L. (1890).

[Ueber die Composition der Systeme von n^2 Grössen mit sich selbst. *Monatsb. . . . Akad. d. Wiss.* (Berlin), pp. 1081–1088: or *Werke*, iii. pp. 463–473.]

The result which concerns us here is essentially the same as the second theorem of Muir's paper of 1885 (see above, p. 30); but it is very important to note that Kronecker himself here refers it back to a paper of his of the year 1873.*

HORTA, F. DA P. (1890).

[Nota sobre os determinantes. *Jorn. de sci. math. phys. e nat.*, (2) v. pp. 67–73.]

The subject here is the enumeration and the evaluation of the various kinds of symmetric determinants and skew determinants, the suggesting sources being Muir's similar papers of 1881. The conditioning equations of symmetry

$$a_{rs} = a_{sr} \qquad (1)$$

$$a_{rs} = a_{n+1-s,\,n+1-r} \qquad (2)$$

$$a_{rs} = a_{n+1-r,\,n+1-s} \qquad (3)$$

with respect to (1) the primary diagonal, (2) the secondary diagonal, and (3) the centre, are also considered as holding simultaneously in pairs, and thus originating a fourth form.† Skewness is dealt with

* KRONECKER, L. Ueber die verschiedenen Sturm'schen Reihen und ihre gegenseitigen Beziehungen. *Monatsb. . . . Akad. d. Wiss.* (Berlin), pp. 117–154 (Art. vii.).

† As the form is the same whether the originating pair of equations be (1) (2), or (1) (3), or (2) (3), the name 'bisymmetric' is fully appropriate. An example of about this date will be found in *Math. u. naturw. Berichte aus Ungarn*, ix. § 14.

in the same way. Note is even taken of the form which is symmetric with respect to one diagonal and skew with respect to the other.

A mode of development incidentally given will be fully understood from the example

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \left| b_2 c_3 d_4 \right| + b_2 \begin{vmatrix} . & a_3 & a_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + c_3 \begin{vmatrix} . & a_3 & a_4 \\ b_1 & . & b_4 \\ d_1 & d_3 & d_4 \end{vmatrix} + d_4 \begin{vmatrix} . & a_2 & a_3 \\ b_1 & . & b_3 \\ c_1 & c_2 & . \end{vmatrix}$$

$$+ \begin{vmatrix} . & a_2 & a_3 & a_4 \\ b_1 & . & b_3 & b_4 \\ c_1 & c_2 & . & c_4 \\ d_1 & d_2 & d_3 & . \end{vmatrix},$$

the related equality which deals with the mere number of terms being

$$4\,! = 3\,! + (4+3+2) + 9.$$

The development bears some resemblance to Cayley's of 1847, where the corresponding equality is

$$4\,! = 1 + 6{\cdot}1 + 4{\cdot}2 + 9.$$

KRONECKER, L. (1890): NETTO, E. (1891).

[Anwendung der Modulsysteme auf Fragen der Determinantentheorie. *Crelle's Journ.*, cvii. pp. 254–261; cviii. pp. 144–146.]

The fundamental theorem here—considered 'noteworthy' by Kronecker himself—is in effect that *If the product* $|\,a_{1n}\,| \cdot |\,b_{1n}\,|$ *with* -1 *affixed to each of its diagonal terms be denoted by* $|\,P_{1n}|$, *and the corresponding result, when the order of the factors is reversed, by* $|\,Q_{1n}\,|$, *then each of the* P's *is expressible as a linear homogeneous function of the* Q's, *with coefficients involving only elements of* $|\,a_{1n}\,|$. The proofs given need not be reproduced, as the theorem is much simplified when increased in definiteness by including the specifica-

We may add that *bisymmetric* n-*line determinants have at most* $\frac{1}{4}(n+1)^2$ *distinct elements when* n *is odd, and* $\frac{1}{2}n(\frac{1}{2}n+1)$ *when* n *is even*; that *the corresponding numbers in the case of centro-symmetric determinants are* $\frac{1}{2}(n^2+1)$, $\frac{1}{2}n^2$,; and that, if B_n and C_n represent the numbers for the two forms, $C_n = B_n + B_{n-2}$.

tion of the functions referred to, these being the elements of the product

$$|\,a_{1n}\,|\cdot|\,Q_{1n}\,|\cdot|\,a_{1n}\,|,$$

where $|\,a_{1n}\,|$ is the reciprocal determinant of $|\,a_{1n}\,|$, and the multiplication is row-by-column throughout. For example, if the initiating determinants be

$$|\,a_1b_2c_3\,| \quad \text{and} \quad |\,\lambda_1\mu_2\nu_3\,|,$$

we have

$$P_{11} = \begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ \hline Q_{11} & Q_{21} & Q_{31} & A_1 \\ Q_{12} & Q_{22} & Q_{32} & A_2 \\ Q_{13} & Q_{23} & Q_{33} & A_3 \end{array} \div |\,a_1b_2c_3\,|, \qquad P_{12} = \begin{array}{ccc|c} a_1 & a_2 & a_3 & \\ \hline Q_{11} & Q_{21} & Q_{31} & B_1 \\ Q_{12} & Q_{22} & Q_{32} & B_2 \\ Q_{13} & Q_{23} & Q_{33} & B_3 \end{array} \div |\,a_1b_2c_3\,|, \text{ etc.}$$

Since the theorem is one concerning *any* element of a determinant, the suggestion occurs that we are dealing with Cayleyan matrices: and, as a matter of fact, it is included in the simple identity

$$uv-1 \;=\; u\,(vu-1)\,u^{-1},$$

where u and v are any matrices of the same order.

PRANGE. A. J. N. (1890).

[Een en ander over nieuwere algebra naar anleiding van een onlangs verschenen werk. *Nieuw Archief voor Wiskunde*, xvii. pp. 158–175.]

Determinants are here employed in solving (pp. 160–166) several irrational equations, beginning with $(a_1x+b_1)^{\frac{1}{2}} \pm (a_2x+b_2)^{\frac{1}{2}} = c$, in resolving $ax^2+2bxy+cy^2+2dx+2ey+f$ into factors (pp. 167–170), and in the solution of the biquadratic equation (pp. 170–172). Taking these examples as evidence of the usefulness of the instrument, the author then devotes a couple of pages to explain wherein the usefulness consists.

MUIR, T. (1891).

[On some hitherto unproved theorems in determinants. *Proceed. R. Soc. Edinburgh*, xviii. pp. 73–82.]

The theorems referred to are those of Cayley's second paper of 1843 (*Hist.*, ii. pp. 14–16). The first theorem is, however, very

imperfectly treated ; as regards the one ' unproved ' part of it—the determination of the Θ's—it is overlooked, for example, that these coefficients are to be independent of the first row of the array. In establishing the two other theorems the last is extended, and it is thus made possible to include the two in one, namely, *If an* m-*by*-(m+h) *array be evanescent, then the array produced by multiplying row-wise by a determinant of the* $(m+h)^{th}$ *order and the array produced by multiplying column-wise by a determinant of the* m^{th} *order are both evanescent.*

CARVALLO, E. (1891) : NIEMÖLLER, F. (1891).

[Théorie des déterminants dans l'esprit de Grassmann. Multiplication des déterminants. *Nouv. Annales de Math.*, (3) x. pp. 219–224, 341–345.]

[Anwendung der linealen Ausdehnungslehre von Grassmann auf die Theorie der Determinanten. Sch.-Progr. 22 pp. Osnabrück.]

The first of these is a simple exposition of the elementary properties, the definition used being that based on ' algebraic keys ' or ' alternate units.'

The second differs merely in being more ambitious and more extensive, with the object of including as many so-called ' applications ' as possible.

Neither author refers to Scott (1881).

GRASSMANN, R. (1891).

[Die Ausdehnungslehre, ... ix.+132 pp. Stettin.]

The brother of the author of the theory here gives a fresh account of it, the contents of his book so far as they go being in the main in keeping with those of the Ausdehnungslehre of 1862.

GERHARDT, K. I. (1891).

[Leibnitz über die Determinanten. *Sitzungsb. . . . Akad. der Wiss.*, (Berlin) pp. 407–423.]

Gerhardt here edits, with an explanatory introduction, four short manuscripts of Leibnitz, ' hitherto unpublished.' The contents of

the latter do not, however, warrant any alteration in what we have already written on the subject (*Hist.*, i. pp. 6–10). Indeed, the essential part of the third manuscript is not now printed for the first time, having been published under Gerhardt's eye in 1863 and been reprinted by us in 1886 (*Hist.*, i. p. 10).

IGEL, B. (1892): ESCHERICH, G. v. (1892).

[Zur Theorie der Determinanten. *Monatshefte f. Math. u. Phys.*, iii. pp. 55–68.]

[Ueber einige Determinanten. *Monatshefte f. Math. u. Phys.*, iii. pp. 68–80.]

Here there is brought to light (pp. 55–60) an interesting connection between Zehfuss' theorem of 1858, Sylvester's of 1851, and Schläfli's of 1851 (*Hist.*, ii. pp. 102–104, 52–53, 196–197). We know, for example, that if the two determinants are of the same order, $|\ a_1b_2c_3\ |$ and $|\ a_1\beta_2\gamma_3\ |$ say, Zehfuss' composite determinant is of the 9th order, and if, further, the two be identical, the composite is equal to $|\ a_1b_2c_3\ |^6$. Now by the performance of three row-subtractions on the composite nine elements are made to vanish, and by the performance of three column-additions other nine go likewise, the result being that the determinant is transformable into

$$\begin{vmatrix} a_1^2 & 2a_1a_2 & 2a_1a_3 & a_2^2 & 2a_2a_3 & a_3^2 \\ a_1b_1 & a_1b_2+a_2b_1 & a_1b_3+a_3b_1 & a_2b_2 & a_2b_3+a_3b_2 & a_3b_3 \\ b_1^2 & 2b_1b_2 & 2b_1b_3 & b_2^2 & 2b_2b_3 & b_3^2 \\ a_1c_1 & a_1c_2+a_2c_1 & a_1c_3+a_3c_1 & a_2c_2 & a_2c_3+a_3c_2 & a_3c_3 \\ b_1c_1 & b_1c_2+b_2c_1 & b_1c_3+b_3c_1 & b_2c_2 & b_2c_3+b_3c_2 & b_3c_3 \\ c_1^2 & 2c_1c_2 & 2c_1c_3 & c_2^2 & 2c_2c_3 & c_3^2 \end{vmatrix} \cdot \begin{Vmatrix} |\ a_2b_1\ | & |\ a_3b_1\ | & |\ a_3b_2\ | \\ |\ a_2c_1\ | & |\ a_3c_1\ | & |\ a_3c_2\ | \\ |\ b_2c_1\ | & |\ b_3c_1\ | & |\ b_3c_2\ | \end{Vmatrix}.$$

We thus see that the second of the two factors so reached must be equal to $-|\ a_1b_2c_3\ |^2$, which is in agreement with Sylvester's theorem regarding the second compound of a determinant, and that the first factor must equal $-|\ a_1b_2c_3\ |^4$ as originally found by Schläfli.

The second paper provides a gradational proof of the fundamental theorem of the first paper, and extends one of the other theorems.

No reference is made to either Zehfuss or Schläfli in either of the papers,—a neglect, we may add, which not a few subsequent writers were equally guilty of.

DITTMAR, O. (1892).

[Neue Permutationsverfahren und Determinantenberechnungen. Progr. 19 pp. Wimpfen.]

This concerns the obtaining of permutations in pairs, each pair consisting of a permutation and its reverse. It leads to a generalization like that of Bonolis (1882).

DICKSTEIN, S. (1892).

[Sur les découvertes mathématiques de Wronski. *Bibliotheca Math.*, (2) vi. pp. 48–52, 85–90.]

This historical note, besides giving useful lists of writings, has a special section on 'combinatory sums' (pp. 49–51) and another on 'aleph functions.' In view of the source of the paper and its author, one would have expected stronger advocacy of Wronski's claims in connection with determinants: nothing more is brought to light than what we had drawn attention to in 1881 and 1887.*

DERUYTS, J. (1892).

[Sur les relations qui existent entre certain déterminants. *Bull. . . . Acad. roy. des sci. . . . de Belgique*, xxiii. pp. 507–521.]

The determinants in question are the m-line determinants of an m-by-n array, and make their appearance as coordinates in the geometry of space of $n-1$ dimensions. The investigation rests throughout on the theory of quantics, and we need therefore only note the result reached, namely, that (p. 516) *all relations connecting the said determinants are deducible from* quadratic *relations by means of multiplications and additions*, and draw attention to its importance in connection with the question of the number of relations that are independent (see below pp. 56-57).

* *Theory of Determinants*, pp. 224–227: *Proceed. R. Soc. Edinburgh*, xiv. pp. 484–485; xv. pp. 503, 542.

MUIR, T. (1892): CAYLEY, A. (1894).

[A problem of Sylvester's in elimination. *Proceed. R. Soc. Edinburgh*, xx. pp. 300–305.]

[Note on Dr. Muir's paper 'A problem . . . .' *Proceed. R. Soc. Edinburgh*, xx. pp. 306–308: or *Coll. Math. Papers*, xiii. pp. 545–547.]

The main result of Muir's examination of Sylvester's classical paper of 1841 (*Hist.*, i. pp. 243-245) is to show not only that a variety of six-line eliminants like Sylvester's exists, but also that three-line eliminants of greater interest are obtainable in the same way. Thus, the original set of equations and their eliminant being

$$\left.\begin{aligned} bz^2 - 2fyz + cy^2 &= 0 \\ cx^2 - 2gzx + az^2 &= 0 \\ ay^2 - 2hxy + bx^2 &= 0 \end{aligned}\right\} \quad \text{and} \quad 2\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^2, \quad \text{or} \quad 2\Delta^2 \text{ say,}$$

he deduces a set in x^2, y^2, z^2 with eliminant $2f^2g^2h^2\Delta^2$; a set in yz, zx, xy, namely,

$$\left.\begin{aligned} a(bg-fh)yz + b(af-gh)zx + c(h^2-ab)xy &= 0 \\ a(f^2-bc)yz + b(ch-gf)zx + c(bg-hf)xy &= 0 \\ a(ch-fg)yz + b(g^2-ca)zx + c(af-hg)xy &= 0 \end{aligned}\right\}$$

with eliminant $abc\Delta^2$; and a set in x, y, z, namely,

$$\left.\begin{aligned} (bg-fh)x + (af-gh)y + (h^2-ab)z &= 0 \\ (f^2-bc)x + (ch-gf)y + (bg-hf)z &= 0 \\ (ch-fg)x + (g^2-ac)y + (af-hg)z &= 0 \end{aligned}\right\}$$

with eliminant Δ^2.

Cayley, who was interested in these results, draws special attention to the last set, pointing out and illustrating from geometry that, when $\Delta = 0$, not only does the determinant of the set vanish, but also all its primary minors, so that there is only one distinct equation in the set. In view of his note of 1856 (*Hist.*, ii. pp. 143–144) he might have been expected to remark also on the fact that the set is the assertion of the vanishing of the remaining three primary minors of

$$\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & . \end{vmatrix}.$$

THE PASSING OF CAYLEY.

It becomes us here to draw attention to the fact that this note of Cayley's was his last contribution to the literature of our subject. As his first contribution had appeared in the beginning of 1841, there lies between the two dates the long period of almost *fifty-four* years. We may note also that the paper of 1841 was his first paper of all, and is thus the opening paper in the first volume of his *Collected Mathematical Papers*, while the other is numbered 963, and is the last but three in the thirteenth volume. Cayley died on 26th January, 1895.

KRETKOWSKI, W. (1892): PRIME, F. (1893).

[O pewnej tożsamości. *Rozpravy Akad.* . . . . (w Krakowie), (2) vi. pp. 151–154.]

[Essai d'une démonstration de la formule de Laplace relative au développement des déterminants. *Journ. de Math. Spéc.*, (4) ii. pp. 3–5.]

The identity dealt with in the first paper is a particular case of that connected with the squaring of an oblong array.

WELTZIEN, C. (1892).

[Ueber das Product zweier Determinanten. *Math. Annalen*, xlii. pp. 598–600.]

For the case of two determinants of the fourth order there is given at full length the four equalities on which the author justifiably bases the general theorem that the sum of the k-line coaxial minors of $|\, u_{1n} \,| \cdot |\, v_{1n} \,|$ is equal to the corresponding sum in $|\, v_{1n} \,| \cdot |\, u_{1n} \,|$, or, as it might be written in a later notation,

$$\mathrm{Saxm}_k(|\, u_{1n} \,| \cdot |\, v_{1n} \,|) = \mathrm{Saxm}_k(|\, v_{1n} \,| \cdot |\, u_{1n} \,|).$$

It is more general than Brioschi's theorem of 1854 (*Hist.*, iii. pp. 96–97, 294), and it is included in the theorem which Sylvester in 1884 put very differently as follows: *The latent roots of* AB *are the same as those of* BA (*Johns Hopkins Univ. Circ.*, iii. pp. 7–9.)*

* Or *Collected Math. Papers*, iv. p. 127.

MILLER, E. (1893).

[Modern higher algebra. *Kansas Univ. Quarterly*, i. pp. 133–136.]

What is here given is a formal proof of a known theorem on a zero determinant, of another on a zero-axial skew determinant, and of a third on Jacobians.

BANG, A. S. (1893).

[Om en Trediegradsligning. *Nyt Tidss. f. Math.*, iv. B, pp. 57–60; ix. B, pp. 94–96.]

Incidentally there is proved here an equality in regard to any three-line determinant, namely,

$$cdh \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = -\mathrm{BFG} + b\mathrm{B}\cdot f\mathrm{F} + f\mathrm{F}\cdot g\mathrm{G} + g\mathrm{G}\cdot b\mathrm{B} - (b\mathrm{B}+f\mathrm{F}+g\mathrm{G})(bfg-cdh) + (bfg-cdh)^2.$$

A useful special case we may put in the form: *If* bfg = cdh, *then*

$$\begin{vmatrix} \mathrm{a} & \mathrm{b} & \mathrm{c} \\ \mathrm{d} & \mathrm{e} & \mathrm{f} \\ \mathrm{g} & \mathrm{h} & \mathrm{k} \end{vmatrix} = \frac{\mathrm{BF}}{\mathrm{g}} + \frac{\mathrm{FG}}{\mathrm{b}} + \frac{\mathrm{GB}}{\mathrm{f}} - \frac{\mathrm{BFG}}{\mathrm{bfg}},$$

the development, be it remarked, containing only three elements and their cofactors.

ALMEIDA, L. C. (1893): TEIXEIRA, J. P. (1893).

[Novas regras para desenvolver os determinantes literaes do terceiro e quarto gráo. *Instituto de Coimbra*, xl. pp. 763–765.]

[Processos expeditos para achar os desenvolvimentos de alguns determinantes. *Jorn. de sci. math. e astron.*, xi. pp. 88–92.]

[Novo methodo de desenvolver os determinantes. *Jorn. de sci. math. e astron.*, xi. pp. 173–186.]

Teixeira's aim here is, like Bonolis' of 1882, to formulate a rule for the purpose of obtaining all the terms of an n-line determinant in the form of *diagonal* terms. The exposition is simpler and much

more detailed than before, distinction being drawn between the cases $n = 4p,\ 4p+1,\ \ldots$.*

LIERS, E. (1893).

[Ueber eine Analogie des Laplace'schen Determinantensatzes. *Archiv d. Math. u. Phys.*, (2) xii. pp. 352–353.]

This is another rediscovery of the generalization brought to light by Muir in 1879, or rather of that case of the latter in which the factors of the expansion are restricted to being minors of the same order. The mode of proof is new, the essential part of it being the application of the ordinary form of Laplace's theorem to an $(n-1)$-line minor of the adjugate.†

MACMAHON, P. A. (1893).

[A certain class of generating functions in the theory of numbers. *Philos. Transac. R. Soc. London*, clxxxv. pp. 111–160.]

Through consideration of 'inversely-symmetric' determinants MacMahon is led to an interesting theorem (pp. 146–150) regarding determinants in general, namely, *In the case of every determinant of even order higher than the second there exist two special relations between its coaxial minors, and each of these two relations can be thrown into a form which exhibits the determinant as an irrational function of its coaxial minors: in the case of a determinant of odd order, on the other hand, no such relations exist, and it is not possible to express the determinant as a function of its coaxial minors.* Unfortunately the proof is lengthy.

* Perhaps the necessary rule of signs may be best formulated as follows: In an n-line determinant the signs of the series of terms composed of elements from lines parallel to the primary diagonal are all positive when n is odd, and are alternately positive and negative when n is even, the leading term of course being first taken: in the case of the series of terms similarly connected with the secondary diagonal each term has the same sign as the corresponding term in the previous series or the opposite sign according as $\frac{1}{2}n(n-1)$ is even or odd. Thus, when n is 3, 4, 5, 6, the signs are

$$+\ +\ +\ ;\quad -\ -\ -$$
$$+\ -\ +\ -\ ;\quad +\ -\ +\ -$$
$$+\ +\ +\ +\ +\ ;\quad +\ +\ +\ +\ +$$
$$+\ -\ +\ -\ +\ -\ ;\quad -\ +\ -\ +\ -\ +\ .$$

† The example given in the umbral notation is very inaccurately printed.

VAHLEN, K. T. (1893).

[Ueber die Relationen zwischen den Determinanten einer Matrix. *Crelle's Journ.*, cxii. pp. 306–310.]

Vahlen's theorem on this subject is Hamburger's rediscovery of 1885, namely, that *the number of independent relations connecting the* k*-line minors of a* k*-by-*n *array is*

$$(n)_k - 1 - k\,(n-k).$$

The array being

$$\left\| \begin{array}{ccccccc} a_{11} & a_{12} & a_{13} & \dots & a_{1k} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2k} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3k} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k1} & a_{k2} & a_{k3} & \dots & a_{kk} & \dots & a_{kn} \end{array} \right\|,$$

he first subjects every element of it to a linear substitution, namely,

$$a_{rs} = \mu_{1r}b_{1s} + \mu_{2r}b_{2s} + \dots + \mu_{kr}b_{ks},$$

obtaining

$$\left\| \begin{array}{ccccccc} b_{11} & b_{12} & b_{13} & \dots & b_{1k} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2k} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3k} & \dots & b_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{k1} & b_{k2} & b_{k3} & \dots & b_{kk} & \dots & b_{kn} \end{array} \right\| \cdot \left| \begin{array}{ccccc} \mu_{11} & \mu_{12} & \mu_{13} & \dots & \mu_{1k} \\ \mu_{21} & \mu_{22} & \mu_{23} & \dots & \mu_{2k} \\ \mu_{31} & \mu_{32} & \mu_{33} & \dots & \mu_{3k} \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{k1} & \mu_{k2} & \mu_{k3} & \dots & \mu_{kk} \end{array} \right|,$$

as Cayley had shown in 1845 (*Hist.*, ii. pp. 33–34). He is thus entitled to assert that for every homogeneous equation connecting determinants of the a-array there exists a perfectly similar equation connecting the corresponding determinants of the b-array, no matter what the elements of the modulus of substitution $|\mu_{1k}|$ may be. Then, on the ground that the k^2 elements of the modulus may be so chosen as to give definite values to k^2 of the b's and so leave only

$$nk - k^2$$

undetermined elements for the $(n)_k - 1$ determinant ratios to be dependent upon, he concludes that between the said ratios there exist (at most) $(n)_k - 1 - k\,(n-k)$ relations.

The relations themselves are obtained in Hamburger's manner by multiplying column-wise each of the determinants of the array

by the adjugate of the first of them, and, as we have already pointed out, are included in Bazin's theorem of 1851 (*Hist.*, ii. pp. 206–208). Thus, the array being

$$\left\| \begin{matrix} a_1 & b_1 & c_1 & f_1 & g_1 & h_1 \\ a_2 & b_2 & c_2 & f_2 & g_2 & h_2 \\ a_3 & b_3 & c_3 & f_3 & g_3 & h_3 \end{matrix} \right\|,$$

there are $(6)_3$ multiplications to be performed. The first is nugatory, the result being

$$|\, a_1b_2c_3\,| \cdot |\, \mathrm{A}_1\mathrm{B}_2\mathrm{C}_3\,| \;=\; |\, a_1b_2c_3\,|^3;$$

so also are the nine, $3(6-3)$, in which the multiplicand has two columns in common with $|\, a_1b_2c_3\,|$, for example,

$$\begin{vmatrix} a_1 & b_1 & f_1 \\ a_2 & b_2 & f_2 \\ a_3 & b_3 & f_3 \end{vmatrix} \cdot \begin{vmatrix} \mathrm{A}_1 & \mathrm{B}_1 & \mathrm{C}_1 \\ \mathrm{A}_2 & \mathrm{B}_2 & \mathrm{C}_2 \\ \mathrm{A}_3 & \mathrm{B}_3 & \mathrm{C}_3 \end{vmatrix} = \begin{vmatrix} |\, a_1b_2c_3\,| & . & . \\ . & |\, a_1b_2c_3\,| & . \\ |\, f_1b_2c_3\,| & |\, a_1f_2c_3\,| & |\, a_1b_2f_3\,| \end{vmatrix}.$$

Then come those in which the multiplicand has only one column in common with $|\, a_1b_2c_3\,|$, an example being

$$\begin{vmatrix} a_1 & f_1 & g_1 \\ a_2 & f_2 & g_2 \\ a_3 & f_3 & g_3 \end{vmatrix} \cdot \begin{vmatrix} \mathrm{A}_1 & \mathrm{B}_1 & \mathrm{C}_1 \\ \mathrm{A}_2 & \mathrm{B}_2 & \mathrm{C}_2 \\ \mathrm{A}_3 & \mathrm{B}_3 & \mathrm{C}_3 \end{vmatrix} = \begin{vmatrix} |\, a_1b_2c_3\,| & . & . \\ |\, f_1b_2c_3\,| & |\, a_1f_2c_3\,| & |\, a_1b_2f_3\,| \\ |\, g_1b_2c_3\,| & |\, a_1g_2c_3\,| & |\, a_1b_2g_3\,| \end{vmatrix},$$

which, on removing the factor $|\, a_1b_2c_3\,|$, becomes

$$|\, a_1f_2g_3\,| \cdot |\, a_1b_2c_3\,| \;=\; |\, a_1b_2f_3\,| \cdot |\, a_1c_2g_3\,| \;-\; |\, a_1b_2g_3\,| \cdot |\, a_1c_2f_3\,|,$$

and lastly there is

$$|\, f_1g_2h_3\,| \cdot |\, \mathrm{A}_1\mathrm{B}_2\mathrm{C}_3\,| \;=\; \begin{vmatrix} |\, f_1b_2c_3\,| & |\, a_1f_2c_3\,| & |\, a_1b_2f_3\,| \\ |\, g_1b_2c_3\,| & |\, a_1g_2c_3\,| & |\, a_1b_2g_3\,| \\ |\, h_1b_2c_3\,| & |\, a_1h_2c_3\,| & |\, a_1b_2h_3\,| \end{vmatrix},$$

in which so many as eleven of the determinants of the array are involved. The number of relations thus obtained is

$$(6)_3 - 1 - 3\,(6-3),$$

i.e.

$$10.$$

In proof of the assertion that the ten are mutually independent, it is adduced that each of them begins with a determinant which occurs in none of the others.

ZANTSCHEWSKY, I. M. (1894).

[Some theorems in determinants: (In Russian.) *Mat. Sbornik* . . . (Moscow), xvii. pp. 585–597.]

This may be viewed as a study of the subject of Sylvester's paper of 1851 (*Hist.*, ii. pp. 61–62), namely, the expression of a product as an aggregate of like products. The author takes up in order (1) the case where, in the obtaining of the expansion, only one row is repeatedly removed, (2) the case where a group of rows is removed, and (3) the case where the orders of the two factors are not the same.

FROBENIUS, G. (1894).

(See under this heading in Chap. XXI.)

MORLEY, F. (1894).

[Three notes on permutations. *Bull. New York Math. Soc.*, iii. pp. 142–148.]

An attractive paper embodying 'a plea for the chessboard in teaching determinants,' 'a special rule of signs,' and a commentary on 'the enumeration of positions' as dealt with by Cauchy in 1841 (*Hist.*, i. pp. 250–253). Under this last the author properly calls attention to a closely related paper of Jacobi's dated only ten days later than Cauchy's.*

HURWITZ, A. (1894).

[Zur Invariantentheorie. *Math. Annalen*, xlv. pp. 381–404.]

This bears indirectly on determinants at several points, for example, on Cayley's determinantal operator Ω (pp. 385–387); but perhaps only once directly, namely, under 'Potenztransformation,' where the theorem formulated (p. 391) regarding the determinant of such a transformation is in essence identical with Schläfli's of 1851.

* Jacobi, C. G. J. Zur combinatorischen Analysis. *Crelle's Journ.*, xxii. pp. 372–374: or *Gesammelte Werke*, iii. pp. 453–457.

MUIR, T. (1894).

[On the expressibility of a determinant in terms of its coaxial minors. *Philos. Magazine*, (5) xxxviii. pp. 537–541.]

MacMahon's theorem of the preceding year on this subject was an easy deduction from another, to the effect that *the number of relations between the coaxial minors of an* n*-line determinant is* 2^n-n^2+n-2; but his proof of the latter was rather lengthy. The want Muir now supplies in three steps of a quite elementary character. First there is the self-evident proposition: *If the rows of an* n*-line determinant be multiplied by* $x_1, x_2, \ldots, x_n$ *respectively and the columns be then divided by* $x_1, x_2, \ldots, x_n$ *respectively, the determinant is unaltered in value, and each of the minors of the new form is, save for a connecting factor, equal to the corresponding minor of the original form, the multiplier in question being*

$$x_p x_k x_l \ldots / x_r x_s x_t \ldots$$

if the minor belong to the h^{th}, k^{th}, l^{th}, ... *rows and* r^{th}, s^{th}, t^{th}, ... *columns.* To this there is the equally evident corollary that *the connecting multiplier in the case of the coaxial minors, as in the case of the whole determinant, is* 1: in other words, *the coaxial minors remain unaltered by the transformation.* Next, it is seen that *the* x*'s may be so chosen that all the elements of any row except the element common to the row and the diagonal shall be* 1. For example, after the multiplications and divisions, the first row being

$$a_{11}, \quad a_{12}\frac{x_1}{x_2}, \quad a_{13}\frac{x_1}{x_3}, \quad \ldots, \quad a_{1n}\frac{x_1}{x_n},$$

may be made to take the form

$$a_{11}, \quad 1, \quad 1, \quad \ldots, \quad 1$$

by giving the x's the values

$$1, \quad a_{12}, \quad a_{13}, \quad \ldots, \quad a_{1n}$$

in order. This means that 2^n-1 quantities, namely, the determinant and its coaxial minors, can be expressed in terms of $n^2-(n-1)$ others, namely, the modified elements which are not equal to 1; and therefore that after eliminating the latter we have 2^n-n^2+n-2 relations connecting the former. The determinant taken as an example is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & a & b \\ 1 & a^{-1} & 1 & c \\ 1 & b^{-1} & c^{-1} & 1 \end{vmatrix}.$$

Calling it Δ, and its four coaxial minors of the third order A, B, C, D, we have

$$A = a - 2 + \frac{1}{a}, \qquad B = b - 2 + \frac{1}{b},$$

$$C = c - 2 + \frac{1}{c}, \qquad D = \frac{ac}{b} - 2 + \frac{b}{ac},$$

$$\Delta = \frac{(1-a)(1-b)(1-c)(ac-b)}{abc},$$

whence one of the relations in question is found to be

$$\left.\begin{aligned} &\Sigma A^2B^2C^2 - \Sigma A^3BCD - 2\Sigma A^3BC + 4\Sigma A^2B^2C - 6\Sigma A^2BCD \\ &\quad + \Sigma A^4 - 4\Sigma A^3B + 4\Sigma A^2BC + 6\Sigma A^2B^2 - 40ABCD \end{aligned}\right\} = 0.$$

MUIR, T. (1894): NEUBERG, J. (1899).

[On a theorem regarding the difference between any two terms of the adjugate determinant. *Proceed. R. Soc. Edinburgh,* xx. pp. 323–327.]

[Question 14167. *Educ. Times,* lii. p. 199: *Math. from Educ. Times,* lxxii. pp. 39–40.]

The theorem is that the difference in question contains the original determinant as a factor. More than one proof is given. As an illustration of the most direct of them, applied to the case where the original determinant is $|\, a_1b_2c_3d_4e_5 \,|$, we have

$$\begin{aligned} A_5B_1C_4D_3E_2 - A_2B_4C_1D_5E_3 &= A_5B_1C_4D_3E_2 - A_5B_4C_1D_3E_2 \\ &\quad + A_5B_4C_1D_3E_2 - A_2B_4C_1D_3E_5 \\ &\quad + A_2B_4C_1D_3E_5 - A_2B_4C_1D_5E_3 \\ &= A_5D_3E_2\,|\,B_1C_4\,| + B_4C_1D_3\,|\,A_5E_2\,| + A_2B_4C_1\,|\,D_3E_5\,|, \end{aligned}$$

where $|\,B_1C_4\,|$, $|\,A_5E_2\,|$, $|\,D_3E_5|$ all contain $|\,a_1b_2c_3d_4e_5\,|$ as a factor, its cofactor in the full expression being

$$|\,a_2d_3e_5\,|\cdot A_5D_3E_2 + |\,b_1c_3d_4\,|\cdot B_4C_1D_3 - |\,a_1b_2c_4\,|\cdot A_2B_4C_1.$$

Attention is drawn to the form of the cofactor in the case where the original determinant is of the third order: and finally there is established the more general theorem that *the difference of any two*

terms of any minor of the adjugate determinant is a multiple of the original determinant.

NETTO, E. (1894).

[Erweiterung des Laplace's Determinanten-Zerlegungssatzes. *Crelle's Journ.*, cxiv. pp. 345–352.]

The ' Erweiterung ' in question is a rediscovery, being merely the result obtainable at once on applying the Law of Extensible Minors (1881) to Laplace's expansion-theorem ; and being, further, in the case where there are only two factors in the terms of the expansion, the result reached in similar fashion by Muir in 1879 (*Hist.*, iii. pp. 79–81). Netto properly views as deducible from it Sylvester's theorem of 1851 (*Hist.*, ii. pp. 60–61).

Of other results one established like the ' Erweiterung ' is specially noteworthy as giving the product of $n-1$ determinants of the n^{th} order as an aggregate of terms each of which is the product of n determinants of the $(n-1)^{\text{th}}$ order. For example, when n is 4, we have

$$| a_1b_2c_3d_4 | \cdot | \alpha_1\beta_2\gamma_3\delta_4 | \cdot | A_1B_2C_3D_4 |$$

$$= \begin{vmatrix}
a_1 & a_2 & a_3 & a_4 & . & . & . & . & . & . & . & . \\
b_1 & b_2 & b_3 & b_4 & . & . & . & . & . & . & . & . \\
c_1 & c_2 & c_3 & c_4 & . & . & . & . & . & . & . & . \\
a_1 & a_2 & a_3 & a_4 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & . & . & . & . \\
b_1 & b_2 & b_3 & b_4 & \beta_1 & \beta_2 & \beta_3 & \beta_4 & . & . & . & . \\
d_1 & d_2 & d_3 & d_4 & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & . & . & . & . \\
. & . & . & . & a_1 & a_2 & a_3 & a_4 & A_1 & A_2 & A_3 & A_4 \\
. & . & . & . & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & C_1 & C_2 & C_3 & C_4 \\
. & . & . & . & \delta_1 & \delta_2 & \delta_3 & \delta_4 & D_1 & D_2 & D_3 & D_4 \\
. & . & . & . & . & . & . & . & B_1 & B_2 & B_3 & B_4 \\
. & . & . & . & . & . & . & . & C_1 & C_2 & C_3 & C_4 \\
. & . & . & . & . & . & . & . & D_1 & D_2 & D_3 & D_4
\end{vmatrix}$$

$$= \sum | a_1b_2c_3 | \cdot | a_4\beta_1\gamma_2 | \cdot | \alpha_3\gamma_4D_1 | \cdot | B_2C_3D_4 |,$$

the number of terms here being $4 \cdot 6 \cdot 4$.

It may be worth adding that in his paper on orthogonants, a month or so earlier in date (*Acta Math.*, xix. pp. 105–114), Netto had already devoted a couple of pages to his extension of Laplace's theorem.

FEHR, H. (1895).

[Sur l'emploi de la multiplication extérieure en algèbre. *Nouv. Annales de Math.*, (3) xiv. pp. 74–79.]

The only fresh subject suggested here for treatment is certainly extensive enough, namely, the theory of invariants.

AHRENS, W. (1895, 1897).

[Ein neuer Satz über die Determinanten. *Zeitschrift f. Math. u. Phys.*, xl. pp. 177–180.]

[Ueber Beziehungen zwischen den Determinanten einer Matrix. *Zeitschrift f. Math. u. Phys.*, xlii. pp. 65–80.]

The new theorem referred to is Cayley's of 1843 (*Hist.*, ii. p. 15), namely, that *the vanishing of* n—m+1 *of the* m*-line determinants of an* m*-by-*n *array entails the vanishing of all.* What is new is the proof and the conditions attaching to it, the first condition being that the $n-m+1$ determinants have to be suitably chosen, and the second that each one of them must have a primary minor that does not vanish.

The second paper is considerably more important, containing a painstaking investigation of Sylvester's more general theorem (*Hist.*, ii. p. 52) in which the determinants concerned may be of a lower order, p say, than the m^{th}. An inquiry suggested by this is also dealt with, and special cases illustrative of both matters are given.

MUIR, T. (1895).

[Further note on a problem of Sylvester's in elimination. *Proceed. R. Soc. Edinburgh*, xx. pp. 371–382.]

This is a continuation of the paper * of 1892 on Sylvester's application of dialytic elimination to sets of ternary quadrics. It de-

* The said paper gave rise to a number of others in addition to Cayley's previously mentioned. What fresh interest they possess, however, does not centre in determinants. See *Proceed. R. Soc. Edinburgh*,

TAIT, P. G. xix. (1892), pp. 131–132.
MCLAREN, LORD. xix. (1893), pp. 264–265.
MUIR, T. xxi. (1896), pp. 220–234, 328–341.
NANSON, E. J. xxii. (1897), pp. 150–157, 353–358.

velops more fully the method there set forth, the set of equations being more general and the eliminant of the 12th degree. It must suffice to note that from one form of the latter there is obtained on specializing back to the original set the curious identity

$$\begin{vmatrix} a & h & g & af+2gh \\ h & b & f & bg+2hf \\ g & f & c & ch+2fg \\ af+2gh & bg+2hf & ch+2fg & abc+8fgh \end{vmatrix} = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}^2,$$

where the determinant on the left is equal to the square of one of its own primary minors.

STERNECK, R. D. v. (1895).

[Beweis eines Satzes über Determinanten. *Monatshefte f. Math. u. Phys.*, vi. pp. 205–207.]

Here the multiplication-theorem is used, as on p. 117 of Muir's text-book (1882), to prove Zehfuss' theorem of 1858. The proof is no longer sketchy or merely suggestive, but is fully worked out.

JENKINS, M. (1895): DAVIS, E. W. (1897).

[On a shortened rule for ascertaining the sign of a given term of a determinant: and on some problems in which the application of the rule occurs. *Messenger of Math.*, (2) xxv. pp. 60–68.]

[On the sign of a determinant's term. *American Journ. of Math.*, xix. p. 383.]

The main interest of Jenkins' paper lies in its first part (pp. 60–64): and although the 'new' rule is only a rediscovery of that given by Bellavitis in 1857 (*Hist.*, ii. pp. 96–97), the introductory discussion of sign-rules is sound and interesting.

Davis' rule is diagrammatic in character, and may therefore be classed with Tanner's of 1879.

AMIGUES, E. (1895).

[Théorème d'Algèbre. *Nouv. Annales de Math.*, (3) xiv. pp. 496–497.]

The so-called theorem would seem to be that if the elements of a determinant be letters carrying a single suffix, and the suffixes

of each row form an equidifferent progression, the difference being constant for all rows, the terms of the determinant are isobaric.

JACOBI, C. G. J.; STÄCKEL, P. (1896).

[Ueber die Bildung und die Eigenschaften der Determinanten. *Ostwald's Klassiker . . . Nr.* 77, pp. 1–49.]

This is a translation of the *De formatione . . .* of 1841. The editor appends six pages of useful notes (pp. 66–72).

GASCÓ, L. G. (1896): GUIMARÃES, R. (1897).

[Reglas prácticas para el desarrollo de las determinantes de cuarto grado. *Archivo de Mat.*, i. pp. 11–15.]

[Règle pratique pour développer les déterminants du 4^{me} ordre. *Assoc. franç . . . des Sci.* (St. Etienne), ii. pp. 129–131.]

Gascó's first rule does not really differ from Teixeira's of 1893, and the others are of less moment.

PASCAL, E. (1896).

[Sopra le relazioni fra i determinanti formati coi medesimi elementi. *Rendic. . . . Istituto Lombardo* (Milano), (2) xxix. pp. 436–438.]

This concerns the determinants arising from $|\ a_{1n}\ |$ by substituting for each of k columns a permutation of the elements of that column. The relation given includes Bagnera's second relation of 1886, and is obtained by the obvious expedient of utilizing a vanishing compound whose elements are primary minors of an oblong array.

PASCAL, E. (1896).

[Su di un teorema del sig. Netto relativo ai determinanti, e su di un altro teoremo ad esso affine. *Rendic. . . . Accad. dei Lincei* (Roma), (5) v. (1) pp. 188–191.]

Netto's theorem of 1894 is here properly viewed as merely the extensional of Laplace's expansion-theorem, and the mode of proof adopted is that originally used in establishing the general law of extensible minors. The other theorem is got by first applying the law of complementaries to Laplace's theorem and then taking the extensional of the result.

SCHICHT, F. (1896): BRAND, E. (1896).

[Beitrag zur Theorie der Determinanten. (Sch.-Bericht.) 2 pp. Prag-Altstadt.]

[Dérivée d'un déterminant. *Journ. de Math. Spéc.*, xx. pp. 102–104.]

The first of these merely re-establishes Cauchy's theorem of 1812 regarding the adjugate of a product. The second differs from Teixeira's of 1880 mainly in notation.

DERUYTS, J. (1896).

[Quelques propriétés du déterminant d'un système transformable. *Bull. . . . Acad. de Belgique*, (3) xxxii. pp. 433–445.]

The real interest here is connected with the Theory of Quantics.

BONOLIS, A. (1896).

[Sul prodotto delle matrici. *Giornale di Mat.*, xxxiv. pp. 375–9.]

The well-known theorem regarding the product of two m-by-n arrays is here established by first expressing the product as a determinant of the $(m+n)^{\text{th}}$ order and then transforming it into a unit determinant bordered by the two arrays (*Hist.*, ii. p. 200). The proof is fully worked out.

MUIR, T. (1896).

[The expression of any bordered skew determinant as a sum of products of Pfaffians. *Proceed. R. Soc. Edinburgh*, xxi. pp. 342–359.]

Three paragraphs of this (§§ 4, 5, 6, pp. 345–349) concern a theorem regarding determinants in general, namely, the expansion of any determinant in terms of any specified diagonal elements and determinants from which the said elements are excluded. For example,

$$|a_1b_2c_3d_4e_5| = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & . & c_4 & c_5 \\ d_1 & d_2 & d_3 & . & d_5 \\ e_1 & e_2 & e_3 & e_4 & . \end{vmatrix} + \Sigma(c_3 \begin{vmatrix} a_1 & a_2 & a_4 & a_5 \\ b_1 & b_2 & b_4 & b_5 \\ d_1 & d_2 & . & d_5 \\ e_1 & e_2 & e_4 & . \end{vmatrix})$$

$$+ \Sigma(c_3d_4 \begin{vmatrix} a_1 & a_2 & a_5 \\ b_1 & b_2 & b_5 \\ e_1 & e_2 & . \end{vmatrix}) + c_3d_4e_5 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix},$$

where the elements singled out are c_3, d_4, e_5. The mode of formation of the right-hand member is (1) to take the various combinations of c_3, d_4, e_5, namely,

$$1\,;\quad c_3,\ d_4,\ e_5\,;\quad c_3d_4,\ c_3e_5,\ d_4e_5\,;\quad c_3d_4e_5\,:$$

and (2) to take as cofactors of these their cofactors in the given determinant, only replacing in the latter the elements c_3, d_4, e_5 by 0 in every case.

The five different expansions of $|\,a_1b_2c_3d_4e_5\,|$ which arise from selecting e_5; d_4, e_5; c_3, d_4, e_5; b_2, c_3, d_4, e_5; a_1, b_2, c_3, d_4, e_5 are given, and the mode of deriving either of a consecutive pair of these expansions from the other is pointed out.

WELTZIEN, C. (1897).

[Über Produkte und Potenzen von Determinanten. Sch.-Progr. 23 pp. Berlin.]

[Über Potenzen von Determinanten. *Math. Annalen*, l. pp. 282–284.]

Weltzien here makes a further advance, his theorem now in regard to 'products' being that *the sum of the* k-*line coaxial minors of the determinant which is the product of* $|\,a_{1n}\,|$, $|\,b_{1n}\,|$, $|\,c_{1n}\,|$, . . . *is not altered when the factors are cyclically permuted.* For his other subject he takes advantage of a start made by d'Ocagne,* and establishes a recurrence-formula for the formation of the elements of any power of a determinant. *If we denote the* (h, k)th *element of* $|\,a_{1n}\,|^p$, $|\,a_{1n}\,|^{p-1}$, $|\,a_{1n}\,|^{p-2}$, . . . *by* e_p, e_{p-1}, e_{p-2}, . . . *respectively*, and *by* σ_r, *the sum of the* r-*line coaxial minors of* $|\,a_{1n}\,|$, *the formula is*

$$e_p = \sigma_1 e_{p-1} - \sigma_2 e_{p-2} + \sigma_3 e_{p-3} - \ldots .$$

The mode of procedure, which is carried out with patience and skill, is to begin with the simplest case and to advance from case to case until conviction is reached of the truth of the general theorem.

The so-called recurrence-formula, however, is simply the 'relation' established by Muir in 1885 and used by him in proving

* Mémoire sur les suites récurrentes. *Journ. de l'École Polyt.*, cah. 64 (1894), pp. 151–224.

Cayley's theorem in matrices. (See above, pp. 29-31.) Strictly speaking, indeed, it is included in Cayley's theorem, and therefore dates back to 1857.

NANSON, E. J. (1897).

[On the relations between the coaxial minors of a determinant. *Philos. Magazine*, (5), xliv. pp. 362–367.]

This is a marked advance on the second part of Muir's paper of 1894. (See above pp. 58-59.) By a different process of elimination the relations in question in the case of a general determinant of the 4th order are obtained in the form

$$\begin{Vmatrix} DL & CQ & BR & AQRL + 2BCD \\ CP & DM & AR & BRPM + 2CAD \\ BP & AQ & DN & CPQN + 2ABD \\ AL & BM & CN & DLMN + 2ABC \\ 1 & 1 & 1 & k \end{Vmatrix} = 0,$$

where the eleven letters A, B, . . . represent simple functions of the coaxials, and k is the one which involves the full determinant. The five equations are equivalent to only two that are mutually independent.

HATHAWAY, A. S. (1897).

[Alternate processes. *Proceed. Indiana Acad. of Sci.*, pp. 117–127.]

Note is taken of this because it is a return to the old subject of those alternating and symmetric functions whose individual terms are got from a typical term by making changes in the order of the variables,—the subject which Cauchy in 1812 took as the starting point of his first great memoir on determinants and to which he returned with fresh views in 1841. Here, however, the 'process' of formation includes the dividing of the aggregate by the number of terms. Actual determinants are of course reached when the typical term is of the form

$$\phi_1(x_1) \cdot \phi_2(x_2) \cdot \ldots \cdot \phi_n(x_n)$$

and the number of terms is $n!$.

TRAVERSO, N. (1897).

[Su una formola d'analisi combinatoria. *Periodico di Mat.*, xiii. pp. 43–47.]

The problem here dealt with is to find the number of ways in which it is possible from the elements of an m-by-n array to form a term containing h elements belonging to each row of the array and k to each column.

MUIRHEAD, R. F. (1897): A. C. (1897).

[Question 13651. *Educ. Times*, l. p. 434 : *Math. from Educ. Times*, lxix. pp. 52–54; lxxi. pp. 121–122.]

[Question 1139. *Mathesis*, (2) vii. p. 237 ; x. pp. 229–230.]

The subject of the first question here is that referred to above (p. 36) as having interested Sylvester in 1850 (*Hist.*, ii. p. 52), namely, the coevanescence of the k-line minors of a p-by-q array. A careful proof is given by Nanson of a theorem closely akin to Sylvester's, that *all of the minors will vanish if* $(p-k+1)(q-k+1)$ *of them, properly chosen, vanish.*

The identity involved in the second question is

$$|A_1^2B_2^2C_3^2| = |a_1b_2c_3|^2 \cdot \{|a_1^2b_2^2c_3^2| - 2|b_1c_1 \cdot c_2a_2 \cdot a_3b_3|\};$$

unfortunately the proof given is merely an unattractive verification.

MUIR, T. (1898).

[The relations between the coaxial minors of a determinant of the fourth order. *Transac. R. Soc. Edinburgh*, xxxix. pp. 323–339.]

This continues the subject of Nanson's paper of the preceding year, the *explicit* expression of $|a_1b_2c_3d_4|$ in terms of its coaxial minors being now effected. The process is based (§§ 2, 3) on a theorem, not recognized as Cunningham's of 1874, giving a devertebrated determinant in terms of the coaxial minors, namely,

$$\begin{vmatrix} . & a_2 & a_3 & a_4 & \dots \\ b_1 & . & b_3 & b_4 & \dots \\ c_1 & c_2 & . & c_4 & \dots \\ d_1 & d_2 & d_3 & . & \dots \\ . & . & . & . & \dots \end{vmatrix} = |a_1b_2c_3d_4\dots| - \Sigma\{a_1|b_2c_3d_4\dots|\} + \Sigma\{a_1b_2|c_3d_4\dots|\} - \dots,$$

it being thereby clear that a relation between the coaxial minors is assured as soon as we obtain an equation connecting the devertebrated coaxial minors and the diagonal elements. In this way it is found that

$$\begin{aligned} &|\, a_1b_2c_3d_4\,| \\ &= \Sigma a_1\,|\, b_2c_3d_4\,| + \Sigma\,|\, a_1b_2\,|\,|\, c_3d_4\,| - 2\Sigma a_1b_2\,|\, c_3d_4\,| + 6a_1b_2c_3d_4 \\ &+ \frac{\gamma_1\gamma_2}{2\beta_1} + \frac{\gamma_1\gamma_3}{2\beta_2} + \frac{\gamma_2\gamma_3}{2\beta_3} - \frac{1}{2\beta_1}\sqrt{\gamma_1{}^2 + 4\beta_1\beta_2\beta_4}\sqrt{\gamma_2{}^2 + 4\beta_1\beta_3\beta_5} \\ &+ \frac{1}{2\beta_2}\sqrt{\gamma_1{}^2 + 4\beta_1\beta_2\beta_4}\sqrt{\gamma_3{}^2 + 4\beta_2\beta_3\beta_6} \\ &- \frac{1}{2\beta_3}\sqrt{\gamma_2{}^2 + 4\beta_1\beta_3\beta_5}\sqrt{\gamma_3{}^2 + 4\beta_2\beta_3\beta_6}, \end{aligned}$$

where the β's are the devertebrated coaxial minors of the 2nd order, and the γ's are those of the 3rd order. As the substitutions still to be made are of the type

$$\beta_1 = |\, a_1b_2\,| - a_1b_2, \qquad \gamma_1 = |\, a_1b_2c_3\,| - \Sigma a_1\,|\, b_2c_3\,| + 2a_1b_2c_3,$$

and as the expression thus to be transformed produces ultimately only six terms of $|\, a_1b_2c_3d_4\,|$ its ungainliness is fully apparent.*

By reason of the form of the result it is suggested that Nanson's four-line determinant (see above p. 66) must be resolvable, and its resolution into four linear factors is actually effected (§§ 7–11), it being found possible to put it into the special bi-axisymmetric form known to us from Bellavitis (1857) and Ferrers (1861).†

In a later part of the paper (§ 17) the other relation, namely, that from which $|\, a_1b_2c_3d_4\,|$ is excluded, is given in the symmetrical form

$$\begin{aligned} 0 = \cos^{-1}\frac{\gamma_1}{2(-\beta_1\beta_2\beta_4)^{\frac{1}{2}}} + \cos^{-1}\frac{\gamma_2}{2(-\beta_1\beta_3\beta_5)^{\frac{1}{2}}} \\ + \cos^{-1}\frac{\gamma_3}{2(-\beta_2\beta_3\beta_6)^{\frac{1}{2}}} + \cos^{-1}\frac{\gamma_4}{2(-\beta_4\beta_5\beta_6)^{\frac{1}{2}}}. \end{aligned}$$

* These troublesome six terms are

$$-a_3b_4c_2d_1 - a_4b_3c_1d_2 - a_2b_3c_4d_1 - a_3b_2c_1d_4 - a_3b_1c_4d_2 - a_2b_4c_1d_3.$$

† See below in Chap. XXII. under Muir, 1898.

METZLER, W. H. (1898).

[A theorem in determinants. *American Journ. of Math.*, xx. pp. 273–276.]

From our account of Binet's memoir of 1812 Metzler extracts the equality (*Hist.*, i. p. 86)

$$\Sigma x_1 \cdot \Sigma |\, y_1 z_2 \,| + \Sigma y_1 \cdot \Sigma |\, z_1 x_2 \,| + \Sigma z_1 \cdot \Sigma |\, x_1 y_2 \,| = \Sigma |\, x_1 y_2 z_3 \,|$$

in regard to a 3-by-n array, and the equality (p. 92)

$$\Sigma t_1 \cdot \Sigma |\, x_1 y_2 z_3 \,| - \Sigma x_1 \cdot \Sigma |\, y_1 z_2 t_3 \,| + \Sigma y_1 \cdot \Sigma |\, z_1 t_2 x_3 \,| - \Sigma z_1 \cdot \Sigma |\, t_1 x_2 y_3 \,| = 0$$

in regard to a 4-by-n array: and taking a 4-by-6 array proves that

$$\Sigma M_2(1, 2) \cdot \Sigma M_2(3, 4) - \Sigma M_2(1, 3) \cdot \Sigma M_2(2, 4) + \ldots + \Sigma M_2(3, 4) \cdot \Sigma M_2(1, 2) = 2\Sigma M_4(1, 2, 3, 4),$$

where $\Sigma M_r(\alpha, \beta, \gamma, \ldots)$ is used by us temporarily to denote the sum of all the r-line minors formable from the rows $\alpha, \beta, \gamma, \ldots$. He then formulates a general theorem in regard to an m-by-n array, the left-hand member being an aggregate of $(m)_r$ products, of which the first factor is the sum of the r-line minors formable from r rows and the second the like sum formable from the $m-r$ complementary rows, and the right-hand member is a multiple of $\Sigma M_m(1, 2, \ldots, m)$.

ANISSIMOFF, W. (1898).

[Sur une formule nouvelle rélative aux déterminants, et son application à la théorie des équations différentielles linéaires. *Math. Annalen*, li. pp. 388–400.]

The 'formula' here carefully established (pp. 389–393) is Schläfli's of 1851 regarding a particular power of a determinant (*Hist.*, ii. pp. 52–53); and the 'application' is to a theorem already dealt with by Malmsten and Tissot regarding the particular integrals of a linear differential equation (*Hist.*, ii. pp. 220, 222).

TRAVERSO, N. (1898).

[Sopra la differenza di due determinanti qualunque. *Periodico di Mat.*, xiii. pp. 146–148.]

The author here again (see above, p. 67) proposes for himself a widely-stretched generalization, namely, from any determinant

$|\, a_{1n} \,|$ another determinant having been formed by changing a number of elements a_{hk}, a_{pq}, ... into b_{hk}, b_{pq}, ... he seeks to obtain an expression for the difference between the original and the derived determinant.

STUDNIČKA, F. J. (1898, 1899).

[O determinantní symbolice vůbec a pridružených determinantech vyšších řádů zolášt. *Věstník České Akad.* . . . , vii. pp. 73–81.]

[Dodatek k symbolice determinantní. *Věstník České Akad.* . . . , vii. pp. 159–164.]

[Beiträge zur Determinanten-Theorie. *Monatshefte f. Math. u. Phys.*, x. pp. 1–17.]

The notation proposed here for $|\, a_{1n} \,|$ is a complication of others already on record : ' adjugate ' is unfortunately used as equivalent to ' compound,' so that $\mathrm{A}(\Delta_n)$ comes to be proposed as the symbol for the adjugate of Δ_n, and $\mathrm{A}_h{}^k(\Delta_n)$ for the so-called ' adjugate ' of the h^{th} class and k^{th} order : and known theorems regarding such matters are tediously verified for the case where Δ_n is a simple 4-line or 5-line recurrent. The iteration of the symbol C, standing for " complementary of," is also dealt with, the result being

$$\left.\begin{aligned} \mathrm{C}^{2k}(a_{rs}) &= a_{rs}(\Delta_n)^{\frac{(n-1)^{2k}-1}{n}} \\ \mathrm{C}^{2k+1}(a_{rs}) &= \mathrm{A}_{rs}(\Delta_n)^{\frac{(n-1)^{2k}-1}{n}\cdot\frac{n-1}{n-2}} \end{aligned}\right\}.$$

As a further illustration, the adjugate of $|\, a^0b^1c^2 \ldots |_n$ is considered, and by removing from it the $(n-2)^{\text{th}}$ power of $|\, a^0b^1c^2 \ldots |_n$ a known equivalent of $|\, a^0b^1c^2 \ldots |_n$ is obtained.

STUDNIČKA, F. J. (1898).

[Nová poučka determinantní. *Věstník České Akad.* . . . , vii. pp. 167–169.]

The theorem in question is that *if* k *columns of a determinant be such that the elements of each of their rows form an arithmetical progression of the* $(\mathrm{k}-2)^{th}$ *order at most, the determinant vanishes* : for example, when k is 4, we have

$$\begin{vmatrix} 1 & 4 & 9 & 16 & a_5 & a_6 \\ 4 & 9 & 16 & 25 & b_5 & b_6 \\ 9 & 16 & 25 & 36 & c_5 & c_6 \\ 36 & 25 & 16 & 9 & d_5 & d_6 \\ 25 & 16 & 9 & 4 & e_5 & e_6 \\ 16 & 9 & 4 & 1 & f_5 & f_6 \end{vmatrix} = 0.$$

Strangely enough, the method adopted for verification is the repeated use of Chio's condensation-theorem of 1853.

STUDNIČKA, F. J. (1898).

[Jak vyjádří se subdeterminant k-tého stupně součinu dvou determinantů n-tého stupně pomocí příslušných subdeterminantů jejich. *Věstník České Akad.* . . . , vii. pp. 239–247.]

The outcome is the familiar expression obtained for a minor of the product-determinant from viewing the said minor as the product of two oblong arrays.

MASSIP, L. (1898): BITALE, I. B. (1898): HATZIDAKIS, J. N. (1898).

[Démonstration du théorème de Janni. *Journ. de Math. Spéc.*, xxii. pp. 102–103.]

[Γενικωτέρα μορφὴ τῶν θεωρημάτων τοῦ G. Fouret. Ἀθηνᾶ, x. pp. 401–408.]

[Περὶ γενικῶν τινῶν ἀναπτυγμάτων ὁριζουσῶν καὶ τῶν ἀροτάσεων τοῦ G. Fouret. Ἀθηνᾶ, x. pp. 558–559.]

The theorem dealt with in the first paper here is that of 1874 regarding change of sign in elements whose place-numbers are unlike. The other papers concern Fouret's rediscoveries of 1886.

WAELSCH, E. (1898).

[Ueber eine geometrische Behandlungsweise der Elemente der Determinantentheorie. *Monatshefte f. Math. u. Phys.*, ix. pp. 207–214.]

Not unnaturally the arrangement of the elements of a determinant in square array suggests that the *point* in geometry may be

looked on as an analogue of the *element*; but the extending of the analogy, as here attempted, leads to artificialities. The set of points corresponding to the elements which form a *term* is viewed in geometry as an entity called a '*Normaleck*': a 'Normaleck' is classed as odd or even according as an odd or even number of 'ascents' must be made in passing in a certain direction—north-west, say—from each point of it to every other point, an *ascent* being thus the analogue of a *derangement*: and so on. The outcome of thus pressing the analogy is not encouraging.

STEPHANOS, C. (1898).

[Sur un mode de composition des déterminants et des formes bilinéaires. *Giornale di Mat.*, xxxvi. pp. 376–379.]

There is here included another way of arriving at Zehfuss' theorem of 1858, but the theorem with which it is associated is formulated for the first time. In different phraseology it is that *each latent root of the* mn*-line product for* $|\,a_{1n}\,|^m \cdot |\,b_{1m}\,|^n$ *is the product of a latent root of* $|\,a_{1n}\,|$ *and a latent root of* $|\,b_{1m}\,|$.

NANSON, E. J. (1898, 1899).

[Questions 13925, 14258. *Educ. Times*, li. p. 328; lii. p. 301.]

The first proposition here is that *the* $(n)_4$ *equations of the type*

$$|\,a_1b_2\,| \cdot |\,a_3b_4\,| - |\,a_1b_3\,| \cdot |\,a_2b_4\,| + |\,a_1b_4\,| \cdot |\,a_2b_3\,| = 0$$

which connect the minors of the array

$$\begin{matrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{matrix}$$

are equivalent to only $(n-2)_2$ *equations.* The second proposition is that *the vanishing of all the* $(m+1)$*-line minors that involve the elements of a single non-vanishing* m*-line minor of* $|\,a_{1n}\,|$ *is dependent on* $\frac{1}{2}(n-m)(n-m+1)$ *conditions*, the proposer's fresh point being that the *m*-line minor need not be coaxial with $|\,a_{1n}\,|$.

MUIR, T. (1899).

[Determination of the sign of a single term of a determinant. *Proceed. R. Soc. Edinburgh*, xxii. pp. 441–477.]

This paper, extending to 37 pages, is an orderly arranged account

of the subject. After an explanatory and historical introduction (pp. 441–446) regarding five of the rules in use, a chapter is devoted to ' inverted-pairs ' (pp. 446–464), considerable space being given to the problem of finding in how many ($V_{n,\delta}$) of the permutations of n things the number of inverted-pairs is δ. Besides other matters the difference-equation of $V_{n,\delta}$ is found, a table of its values is given, and the problem is shown to be identical with two already known problems in combinations. ' Moves ' and ' interchanges ' are then shortly dealt with, and the nature of their connections with inverted-pairs made clear. Lastly, the properties of ' circular substitutions ' are treated of at some length (pp. 468–475), and especially those connecting them with interchanges. It may be noted too that under the first and fourth headings a subject which receives attention is that of conjugate permutations. For ease in reference the more important properties are carefully formulated throughout, and a two-page list of contents is given at the end.

MÜLLER, E. (1899).

[Beweis einiger Determinantensätze mittels der Grassmann'schen Ausdehnungslehre. *Zeitschrift f. Math. u. Phys.*, xliv. pp. 28–40.]

The proofs with which Scott and others have already made us familiar are supplemented here, the fresh theorems for treatment being the ' extensional ' of Laplace's expansion-theorem and Sylvester's theorem of March 1851.

LERCH, M. (1899).

See under this heading in Chap. VI.

STUDNIČKA, F. J. (1899).

[Nové doplňky nauky determinantní jakož i upotřebení jejího. *Věstník České Akad.* . . . , viii. pp. 59–67, 197–208.]

The first of the contributions here (pp. 59–61) really only amounts to saying that a determinant will vanish if one of its compounds vanishes ; and the fifth (pp. 197–199) discusses the consequences of the vanishing of the central secondary minor of a determinant.

STEPHANOS, C. (1899).

[Sur une extension du calcul des substitutions linéaires. *Comptes rendus . . . Acad. des Sci.* (Paris), cxxviii. pp. 593–596.]

In the account here given of the author's 'conjonction' and 'composition bi-alternée' of two bilinear forms, as contrasted with the ordinary 'composition' of Frobenius, several determinant theorems make their appearance, most of them, however, already made known by Frobenius, Rados, or Stephanos himself. (See above, p. 72.) Those of freshest interest concern latent roots.

MUIR, T. (1899).

[On a development of a determinant of the mn^{th} order. *Transac. R. Soc. Edinburgh*, xxxix. pp. 623–628.]

The main result here is the expression of a determinant of the mn^{th} order as an aggregate of determinants or of permanents of the m^{th} order, each of whose elements is a determinant of the n^{th} order. For example, when mn is 6 we have

$$|\, a_1b_2c_3d_4e_5f_6 \,| \;=\; \sum \begin{vmatrix} |\, a_1b_2c_3 \,| & |\, a_4b_5c_6 \,| \\ |\, d_1e_2f_3 \,| & |\, d_4e_5f_6 \,| \end{vmatrix},$$

and

$$|\, a_1b_2c_3d_4e_5f_6 \,| \;=\; \sum \begin{vmatrix} |\, a_1b_2 \,| & |\, a_3b_4 \,| & |\, a_5b_6 \,| \\ |\, c_1d_2 \,| & |\, c_3d_1 \,| & |\, c_5d_6 \,| \\ |\, e_1f_2 \,| & |\, e_3f_1 \,| & |\, e_5f_6 \,| \end{vmatrix}_{+},$$

there being included under the sign of summation 10 terms in the former case and 15 terms in the latter. As a first step in the establishment it is proved that *if the rows of a determinant of the* mn^{th} *order be separated by horizontal lines into* n *sets of* m *rows each, and the columns be similarly divided, the result may be viewed as roughly representing a compound determinant of the* n^{th} *order, each of whose elements is a determinant of the* m^{th} *order and a minor of the original determinant, and each term of the compound determinant thus arising will produce* $(\text{m}\,!)^{\text{n}}$ *terms differing at most only in sign from terms of the original determinant.* In the next place it is shown that the number of ways of breaking up mn things into n sets of m each is

$$(mn)!/(m!)^n n!\,.$$

In the third place a theorem regarding inverted-pairs is established which readily leads to the following: *If the first* mn *integers be separated into* n *successive groups of* m *each, say the groups*

$$A_1, A_2, \ldots, A_n,$$

then, when m *is even, any one of the* n! *permutations of* $A_1A_2 \ldots A_n$ *has an even number of inverted-pairs of the* mn *integers more than the standard permutation* $A_1A_2 \ldots A_n$ *has: and when* m *is odd, has an odd or even number of inverted-pairs more, according as the suffixes of the* A's *have an odd or even number of inverted-pairs.* On these three results the validity of the new expansion is based.

LOVETT, E. O. (1899).

[A theorem in determinants. *Annals of Math.*, xii. pp. 161–163.]

This is a much belated rediscovery, the theorem being the so-called Sylvester theorem of 1839 regarding an aggregate of products of pairs of determinants.

BES, K. (1899).

[Théorie générale de l'élimination, d'après la méthode Bezout, suivant un nouveau procédé. *Verhand. . . . Akad. van Wetens.* (Amsterdam), (1) vi. no. 7, 121 pp.]

This is a clearly and simply written exposition. The investigation of a given set of m linear homogeneous equations in n unknown x's is carried out with the help of the n similar equations in m unknown ξ's whose $(r, s)^{\text{th}}$ coefficient is the same as the $(s, r)^{\text{th}}$ coefficient in the given array. From this standpoint an array ('assemblant') is considered not to vary on the change of rows into columns, and the cases that fall to be examined are (1) that in which neither set of equations can be satisfied, (2) that in which only one can be satisfied, and (3) that in which both can be satisfied, —a division held to be equivalent to (1) $m=n$, (2) array non-evanescent, (3) array evanescent. The second chapter (pp. 34–58) concerns two binary quantics, the third concerns three ternary quantics, and the fourth generalizes.* Although the procedure here agrees in

* See also Bes, K., On the formation of the resultant. *Proceed. R. Acad. Amsterdam*, ii. (1899), pp. 85–88.

essence with Cayley's of 1848, and therefore cannot fairly be said to be new, it is manifestly the author's own, and the account of it does not, like Cayley's or even Salmon's, induce neglect from want of helpful detail.

NANSON, E. J. (1900).

[On certain determinant theorems. *Crelle's Journ.*, cxxii. pp. 179–185.]

Although this paper professes to show merely that most of Netto's theorems of 1894 are immediate corollaries of Muir's law of 1881, it in reality does much more. First of all, the Law of Complementaries and the Law of Extensible Minors are formulated and established anew. He then shows the effectiveness of these laws in arriving at determinant equalities, and in binding an assemblage of such equalities into a compact whole. The particular theorems which he so knits together by way of illustration are Laplace's expansion-theorem (1772), Cauchy's theorem regarding the adjugate (1812), Jacobi's regarding a minor of the adjugate (1831), Sylvester's regarding compounds (1851), Franke's regarding a minor of a compound (1862), Muir's extension of Laplace's theorem (1879), and Netto's similar extension (1894). He also makes his own contribution to the list, namely, an expansion like Laplace's for $|\, a_{1n}\,|^{r-1}$ when $r > 2$. This he obtains by taking the complementary of that form of Laplace's theorem in which each term consists of r factors; for example, knowing that

$$|\, a_1b_2c_3d_4e_5\,| \;=\; \sum\{|\, a_1b_2\,|\cdot|\, c_3d_4\,|\, e_5\},$$

he deduces

$$|\, a_1b_2c_3d_4e_5|^2 \;=\; \sum\{|\, a_1b_2c_3d_4\,|\cdot|\, a_1b_3e_5\;|\cdot|\, c_3d_4e_5\,|\}.$$

We may recall that by taking the extensional of this, say the extension f_6g_7, there is obtained

$$|\, a_1b_2c_3d_4e_5f_6g_7|^2\cdot|\, f_6g_7| \;=\sum\{|\, a_1b_2c_3d_4f_6g_7\,|\cdot|a_1b_2e_5f_6g_7\,|\cdot|c_3d_4e_5f_6g_7|\}$$

as observed by Pascal in 1896.

CHAPTER II.

TEXTS-BOOKS, FROM 1880 TO 1900.

ALTHOUGH the number of text-books and other publications of a text-book character is not appreciably different from what it was in the preceding period, there is considerable convenience to the reader in having them segregated from their former fellows, whose number has greatly increased. About a dozen languages are represented, being one or two fewer than before: on the other hand, Serbia appears for the first time with a good text-book in her own language, and Spanish works are effectively supplemented by one from Argentina. German still heads the list with double the number in either French or English.

Fortunately, in accordance with their dates, some critical papers on the subject of text-books, and generally on the teaching of determinants, fall to be noticed at the outset.

HOFFMANN, J. C. V., DIEKMANN, J., ERLER, ., BARDEY, E., GERLACH, (1880-1882): DÖTSCH, G. (1880): JENRICH, P. (1882).

[Determinanten oder nicht? Eine Gefahr! *Zeitschrift f. math. u. naturw. Unterricht*, xi. pp. 343–360; xii. pp. 84–85, 95–99, 193–196, 413–417, 425–427; xiii. pp. 111–114, 171–186, 356, 442–443.]

[Zum Unterricht in der Determinantenlehre. *Blätter f. d. bair. Gymn.-u.-Realschulwesen*, xvi. pp. 25–28.]

[Beiträge zur Methodik des mathematischen Unterrichts mit Berücksichtigung der Determinantenfrage. Sch.-Prog. 41 pp. Magdeburg.]

What we have here is a lengthened discussion of the position and value of determinants in the mathematical section of the school

curriculum, together with suggestions as to the best mode of introducing the subject to the learner. On all these points the writer of the leading paper shows himself pedagogically sound and experienced.

CARDENAS, A. RUIZ DE (1878): FRASCARA, G. A. (1880): HOPPE, R. (1880).

[Primi notizie intorno al calcolo dei determinanti. Annali del R. Ist. Tecn. e Naut., viii. pp. 87–117.]

[Sui Determinanti. 46 pp. Genova.]

[Elemente der Determinantentheorie. *Archiv d. Math. u. Phys.*, lxv. pp. 65–72.]

The first two writings are quite elementary expositions, not calling for any special note.*

Hoppe gives a very condensed presentation of the elements suitable for teachers.

KLEMPT, D. A. (1880).

[Lehrbuch zur Einführung in die moderne Algebra. xii+260 pp. Leipzig.]

The subjects dealt with are quite similar to those of Salmon's *Modern Higher Algebra*. The book, however, is intended for less advanced students; and to such, by reason of its fuller exposition and its large number of simple exercises, it ought to be serviceable.

SCOTT, R. F. (1880).

[A Treatise on the Theory of Determinants, and their applications to Analysis and Geometry. xi+251 pp. Cambridge.]

This is noteworthy as giving for the first time to the English-speaking student a reasonably full account of the subject. No branch of any note is neglected, even a chapter on cubic determinants being included.† At the end there is given a valuable

* For help in learning about them I am indebted to the kindness of Professors G. Loria and G. Lazzeri.

† Curiously enough Sylvester's important theorem of 1839, 1851 (*Hist.*, ii. pp. 61–62) has been found omitted from it.

collection of exercises, about a hundred in number, and most of them of the kind deserving record as being statements of more or less useful properties. There is appended also a bibliographical list containing about 200 titles. A special feature of the exposition is the prominence given to the use of so-called alternate numbers, the resolution of a determinant into a product of such numbers being employed at the very outset to evaluate the determinant

$$\begin{vmatrix} . & \cos(a_1+a_2) & \cos(a_1+a_3) & \dots \\ \cos(a_2+a_1) & . & \cos(a_2+a_3) & \dots \\ \cos(a_3+a_1) & \cos(a_3+a_2) & . & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n,$$

and being repeatedly recurred to later in the establishment of the general properties.

HESSE, O. (1880): MANSION, P. (1880): BALTZER, R. (1881): BRUNO, F. FAÀ DI (1881).

[Die Determinanten, elementar behandelt. (Translation into Polish by Żdziarski.) Warsaw.]

[Éléments de la Théorie des Déterminants, ... 3e éd. 64 pp. Mons.]

[Theorie und Anwendung der Determinanten. 5te verbesserte und vermehrte Auflage. vi+279 pp. Leipzig.]

[Einleitung in der Theorie der Binären Formen. . . . . Deutsch von T. Walter. viii+379 pp. Leipzig.]

Mansion very considerably enlarges and improves his German edition of 1878, the chapter on elimination, etc. (pp. 41–63), naturally receiving most attention.

The new edition of Baltzer, the last of all, has received only slight enlargement, but the same careful revision and amendment as its predecessor. Most of the alterations are in the section on 'applications.'

In the German edition of Bruno, produced with the help of Noether, there is quite a number of alterations and extensions, the value of the book being increased thereby.

GRÖLL, R. (1881): KAISER, H. (1882).

[Die Determinanten für den Schulgebrauch. Sch.-Progr. 20 pp. Quackenbrück.]

[Die Anfangsgründe der Determinanten, in Theorie und Anwendung. Zur Selbstbelehrung leichtfasslich bearbeitet. viii+41 pp. Wiesbaden.]

The first author here teaches by means of the simplest particular instances, and is not sparing in explanation: the second by means of general propositions accompanied by the minimum of illustration. Neither pamphlet seems to justify its existence in view of others of like kind.

BURNSIDE, W. S AND PANTON, A. W. (1881).

[The Theory of Equations, with an introduction to the theory of binary algebraic forms. (Chap. XI. Determinants, pp. 221–265.) xvi+387 pp. Dublin.]

This is in some points better than the usual chapter given in such a book. The exercises, for example, are chosen so as to be of value apart from the practice which they afford the reader, and sometimes involve fresh results.

THOMSON, W. (1881).

[An Introduction to Determinants, with numerous examples. For use in Schools and Colleges. viii+101 pp. Edinburgh.]

Another want felt in English is here supplied, namely, an elementary exposition accompanied by a variety and sufficiency of easy exercises to enable the less advanced student to handle determinants with reasonable facility.

MUIR, T. (1881).

[A Treatise on Determinants. With graduated sets of exercises. For use in Colleges and Schools. viii+240 pp. London.]

This is a students' text-book intended to make readers interested in the theory for itself, and not merely as a pleasant auxiliary to other branches of elementary mathematics. No separate chapter

or section is devoted to the 'applications': examples of such are only given at suitable points in illustration of the theory: geometry is not at all referred to. There is thus made possible a fuller statement of the theory, the 240 available pages being allocated to a simply-written chapter on determinants of the 2nd, 3rd and 4th orders (pp. 1–23), a chapter on general determinants (pp. 24–148), a chapter on special forms (pp. 149–227), and a chapter containing a historical and bibliographical summary. Nineteen sets of exercises are given at intervals,—436 exercises in all.

As an instance of fresh matter in regard to general determinants, we may note a proof of the Law of Complementaries (p. 142). M_r being used to denote an r-line minor of $|\, a_{1n} \,|$, the known equality can be put in the form

$$\mathrm{M}_r\mathrm{M}_s\mathrm{M}_t \ldots + \ldots = \mathrm{M}_\rho\mathrm{M}_\sigma\mathrm{M}_\tau \ldots + \ldots,$$

where of course $r + s + t + \ldots = \rho + \sigma + \tau + \ldots$. Now this, being true in regard to every determinant, must hold when the corresponding minors $\mathfrak{M}_r$, $\mathfrak{M}_s$, ... of the adjugate are substituted: and, as we have

$$\mathfrak{M}_r = |\, a_{1n} \,|^{r-1} \cdot \mathrm{M}'_{n-r},$$

where M'_{n-r} is the cofactor of M_r in $|\, a_{1n} \,|$, it follows that

$$|\, a_{1n} \,|^{r-1}\mathrm{M}'_{n-r} \cdot |\, a_{1n} \,|^{s-1}\mathrm{M}'_{n-s} \cdot |\, a_{1n} \,|^{t-1}\mathrm{M}'_{n-t} \ldots + \ldots$$
$$= |\, a^{1n} \,|^{\rho-1}\mathrm{M}'_{n-\rho} \cdot |\, a_{1n} \,|^{\sigma-1}\mathrm{M}'_{n-\sigma} \cdot |\, a_{1n} \,|^{\tau-1}\mathrm{M}_{n-\tau} \ldots + \ldots,$$

whence, on division by the lowest power of $|\, a_{1n} \,|$ contained in any term, there results the new identity predicated.

The subject, also, of *vanishing aggregates of products of pairs of determinants* receives (pp. 121–128) fuller attention than in any other text-book, Sylvester's theorem of 1839 being first carefully proved (§ 85), and then his generalization of 1851 (§§ 86, 87). The difficulty of specifying signs in the latter is got over by modifying the enunciation, namely, *The product of two determinants of the same order is equal to the sum of like products obtained from the original by interchanging* q *chosen columns of the one determinant with every set of* q *columns of the other in succession, the interchange of* q *columns with* q *columns being effected by interchanging the first column of the one set with the first column of the other, the second column of the one with the second of the other, and so on.*

SUAREZ A. Y GASCÓ, L. G. (1882): BACAS, D. Y ESCANDÓN, R. (1883).

[Lecciones de coordinatoria con las determinantes y sus principales aplicaciones. xvi+451 pp. Valencia.]

[Teoria elemental de las determinantes y sus aplicaciones al álgebra y á la trigonometria. 196 pp. Madrid.]

To the first Spanish text-book on determinants some special interest attaches, an unusually large portion of it being occupied with the auxiliary subject of 'coordinatoria,'—that is to say, combinations and permutations. The contents of the book are (1) seven lessons (146 pages) on 'coordinatoria,' (*b*) ten lessons (242 pages) on determinants, and (*c*) three short lessons on applications. In all cases the exposition is made with great care, and in fullness it even exceeds that of Dostor.

The other text-book is more ordinary in character, and less accurate. It takes over from Dostor two pages on 'déterminantes múltiples,' and some of its 'applications to trigonometry' are less appropriate than Dostor's.

SHAPOSHNIKOFF, . (1882).

[Elements of the Theory of Determinants. (In Russian.) 35 pp. Moscow.] *

MANSION, P. (1882, 1883, 1886): DOSTOR, G. (1883).

[Introduction à la Théorie des Déterminants, ... 2^{e} édition. 32 pp. Gand.]

[Éléments de la Théorie des Déterminants, ... 4^{e} édition. iv+80 pp. Paris.]

[Elemente der Theorie der Determinanten, ... 2te vermehrte Auflage. xxiv+56 pp. Leipzig.]

[Éléments de la Théorie des Déterminants, ... 2^{e} édition. xxxiv +361 pp. Paris.]

In these new editions Mansion enlarges and improves both the *Introduction* and the *Éléments*, the latter being now almost quite

* Not seen.

double its original size. Save for the arrangement, the German edition of 1886 does not differ from the French edition of 1883. Dostor's book is also slightly larger.

EGIDI, G., KRÖBER, ., BUNKOFER, W., VALENTA, J. (1883).

[Trattato Elementare dei Determinanti e loro Applicazioni, estratto dalle . . . xvi+196 pp. Roma.]

[Die Determinanten für den Unterricht bearbeitet. Sch.-Progr. 30 pp. Bischweiler.]

[Die ersten Elemente der Determinantentheorie. 27 pp. Tauberbischoffsheim.]

[The Theory of Determinants (in Czech). Sch.-Progr. Prag.]

The first of these is a well-intentioned contribution, but shows no advance on previous Italian text-books either in teaching skill or in freshness, being little more than a series of conscientiously framed definitions, theorems and corollaries. As a specimen we may note that the equality which for the 4th order is

$$2\begin{vmatrix} a & b & c & d \\ -a & b & x & y \\ -a & -b & c & z \\ -a & -b & -c & d \end{vmatrix} = 2^4abcd$$

is carefully formulated as a theorem, the wording of which occupies seven well-filled lines.

The second is a fair specimen of the school 'program' of the purely class-room type. The third is equally elementary, but less workmanlike. The fourth has not been seen.

LEBOULLEUX, L. (1880, 1884).

[Traité élémentaire des déterminants. 175 pp. Paris.]*

[Traité élémentaire des déterminants, contenant 299 exercices. viii+184 pp. Genève.]

A very elementary 'treatise.' To two-line determinants 18 pages are devoted, to three-line 33, to n-line 66, and then follow 64 pages

* This I have only seen referred to. The other makes no mention of a previous edition.

of mixed matter, mainly worked-out exercises. The author is a willing and profuse guide, but not always safe.

LONGCHAMPS, G. DE (1883).

[Algèbre. (Déterminants, etc., pp. 64–111, 637–664.) xii+671 pp. Paris.]

Longchamps' exposition is of the type usual in such a book, save as regards the very full consideration given to linear equations (pp. 88–111).

ALBUQUERQUE, J. A., SCHRADER, ., REUSCHLE, C. (1884): KAISER, H. (1885).

[Primeiros principios da theoria dos determinantes, para usos dos lyceos. viii+55 pp. Paris.]

[Die Determinanten im Schulunterricht, und ihre Anwendung auf Gegenstände der analytischen Geometrie. Sch.-Progr. 23 pp. Halle a. S.]

[Determinantentheorie. Erstes Kapitel, Einführung in die Determinantentheorie. 16 pp. Stuttgart.]

[Die Determinanten für den ersten Unterricht in der Algebra. 23 pp. Wiesbaden.]

The first of these is an elementary text-book with suitable exercises, somewhat resembling in its aims Mansion's *Éléments*, and therefore likely to be useful for Portuguese schools. The second is a school-program with only six pages expository of determinants, and fourteen devoted to coordinate geometry. The third and fourth resemble school-programs of the most elementary type.

SALMON, G. (1885).

[Lessons introductory to the Modern Higher Algebra. 4th ed. xvi+360 pp. Dublin.]

To this, the last, edition the most noticeable extensions concern binary quantics: in determinants proper only about a score of exercises are added. In connection with Schläfli's expressions for $|a_1b_2|^3$, $|a_1b_2|^6$, ... it is pointed out how, from considering the eliminant of

$$\left.\begin{aligned}(b_1x+b_2y)(c_1x+c_2y) &= 0\\ (c_1x+c_2y)(a_1x+a_2y) &= 0\\ (a_1x+a_2y)(b_1x+b_2y) &= 0\end{aligned}\right\},$$

a known identity may be reached: also how the 3-line minors of the array

$$\left\|\begin{array}{cccccc} a_1^2 & b_1^2 & c_1^2 & b_1c_1 & c_1a_1 & a_1b_1 \\ 2a_1a_2 & 2b_1b_2 & 2c_1c_2 & b_1c_2+b_2c_1 & c_1a_2+c_2a_1 & a_1b_2+a_2b_1 \\ a_2^2 & b_2^2 & c_2^2 & b_2c_2 & c_2a_2 & a_2b_2 \end{array}\right\|$$

are ascertained to be ternary products of $|\,b_1c_2\,|$, $|\,c_1a_2\,|$, $|\,a_1b_2\,|$.

SICKENBERGER, A. (1885): TARTINVILLE, A. (1885, 1886).

[Die Determinanten in genetischer Behandlung. Eine Einführung in die Lehre von den Determinanten. Sch.-Prog. 80 pp. München.]

[Première leçon sur les déterminants. Paris.]

[Seconde leçon sur les déterminants. Paris.]

This school-program of the expository type is better and much more extensive than any of its immediate predecessors. The author devotes 12 pages to determinants of the second degree, 38 to those of the third degree, and 30 to determinants in general. Though explanation may be overdone, the method is in general sound.

The two 'leçons' I have not seen.

GORDAN, P. (1885).

[Vorlesungen über Invariantentheorie. Herausg. von Dr. Georg Kerschensteiner. Erster Band: Determinanten. xii+202 pp. Leipzig.]

This volume is divided equally between 'theory' and 'applications.'

The first half, though freshly written, shows no improvement on previous expositions, the explanations and illustrations being sometimes a little overdone, and yet not too effective. While dealing with the transposition of rows, attention is given (pp. 28–31) to the problem of transforming a determinant so as to have in the diagonal

any given set of elements that may form a term,—a matter at the basis of such proposals as those of Bonolis, Teixeira, etc.

The second half is considerably more interesting, the treatment of the solution of linear equations (pp. 101–120) being very readable, and the improvement being still more marked in the chapters dealing with the resultant of two binary quantics (pp. 131–166, 174–186) and with the discriminant (pp. 185–196).

CARR, G. S. (1886): HAGEN, J. G. (1891).

[Determinants, ... *Synopsis of Elem. Results in Pure Math.*, pp. 144–158, ...]

[Theorie der Determinanten, ... *Synopsis der höheren Math.*, i. pp. 215–228, 189–193, ...]

The interest of the first of these synopses does not lie in the necessarily meagre collection of 'results,' but in an unexpected subject-index (pp. 841–935) to the mathematical papers published in thirty-two serials during the preceding part of the century, also in a table showing at a glance the current volume-number of any one of the serials for each year from 1801 to 1885.

The second synopsis is meant for more advanced students, and is most carefully drawn up: in usefulness, however, it naturally falls short of a good text-book.

YOUNG, T. E. (1886): JANNI, V. (1886): DÖLP, H. (1887): HATTENDORF, K. (1887): SICKENBERGER, A. (1887).

[A note on determinants. *Journ. Inst. of Actuaries*, xxvi. pp. 149–152.]

[Lezioni di Algebra Complementare. Terza edizione, corretta ed ampliata. (viii)+272 pp. Napoli.]

[Die Determinanten . . . . 3te Auflage. 94 pp. Darmstadt.]

[Einleitung in die Lehre . . . . 2te Ausgabe. 60 pp. Hannover.]

[Die Determinanten . . . . 2ter Abdruck. viii+80 pp. München.]

The first of these is an expository trifle. The second is an improved and enlarged edition of a text-book first published in 1876, but known to us only from being quoted by Günther and Pascal (*Hist.*, iii. p. 460). The others are practically reprints.

CAPELLI, A. E GARBIERI, G. (1886).

[Corso di Analisi Algebrica. Vol. I. Teorie Introduttorie. vii+511 pp. Padova.]

The fourth chapter of this 'Course' (pp. 273–363) is devoted to the pure theory of determinants. It is sound in quality, like other previous Italian expositions of the same type—Rubini's of 1863, for example, and Janni's of 1876—gives proper evidences of improvement and progress, and though merely part of a book on general algebra is almost as full in its treatment as special text-books of a few years before. A portion also of the next chapter ('Systems of Linear Forms') is, strictly speaking, an addendum, the subject being the evanescence of arrays (pp. 387–400).

HANUS, P. H. (1886): PECK, W. G. (1887).

[An Elementary Treatise on the Theory of Determinants. A text-book for colleges. viii+218 pp. Boston, U.S.A.]

[Elementary Treatise on Determinants. iv+47 pp. New York and Chicago.]

Although, avowedly, the author of the first of these text-books makes free use of a number of previous works of the same kind, and specially mentions those of Muir and of Burnside and Panton, the result is a fresh whole, more homogeneous, indeed, than the works of some authors who are less frank. It consists of an introductory chapter of 16 pages, a chapter of 65 pages on general determinants, and a long chapter readily separable into two, namely, one on the use of determinants in dealing with a variety of questions regarding equations (pp. 82–146), and the other on determinants of special form (pp. 146–217). We may note in passing that in the last of these, under the head of continuants, there is reproduced from Günther a so-called proof by means of determinants that

$$\frac{a}{b+\frac{c}{d}} = \frac{ma}{mb+\frac{mc}{d}}.$$

Speaking generally, this first home-made text-book on determinants would be found by United-States students sound and serviceable.

The other book is a quite elementary exposition, and, though filling a gap in America, is of merely local interest.

SCHRADER, W. (1886).

[Beiträge zur Theorie der Determinanten. Neue Sätze und eine neue Bezeichnung. vi+156 pp. Halle a. S.]

Notwithstanding the title, this is in form and reality a *text-book*, the first two chapters (pp. 1–75) dealing with determinants in general, and the third (pp. 75–141) with special types, numbering about a dozen in all. There is also an appendix (pp. 152–155) on the division of one determinant by another. The treatment is not quite commonplace. Indeed, in the title and preface the author makes definite claims to originality, and, though this cannot in general be sustained, it is clear that he sought out paths and results for himself.

His full-length notation for the n-line determinant, which has double-suffixed a's for elements, is

$$\mathrm{D}\,(a_{i,k})^{i_1, i_2, \ldots, i_n}_{k_1, k_2, \ldots, k_n},$$

where the i's are the row-suffixes and the k's the column-suffixes; but he usually shortens it into

$$\overset{n}{\mathrm{D}}\,(a_{i,k}),$$

when the suffixes are in both cases $1, 2, \ldots, n$.

His new theorems on general determinants mainly concern the adjugate, and are to be found in the second chapter (pp. 53–75). Of the few requiring notice, one is in effect that *if* $|\,a_{1n}\,|$ *be multiplied row-wise by* $|\,b_{1n}\,|$, *and the product be multiplied column-wise by the adjugate of the former, the* $(r, s)^{th}$ *element of the result is equal to* $b_{rs}\cdot|\,a_{1n}\,|$. With a superfixed comma to denote the operation of changing rows into columns, it is the same thing to say that *the* $(r, s)^{th}$ *element of*

$$\{|\,a_{1n}\,|\cdot|\,b_{1n}\,|\}'\cdot|\,A_{1n}\,|'$$

is equal to $b_{rs}\cdot|\,a_{1n}\,|$, or that *the elements of* $\{|\,a_{1n}\,|\cdot|\,b_{1n}\,|\}'\cdot|\,A_{1n}\,|'$ *and* $|\,b_{1n}\,|$ *are proportional.* By way of proof it is only necessary to write the $(r, s)^{th}$ element of the product in the form of a so-called bipartite function. It will then also be apparent that the proposition is not affected by interchanging $|\,a_{1n}\,|$ and $|\,A_{1n}\,|$, a fact which, if noted by the author, would have rendered one of his other

propositions (p. 61) unnecessary. He fails too sometimes to recognize an old theorem in a new guise; for example, the fact (p. 73) that the elements of the $(n-m)^{\text{th}}$ compound of $|\, a_{1n} \,|$ are proportional to those of the m^{th} compound of $|\, A_{1n} \,|$ is merely Jacobi's theorem regarding a minor of the adjugate. Lastly, we may note a proposition which is an immediate consequence of the multiplication of two oblong arrays, namely, *If* m *rows of* $|\, a_{1n} \,|$ *and* m *rows of* $|\, A_{1n} \,|$ *be taken, and each determinant of the former array be multiplied by the corresponding determinant of the latter, the sum of the products equals* $|\, a_{1n} \,|^{m}$ *or* 0, *according as the rows taken in the second case are or are not the same selection as in the first case.*

As for the division-problem above referred to, the kind of solution obtained will be understood from the statement that M *and* N *being any* n*-line determinants, the* $(r, s)^{th}$ *element in the quotient of* M by N *is got by replacing the* r^{th} *row of* M *by the* s^{th} *row of* N *and dividing the result by* N. Any want of value in this as a practical solution may be held compensated for by the fact that the outcome is the theorem in compound determinants originally discovered by Bazin in 1851. Schrader illustrates it, and indeed almost all the others, by purely arithmetical examples.

SCHEIBNER, W. (1887).

[Mathematische Bemerkungen. *Berichte . . . Ges. d. Wiss. zu Leipzig: math.-phys. Cl.*, xxiv. pp. 1–13.]

These are unimportant extracts from letters addressed to Baltzer in the early part of the year of his death (1887) and containing suggested amendments to his 5^{th} edition, the passages concerned being § 6_8, § 7_2, § 8_2, § 11_8.

AMORETTI, F. y MORALES, C. M. (1888).

[Teoría elemental de las determinantes, y sus principales aplicaciones al álgebra y la geometria. xviii+180 pp. Buenos Ayres.]

A very useful text-book for Argentine students, the authors being pupils of Valentin Balbin and following the lines of his university lectures. In extent it is similar to that of Bacas and Escandon of 1883, but in other respects it differs considerably for the better.

NEWMAN, F. W. (1888).

[Mathematical Tracts. Part I. Tract IV. On superlinears, pp. 35–53. 80 pp. Cambridge.]

The 'tract' is a short elementary exposition, simply and clearly written; 'superlinears' are determinants: and a 'bilinear tablet' is a determinant of two rows.

HORTA, F. P. (1889).

[Estudo elementar dos determinantes, sequido de uma parte complementar relativa, principalmente, aos determinantes funcionnaes. 176 pp. Lisboa.]

This supplies Portuguese students with a reliable text-book of a more advanced type than Albuquerque's of 1884. Unfortunately, it does not contain a single example for practice. Exactly half of the whole is devoted to 'applications.'

WENTWORTH, G. A., ETC. (1889).

[Algebraic Analysis. Solutions and exercises, illustrating .... Part I. x+418 pp. Boston, U.S.A.]

What is of note here is that in addition to 168 exercises given in sets at intervals to test the learner's progress there is at the end a careful compilation of 300 more not confined to determinants, but dealing also with the so-called applications.

POWEL, A. (1888): DIEKMANN, J. (1889).

[Anwendung der Determinanten in der Schule. Sch.-Progr. 18 pp. Gumbinnen.]

[Anwendung der Determinanten und Elemente der neueren Algebra auf dem Gebiete der niederen Mathematik. 111 pp. Leipzig.]

Diekmann's attitude towards the inclusion of determinants in the school course we have already noted (*Hist.*, iii. p. 59). The present is another effort to show what effective applications can be made of even a very elementary knowledge of the theory.

The other writing I have not seen, but it is said to be of comparatively trivial importance among 'programs.'

PRADO, G. FERNANDEZ DE (1890, 1902).

[Elementos de la teoría de los determinantes, y sus aplicaciones á la eliminación y á la teoría de las formas. (?) pp. Madrid.]

[Elementos. . . . 2ª edición, revisada y ampliada. xii+324 pp. Madrid.]

A clear but stiffly formal exposition, done carefully to order, with no unfamiliar material, and with no exercises for practice. Only a third of the book concerns the theory, and the last third is devoted to the algebra of quantics. The first edition I have not seen.

KNESER, A. (1891).

[Ueber eine Methode zur Darstellung der Determinantentheorie. *Sitzungsb. d. Natur f. Ges.* . . . (Dorpat), ix. pp. 522–532.]

The method is one that has already been more than once brought forward, probably with the greatest fullness by Matzka in 1877. The proofs, including that of the multiplication-theorem, are gradational.

LUCAS, E. (1891).

[Théorie des Nombres. I. xxxiv+520 pp. Paris.]

A brief account of determinants and linear equations occupies chap. xvi. (pp. 281–298): the applications to the Theory of Numbers are fewer and more commonplace than might have been expected, considering the author's devotion to the subject.

CARNOY, J. (1892).

[Cours d'Algèbre Supérieure. xii+537 pp. Louvain.]

The first part (pp. 1–65) is a clear exposition of the theory of determinants, and the two others, dealing with the theory of equations and the theory of binary forms, furnish a number of applications.

MILLER, G. A. (1892): WELD, L. G. (1893).

[Determinants. An introduction to the study of, with examples and applications. (iii)+101 pp. New York.]

[A Short Course of the Theory of Determinants. xiv+298 pp. London.]

The first of these belongs to a series of pocket booklets on technical science, where, doubtless, simplicity and clearness of exposition were held to be the most important requisites.

The second is a text-book not unlike others already in English—for example, those of Muir and Hanus—but aiming at still further simplification of the exposition and exercises.

WEICHOLD, G. (1893).

[Lehrbuch der Determinanten und deren Anwendungen. Erster Theil . . . xvi+380 pp. Stuttgart.]

This is a text-book of no ordinary type. Instruction is conveyed in it by question-and-answer with the aid of numerous solved and unsolved exercises and a wealth of explanations. The page is double-columned: the questions are placed in the first column, the answers in the second, and both are printed in the larger of the two types. The answers being much longer than the questions, the remaining space in the left-hand columns is packed with 'explanations' in a smaller type and with the statement of the exercises. The solutions of the latter are intended for the right-hand column, but often fail to be confined. The thoroughness of the system (the 'System Kleyer') is admirable, the 380 carefully printed pages containing about as much matter as 700 pages of an ordinary book such as Hesse's 'Determinanten.' Much of it, however, is of very doubtful utility: for example, the resolution of

$$\begin{vmatrix} \sin\alpha & \sin 2\alpha & \tan\alpha & \sin^2\alpha \\ \sin\beta & \sin 2\beta & \tan\beta & \sin^2\beta \\ \sin\gamma & \sin 2\gamma & \tan\gamma & \sin^2\gamma \\ \sin\delta & \sin 2\delta & \tan\delta & \sin^2\delta \end{vmatrix}$$

into factors occupies $2\frac{1}{2}$ full pages, and is still curiously unfinished.

KORCZYNSKI, J. (1892): OTT, A. (1893): DÖLP, H. (1893).

[Elementary Theory of Determinants. (In Polish.) Sch.-Progr. 42 pp. Cracow.]

[Ueber Determinanten. Sch.-Progr. 26 pp. Weimar.]

[Die Determinanten, ... 4te Aufl. iv+95 pp. Darmstadt.]

The first two are 'programs' of the class-room type: the third is almost a complete reprint.

GARBIERI, G. (1893).

[Teoria ed applicazioni dei determinanti. 100 pp. Reggio d'Emilia.]

This has been somewhat incorrectly viewed as a second edition of Garbieri's text-book of 1874: it is rather a new work, drawn up with the help of the old, on the lines of his lectures at Genoa.

CESÀRO, E. (1894).

[Corso di Analisi Algebrica, con . . . . viii+500 pp. Torino.]

We have here again a clear and readable account of determinants, occupying the opening chapters (pp. 1–51, . . .) of a book on general algebra. It is naturally, however, less full than that given by Capelli and Garbieri in the first volume of their work bearing the same title (see above, p. 87).

WEBER, H. (1895, 1898, 1898): NETTO, E. (1896, 1900): WELD, L. G. (1896).

[Lehrbuch der Algebra. Bd. I. xv+653 pp. Braunschweig. 2te Aufl. xvi+704 pp. (1898).]

[Traité d'Algèbre Supérieure. Traduit . . . par J. Griess. xi+704 pp. (1898.) Paris.]

[Vorlesungen über Algebra. Bd. I. x+388 pp. Bd. II. iv+519 pp. Leipzig.]

[Merriman and Woodward's Higher Mathematics, pp. 33–69.]

Weber, in seeking to produce an up-to-date work on the lines of Serret's *Cours d'Algèbre Supérieure* devotes, like Serret, a chapter (II.) to determinants, and uses them freely afterwards in dealing

with other subjects. His chapter (IV.) on symmetric functions is especially readable.

Netto has no lecture specially given up to the subject, but for all the so-called 'algebraical applications' his book deserves to be carefully consulted.

Weld's article aims at giving in short compass to fairly advanced readers a general idea of the subject.

CAPELLI, A. (1895, 1898).

[Lezioni di Algebra Complementare, ad uso. . . . xii+527 pp.: 2da ediz. xvi+680 pp. Napoli.]

In view of the character of Capelli's work with Garbieri on this subject (1886) one is not surprised to find here a superior text-book, clear in exposition and fresh in material, nay even occasionally original. A chapter, the third, is wholly devoted to determinants, and there are also of course the usual applications.

PASCAL, E. (1897).

[I Determinanti: teoria ed applicazioni: con tutte le più recente ricerche. viii+330 pp. Milano.]

A very attractive pocket text-book, with pages only a trifle larger than a post-card, but 330 of them, and almost none of them occupied with Geometry. It is divided into two strikingly unequal parts, one of 48 pages on the fundamental properties, and one of 282 pages on special researches and applications; and although, of course, the latter does not really contain 'all the most recent researches' nor go very deeply into any one subject, it is surprising to find how many comparatively fresh matters are touched on and attributed with fair accuracy to the original writers. There are no exercises.*

STOFFÄES, . (1897): DÖLP, H. (1899).

[Théorie des déterminants. Lille.]

[Die Determinanten, ... 5te Aufl. iv+95 pp. Darmstadt.]

* Another noteworthy feature is the price. For 3 lire (about half-a-crown) Italian students receive better value than those of any other nationality. Even the German translation costs about four times as much (10 marks).

LIEBER, H. AND MÜSEBECK, C.;
BUDISAVLJEVIČ, E. (1898).

[Aufgaben über kubische und diophantische Gleichungen, Determinanten und Kettenbrüche, Combinationslehre und höhere Reihen. vi+129 pp. Berlin.]

[Grundzüge der Determinanten-Theorie und der projectivischen Geometrie. x+492 pp. Wien u. Leipzig.]

The section devoted to our subject in the first of these books is § 4 (pp. 16–33). The treatment is quite elementary.

The second book I have not seen.

NETTO, E. (1898).

[Kombinatorik. *Encykl. d. math. Wiss.*, i. pp. 28–46.]

To give within the space of $10\frac{1}{2}$ pages an account of determinants, likely to be of any real use, would seem to be an impossibility, and the task is not made easier if a bibliography is to be included. Nevertheless the effort is here made, and the result might easily have been less worthy. In such a matter, however, as accuracy in the references and in the history, there is unfortunately room for considerable amendment.

GAVRILOVITCH, B. (1899).

[Teoriya Determinanata. xii+278 pp. Beograd.]

A workmanlike text-book, written with a wide and accurate knowledge of others, well chosen as to matter, and up-to-date in construction. Certain to be valuable to Serbian students, and not unworthy of examination by expositors outside Serbia.

BURNSIDE, W. S. AND PANTON, A. W. (1899).

[An Introduction to Determinants, being a chapter from *The Theory of Equations.* 84 pp. Dublin.]

The text-book by the same authors on the Theory of Equations (1881) we have already noted at the proper place. In 1886 a second edition of it appeared with the chapter on determinants increased to 60 pages, and in 1892 a third edition with a further increase of

5 pages. The booklet we have now reached is a reprint from the fourth edition, which shows again considerable growth and improvement. The separate publication of the chapter gives to English-speaking readers for the first time something comparable with Mansion's *Éléments*.

MANSION, P. (1899, 1900).

[Introduction à la théorie des déterminants. 3^{e} éd. 40 pp. Gand.]

[Einleitung in die Theorie der Determinanten, für Gymnasien und Realschulen. Aus der dritten französischen Auflage übersetzt von B. J. Clasen. 40 pp. Leipzig.]

[Elemente der Theorie der Determinanten. . . . Dritte vermehrte Aufl. 103 pp. Leipzig.]

[Éléments de la théorie des déterminants. . . . 6^{e} éd., revue et augmentée. iv+91 pp. Paris.]

Still further improvements and extensions have been made to the 'Eléments,' and the German edition is now closely assimilated to the French.

STUDNIČKA, F. J. (1899).

[Úvod do nauky o determinantech. *Sbornik Jednoty Českých math. v Praze*, Číslo iii. 230 pp.]

This is quite different from Studnička's text-book of 1869 (*Hist.*, iii. p. 29). In the first place it is almost four times as large, and in the second it gives liberal space to the author's own writings of the preceding thirty years. The latter feature is more pronounced in the case of the second of the three sections, which deals with determinants of special form. By its clear and detailed exposition the book is well calculated to spread a knowledge of the subject among Czech-speaking students,—a fact, doubtless, which led to its receiving the imprimatur of the Society of Czech Mathematicians.

CHAPTER III.

DETERMINANTS AND LINEAR EQUATIONS, FROM 1880 TO 1898.

THE number of writings to be reported on here is about a half more than for the period of the preceding volume : on the whole, however, the importance of them is less.

MANSION, P. (1880, 1885).

[Sur un système d'équations linéaires. *Nouv. Corresp. Math.*, vi. pp. 30–32.]

[Note sur la méthode des moindres carrés. *Bull. . . . Acad. R. de Belgique*, (3) ix. pp. 9–14.]

Mansion's first result, which he establishes by means of determinants,* we may formulate thus : If from a given set of linear equations in $x, y, z, \ldots$ there be formed the so-called 'normal' set, and if between the first and each of the others of the normal set x be eliminated, the third set so obtained is the same as would be got by an almost exact reversal of the operations, namely, eliminating x in every possible way from the given set, and then forming the normal set of the set resulting.

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* To the previous list of writings which use determinants in dealing with the Method of Least Squares there fall to be added :

1880. GLAISHER, J. W. L. *Monthly Notices . . . R. Astron. Soc.*, xl. pp. 600–614; xli. pp. 18–23.

1884. ZBROZEK, D. *Pamietnik akad. . . . Krakowie*, ix. pp. 199–218.

1885. NEKRASSOFF, P. A. *Mat. Sbornik* (Moskva), xii. pp. 377–420.

1898. LE GRAND ROY, E. *Archives des sci. phys. et nat.* (Genève), (4) vii. p. 588.

Glaisher's three papers (1874, 1880) contain a fairly complete exposition. Nekrassoff's has the special interest that the determinant of his set of equations is a *circulant*.

His second result is more general, including also Jacobi's of 1841 and Catalan's of 1878 (*Hist.*, i. p. 272 ; iii. p. 115).

LINDELÖF, L. (1880).

(See under this heading in Chap. I.)

BIEHLER, C. (1880).

[Sur les équations linéaires. *Nouv. Annales de Math.*, xix. pp. 311–331, 356–362.]

The treatment here is fuller and more thorough than in the author's thesis of 1878. In dealing with his third case (pp. 325–331), where the number of equations is less than the number of unknowns, he proves a theorem including four of Ventéjols' (1877), but does not mention Rouché (1875).* The fourth section (pp. 356–362) concerns a so-called 'application,' namely, the expression of a quadric as a sum of squares.

SCHMITZ, A. (1880).

[Bemerkungen über die Anwendbarkeit der französischen Methode zur Auflösung linearer Gleichungen. *Zeitschrift f. math. u. naturw. Unterricht*, xi. pp. 428–431.]

As the application of the method ('undetermined multipliers') to a set of n equations necessitates the solution of a set of $n-1$ new equations, and thereafter of a set of $n-2$ equations, and so on, it is clear that if one of these auxiliary sets be intractable through inconsistency or otherwise, the method fails. Further, as the determinants of the auxiliary sets are minors of the original, the conditions for the applicability of the method are easily stated. This is the matter to which attention is here fully drawn by example and reasoning.

Additional examples will be found in a paper by C. Prediger in the *Archiv d. Math. u. Phys.*, lxx. (1883), pp. 319–333.

KRONECKER, L. (1881).

(See under this heading in Chap. I.)

* Rouché's paper of 1880 had not yet been published.

WARNER, J. D. (1881).

[Symmetrical method of elimination in simple equations by use of some of the principles of determinants. *Proceed. American Assoc. Adv. Sci.* (Cincinnati), pp. 50–56.]

Unimportant.

BURNSIDE, W. S. AND PANTON, A. W. (1881).

[The Theory of Equations, ... xvi+387 pp. Dublin.]

It is worth noting that a legitimate solution of the equations

$$\left.\begin{aligned} a_1x + a_2y + a_3z + a_4w &= 0 \\ b_1x + b_2y + b_3z + b_4w &= 0 \end{aligned}\right\},$$

when none of the coefficients nor any determinant of the array is zero, is obtained by forcing into the set the additional pair of equations

$$\frac{a_1^2}{b_1}x + \frac{a_2^2}{b_2}y + \frac{a_3^2}{b_3}z + \frac{a_4^2}{b_4}w = \lambda,$$

$$\frac{b_1^2}{a_1}x + \frac{b_2^2}{a_2}y + \frac{b_3^2}{a_3}z + \frac{b_4^2}{a_4}w = \mu,$$

and using the equality

$$|a_1^3\ a_2^2b_2\ a_3b_3^2\ b_4^3| = |a_1b_2| \cdot |a_1b_3| \ . \ . \ . \ . \ |a_3b_4|.$$

LONGCHAMPS, G. DE (1883).

[Algèbre. (Équations linéaires, . . . pp. 88–111.) xii+671 pp. Paris.]

This is a fuller and more serviceable exposition of the subject than is to be found in any previous text-book of determinants. A chapter is devoted to linear equations in general (pp. 88–100), Rouché's lead (1875, 1880) being carefully followed; and a second is occupied with those that are homogeneous (pp. 101–111).

GIUDICE, F. (1883).

[Equazioni simultanee di primo grado. *Rivista di mat. elem.* (Novara).]

This paper, which has not been found accessible, was considered important by Garbieri (see below, p. 105). The author himself at

a later date drew attention to the fact that it contains the enunciation and a proof of the theorem that *in a set of equations that are* (n−k)*-fold indeterminate the choice of the* n−k *unknowns that may be left arbitrary has no influence on the solutions.* At the same time he also claims that it contains the first introduction of the notion of a 'resolvent system.'

MÉRAY, CH. (1884).

(See under this heading in Chap. I.)

WEYR, ED. (1884).

[O řešení lineárných rovnic. *Časopis pro pěstování Math. a Fys.*, xiv. pp. 101–110, 149–159.]

This discussion is of about the same extent as Longchamps', but would seem to be more independently thought out. The writer does not make use of Rouché's concept, but with the same purpose in view—clearness and brevity of expression—introduces a technicality of his own, defining two or more rows as being "substantially different" when the oblong array formed by them is not evanescent. By way of example we may note his treatment of the question as to the number of independent solutions in the case of a set of n homogeneous equations in n unknowns when the critical minor of the determinant of the set is given (§ 7, pp. 158–159).

SIMMONS, T. C. (1885).

[An application of determinants to the solution of certain types of simultaneous equations. *Math. from Educ. Times,* xliv. pp. 136–143; or *Progreso Mat.*, v. pp. 1–4, 40–44.]

The characteristic of the main type of equations referred to seems to be that an elimination performed on the set produces a vanishing determinant containing only one compound unknown, the finding of which enables the veritable unknowns to be determined. For example, the set

$$\begin{aligned} yz(a_1x + a_2y + a_3z) &= f, \\ zx(b_1x + b_2y + b_3z) &= g, \\ xy(c_1x + c_2y + c_3z) &= h \end{aligned}$$

is thrown into the form

$$\left.\begin{aligned} \left(a_1 - \frac{f}{xyz}\right)x + \quad a_2 y \quad + \quad a_3 z \quad &= 0 \\ b_1 x \quad + \left(b_2 - \frac{g}{xyz}\right)y + \quad b_3 z \quad &= 0 \\ c_1 x \quad + \quad c_2 y \quad + \left(c_3 - \frac{h}{xyz}\right)z &= 0 \end{aligned}\right\},$$

which by elimination provides a cubic in xyz. Numerous illustrative examples of this and other types are given.

GORDAN, P. (1885).

[Vorlesungen . . . I. (§ 8. Systeme lin. Gleich. . . . pp. 101–120.) xii+202 pp. Leipzig.]

The exposition here given resembles Longchamps' in fullness, no previous German text-book on determinants being comparable with it in this respect. The resemblance referred to, however, goes no further : in Gordan's treatment there are considerably fewer formulated theorems and greater freshness of treatment. It is constructed, too, on quite a different plan. Homogeneous equations are taken first and assigned almost all the space (pp. 101–117) : the other class is considered to be a mere specialization, and is dismissed in a page followed by four examples.

JUERGENS, E. (1886).

[Zur Auflösung linearer Gleichungssysteme und numerischen Berechnung von Determinanten. Festschr. 20 pp. Aachen.]

What this actually concerns is the solution of any set of equations in which the known quantities are numbers in the ordinary decimal notation, like 1·7254 . . . , each with the possibility of being made more accurate by the use of additional digits. The special procedure proposed is the replacement of the set by another in which each diagonal coefficient is positive and greater than the sum of the absolute values of the other coefficients of its row. The multipliers for producing the second set from the first are themselves to be determined by the solution of sets of equations. In the next place, determinants of the type connected with such sets of equations

are directed to be evaluated by a process which brings out the result in the form of a product: for example, $|\, a_1b_2c_3d_4 \,|$ in the form

$$\frac{|\, a_1b_2c_3d_4 \,|}{|\, b_2c_3d_4 \,|} \cdot \frac{|\, b_2c_3d_4 \,|}{|\, c_3d_4 \,|} \cdot \frac{|\, c_3d_4 \,|}{d_4} \cdot \frac{d_4}{1}.$$

CAPELLI, A. E GARBIERI, G. (1886).

[Corso di Analisi Algebrica. Vol. I. Teorie introduttorie. vii+511 pp. Padova.]

The chapter on Linear Equations (pp. 367–412), like that on Determinants, is a pleasing feature of this general manual of Algebra, an effort at progress being evident in both the matter and the manner of the exposition. For the first time in a text-book use is made of the term 'characteristic' (Frobenius' 'Rang') in the statement and discussion of the theorems. This of course is a help towards attaining greater accuracy, although on the other hand it at first raises a danger of mistaking newness of form for newness of substance. Of the series of nine formulated results attention is drawn to the second, namely, *By means of successive substitution the set of equations*

$$\left.\begin{matrix} a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = 0 \end{matrix}\right\}_{r=1}^{r=n}$$

can be transformed into an equivalent set of the form

$$\left.\begin{aligned} \beta_{11}x_1 + \beta_{12}x_2 + \beta_{13}x_3 + \ldots + \beta_{1k}x_k + \ldots + \beta_{1n}x_n &= 0 \\ \beta_{22}x_2 + \beta_{23}x_3 + \ldots + \beta_{2k}x_k + \ldots + \beta_{2n}x_n &= 0 \\ \beta_{33}x_3 + \ldots + \beta_{3k}x_k + \ldots + \beta_{3n}x_n &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \beta_{kk}x_k + \ldots + \beta_{kn}x_n &= 0 \end{aligned}\right\},$$

in which the coefficients $\beta_{11}, \beta_{22}, \ldots, \beta_{kk}$ *are non-zero, and* k *is the characteristic of* $|\, a_{1n} \,|$.

Closely connected, of course, with linear equations is the evanescence of arrays, and thus it is that there comes to be appended (pp. 387–400) a series of results on the latter subject. Of these the most general is the one worthy of note, and after some alteration it stands as follows: *the necessary and sufficient condition for the vanishing of an array, oblong or square, is that its rows be connected*

by a linear homogeneous relation with coefficients not all zero. For example, if the array

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \end{matrix}$$

be evanescent, with $| a_1b_2c_3 |$ for its 'critical minor,' then the elements of every column are connected by a linear relation whose coefficients are

$$-| b_1c_2d_3 |, \quad | a_1c_2d_3 |, \quad -| a_1b_2d_3 |, \quad | a_1b_2c_3 |;$$

and, conversely, if such a fixed linear relation exist, the array must manifestly be evanescent. It may also be noted that there is established an analogue to the former of these two theorems in the case of the *rows*: thus, to take the same example, the elements of every row are connected by a linear relation whose coefficients are

$$-(234 + 235 + 236), \quad 134 + 135 + 136,$$
$$-(124 + 125 + 126), \quad 123, \quad 123, \quad 123,$$

where 234, . . . are primary minors of the given 3-by-6 array.

CLASEN, B. J. (1888).

[Sur une nouvelle méthode de résolution des équations linéaires et sur l'application de cette méthode au calcul des déterminants. *Annales . . . soc. sci. de Bruxelles*, xii. B, pp. 251–281; A, pp. 50–59; or *Mathesis*, ix. Suppl. 2, 40 pp.]

In the solution of a set of linear equations with arithmetical coefficients considerable latitude is available in the choice of the series of derived equations which is to end up with the values of the unknowns. The choice made by the writer now reached is made with real skill and deserves attention. For exposition it must suffice to say that with the set of equations

$$\left.\begin{array}{lr} a_1x + b_1y + c_1z + d_1u + e_1v = f_1 & (a_1) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ a_5x + b_5y + c_5z + d_5u + e_5v = f_5 & (a_5) \end{array}\right\}$$

the derived sets would be

$$\left.\begin{aligned} |a_1b_2|y + |a_1c_2|z + |a_1d_2|u + |a_1e_2|v &= |a_1f_2| \quad (\beta_1)\\ -|a_1b_2|x + |b_1c_2|z + |b_1d_2|u + |b_1e_2|v &= |b_1f_2| \quad (\beta_2)\end{aligned}\right\}$$

$$\left.\begin{aligned} |a_1b_2c_3|z + |a_1b_2d_3|u + |a_1b_2e_3|v &= |a_1b_2f_3| \quad (\gamma_1)\\ -|a_1b_2c_3|y + |a_1c_2d_3|u + |a_1c_2e_3|v &= |a_1c_2f_3| \quad (\gamma_2)\\ |a_1b_2c_3|x + |b_1c_2d_3|u + |b_1c_2e_3|v &= |b_1c_2f_3| \quad (\gamma_3)\end{aligned}\right\}$$

$$\left.\begin{aligned} |a_1b_2c_3d_4|u + |a_1b_2c_3e_4|v &= |a_1b_2c_3f_4| \quad (\delta_1)\\ -|a_1b_2c_3d_4|z + |a_1b_2d_3e_4|v &= |a_1b_2d_3f_4| \quad (\delta_2)\\ |a_1b_2c_3d_4|y + |a_1c_2d_3e_4|v &= |a_1c_2d_3f_4| \quad (\delta_3)\\ -|a_1b_2c_3d_4|x + |b_1c_2d_3e_4|v &= |b_1c_2d_3f_4| \quad (\delta_4)\end{aligned}\right\}$$

$$\left.\begin{aligned} |a_1b_2c_3d_4e_5|v &= |a_1b_2c_3d_4f_5| \quad (\epsilon_1)\\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\end{aligned}\right\}$$

the first of any derived set after the β set being got, by addition of multiples, from all of the preceding set and one of the α's, and the others of the set from the first and one of the preceding set. The advantages of the process, which will be best appreciated by an actual trial of it, and which are rather obscured by the use of literal coefficients, are brevity, easy discovery of errors in calculation, and timely detection of inconsistency or incompleteness in the data. Apart from these it is a useful lesson in determinants, comparable with Desnanot's of 1819.

CAPELLI, A. (1888-9, 1892, 1895, 1898).

[Lezioni di Algebra Complementare, date nell' anno accademico 1888–1889. Cap. III. § 6. Napoli.]

[Sopra la compatibilità o incompatibilità di più equazioni di primo grado fra più incognite. *Rivista di Mat.*, ii. pp. 54–58.]

[Lezioni di Algebra Complementare, ad uso. . . . xii+527 pp.; 2da ediz. xvi+680 pp. Napoli.]

Here we find ourselves faced with an important advance in the formulation (p. 165) of the test for consistency. It may be put very shortly thus: *In order that a set of non-homogeneous linear equations may be consistent it is necessary and sufficient that their unaugmented and augmented arrays have the same characteristic.* By way of proof the author first recalls three theorems regarding *homo-*

geneous equations which he and Garbieri had formally established in their text-book of 1886. Then, taking his set of m non-homogeneous equations in n unknowns, and supposing first that the two arrays of the set have the same characteristic, he makes the given equations homogeneous in the usual way by bringing in an $(n+1)^{\text{th}}$ unknown, and proves by use of the aforesaid theorems that a solution is obtainable in which the unknowns have finite values and the $(n+1)^{\text{th}}$ of them is not zero. The sufficiency of the asserted condition is thus established. Next, supposing that the unaugmented array has a lower characteristic than the other, and that the set is nevertheless consistent, he is able to show with the help of similar reasoning that he is thereby led to a contradiction of his hypothesis. The asserted condition is thus seen to be also a necessity.

It remains to note that the treatment of the subject in 1892 is said by the author to be essentially the same as that of the ' Lezioni ' (1888–89).

BUSCHE, E. (1891).

[Ueber Kronecker'sche Aequivalenzen. *Mitteil. d. math. Ges.* (Hamburg), iii. pp. 3–7.]

It is here shown that the number of solutions of a set of n so-called ' equivalences ' in n unknowns is the absolute value of the determinant of the set.

TYLER, H. W. (1891).

(See under this heading in Chap. XIV.)

GARBIERI, G. (1891).

[Introduzione del una teorica dell' eliminazione. *Giornale di Mat.*, xxx. pp. 41–105.]

The matter here given would appropriately form three chapters in a bulky text-book on ' modern higher algebra.' It is mainly of interest as bringing once more into prominence the intimate relation which exists between the solvability of a set of linear equations and the properties of the two arrays associated therewith—the ' unaugmented ' and ' augmented ' arrays of H. J. S. Smith (1861), and the ' V-block ' and ' B-block ' of Dodgson (1867).

The first chapter (pp. 42–56) deals with the evanescence of arrays, a subject not so singled out for exposition since Dodgson's time (1867). After the natural preliminary explanations, the three operations which do not affect the nullity of an array are discussed; 'characteristic' is used as by Capelli; the condition for evanescence is carefully worked out (pp. 44–49); and Frobenius' property of an evanescent array enunciated and proved anew. The concluding pages (51–56) are less relevant, and their contents not recognized to be cases * of Sylvester's theorem regarding a sum of products of pairs of determinants.

The second chapter (pp. 56–70) concerns non-homogeneous linear equations. Its series of carefully formulated propositions culminate in Capelli's test for consistency. The requisite specialization is then made, and justification is given for throwing the burden of further discussion on what follows.

The third chapter, on homogeneous linear equations (pp. 70–105), is in consequence still more scrupulously detailed. After the test for consistency has been established, the conception of a 'reduced' set of equations ('*sistema ridotto*') is brought forward and utilized at considerable length. The term is used to denote a set of equations equivalent to the given set and consisting of the lowest possible number of mutually independent equations. The concluding pages are occupied with the question of the dependence or independence of solutions.

AMIGUES, E. (1892): PAPELIER, G. (1897): GASCÓ, L. G. (1897).

[Démonstration analytique du théorème de M. Rouché relatif à un système d'équations algébriques du premier degré. *Nouv. Annales de Math.*, (3) xi. pp. 47–48.]

[Note sur les équations linéaires. *Revue de Math. Spéc.*, iv. (7e année), pp. 81–85.]

[Resolución por determinantes de los sistemas de ecuaciones lineales. *Archivo de Mat.*, ii. pp. 124–127.]

As a mode of establishing Rouché's theorem it is sought in the first paper here to prove that *having selected from the given set of*

* Muir's *Determinants* (1881), § 88, p. 128. Garbieri attributes them to Giudice (1883).

equations the group with the greatest breadth of non-evanescent array it is justifiable to replace, as if for purposes of solution, each of the remaining equations by the vanishing of one of the 'déterminants caractéristiques.' The proof is not without weakness.

The second paper gives another proof.

COLLET, J. (1893): MEYER, A. (1894).

[Les équations linéaires et leurs applications. *Annales de l'Enseignement Sup.*, v. pp. 351–364.]

[Ligninger af 1ste Grad. *Nyt Tidsskrift for Math.*, v. B, pp. 4–17.]

The exposition given by Meyer is of the classificatory type, non-homogeneous equations being given the leading place, and discussed under four heads with sub-heads and examples (pp. 4–16).

CHAPTER IV.

AXISYMMETRIC DETERMINANTS, FROM 1880 TO 1897.

THE amount of work done on the various kinds of symmetric determinants approximates closely to that done in the preceding period, being however differently distributed amongst them. By reason of this there is a shrinkage in the contents of Chap. V.: nevertheless, for convenience in reference, the classification of this part of Vol. III. has been retained.

SCOTT, R. F. (1880).

[A Treatise on the Theory of Determinants, ... xi+251 pp. Cambridge.]

We have already seen how Scott seems (pp. 15–16) to support his predilection for so-called 'alternate numbers' by using them to show that

$$\begin{vmatrix} . & \cos(a_1+a_2) & \cos(a_1+a_3) & \dots \\ \cos(a_2+a_1) & . & \cos(a_2+a_3) & \dots \\ \cos(a_3+a_1) & \cos(a_3+a_2) & . & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n$$

$$= (-1)^{n-1} \cdot \{\cos 2a_1 \cos 2a_2 \dots\} \cdot \left\{ n-1+\sum \frac{\sin^2(a_h-a_k)}{\cos 2a_h \cos 2a_k} \right\}.$$

Later (pp. 221–222) he gives a similar equality,

$$\begin{vmatrix} . & a_1+a_2 & a_1+a_3 & \dots \\ a_2+a_1 & . & a_2+a_3 & \dots \\ a_3+a_1 & a_3+a_2 & . & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n$$

$$= (-1)^{n-1} \cdot 2^n \cdot a_1 a_2 \dots a_n \cdot \left\{ n-1+\sum \frac{(a_h-a_k)^2}{4a_h a_k} \right\},$$

without indication of procedure. Both, however, are readily evaluated by appropriately transforming them into determinants of the $(n+2)^{\text{th}}$ order; for example, the first into

$$-\begin{vmatrix} 1 & . & \sin\alpha_1 & \sin\alpha_2 & \sin\alpha_3 & \dots \\ . & -1 & \cos\alpha_1 & \cos\alpha_2 & \cos\alpha_3 & \dots \\ \sin\alpha_1 & \cos\alpha_1 & -\cos 2\alpha_1 & . & . & \dots \\ \sin\alpha_2 & \cos\alpha_2 & . & -\cos 2\alpha_2 & . & \dots \\ \sin\alpha_3 & \cos\alpha_3 & . & . & -\cos 2\alpha_3 & \dots \\ . & . & . & . & . & . \end{vmatrix}.$$

He records the fact that a determinant of the form

$$\begin{vmatrix} 1 & \cos\alpha & \cos(\alpha+\beta) & \cos(\alpha+\beta+\gamma) & \dots \\ \cos\alpha & 1 & \cos\beta & \cos(\beta+\gamma) & \dots \\ \cos(\alpha+\beta) & \cos\beta & 1 & \cos\gamma & \dots \\ \cos(\alpha+\beta+\gamma) & \cos(\beta+\gamma) & \cos\gamma & 1 & \dots \\ . & . & . & . & . \end{vmatrix}$$

exemplified by Wolstenholme in 1878, vanishes for all orders above the 2^{nd}; and that

$$\begin{vmatrix} a+b+c+d & a-b-c+d & a-b+c-d \\ a-b-c+d & a+b+c+d & a+b-c-d \\ a-b+c-d & a+b-c-d & a+b+c+d \end{vmatrix} = 16\,(bcd+acd+abd+abc).$$

In regard to the latter it might have been mentioned that if on the right a, b, c, d were replaced by

$$a+b+c+d, \quad a+b-c-d, \quad a-b+c-d, \quad a-b-c+d,$$

and the opposite change made on the left, it would only be necessary to change 16 into $\frac{1}{4}$ in order to retain the equality. Finally, the consequences of bordering

$$\begin{vmatrix} (b+c)^2 & ab & ac \\ ab & (c+a)^2 & bc \\ ac & bc & (a+b)^2 \end{vmatrix}$$

in several different ways are specified, all of the resulting determinants being necessarily divisible by $a+b+c$ as this is a factor of every primary minor of the original (*Hist.*, iii. p. 125).

HAZZIDAKIS, J. N. (1880).

[Ueber eine Eigenschaft der Unterdeterminanten einer symmetrischen Determinanten. *Crelle's Journ.*, xci. pp. 238–247.]

Notwithstanding the title, the real subject here is the conditions under which the cofactors of f_{11}, $f_{11}f_{22}$, $f_{11}f_{22}f_{33}$, . . . in the axisymmetric determinant $|f_{1n}|$, where the elements are rational integral functions of x, constitute a Sturmian series for the examination of the roots of the equation $|f_{1n}|=0$ in the interval from $x=a$ to $x=b$. The conditions found are (1) that none of the f's must vanish identically, and (2) that $F_{\mu\nu}$, $f'_{\mu\nu}$ being the cofactor and derivate of any element, the sum

$$\sum_{\mu\nu} f'_{\mu\nu} F_{1\mu} F_{1\nu}$$

must not vanish in the interval in question. By way of illustration the result is applied to several special cases—the original case of Lagrange, Sylvester's case of 1853 where the f's are all linear, and so on.

MUIR, T. (1881).

[Note on a special symmetrical determinant. *Analyst*, viii. pp. 169–171.]

What is effected here is the generalization of an identity published anonymously in non-determinant form in 1846 (*Hist.*, ii. pp. 114–115). If S_4 be written for $a_1a_2+b_1b_2+c_1c_2+d_1d_2$, the 'next case' is

$$\begin{vmatrix} 2a_1a_2-S_4 & a_1b_2+a_2b_1 & a_1c_2+a_2c_1 & a_1d_2+a_2d_1 \\ a_1b_2+a_2b_1 & 2b_1b_2-S_4 & b_1c_2+b_2c_1 & b_1d_2+b_2d_1 \\ a_1c_2+a_2c_1 & b_1c_2+b_2c_1 & 2c_1c_2-S_4 & c_1d_2+c_2d_1 \\ a_1d_2+a_2d_1 & b_1d_2+b_2d_1 & c_1d_2+c_2d_1 & 2d_1d_2-S_4 \end{vmatrix}$$
$$= -S_4^2\,(a_1^2+b_1^2+c_1^2+d_1^2)\,(a_2^2+b_2^2+c_2^2+d_2^2),$$

and is proved by expanding the determinant in descending powers of S_4. On putting $d_1=d_2=0$, the original case is reobtained: a second interesting specialization is $a_1=a_2=a$, $b_1=b_2=b$, . . . ; and a third results from putting $d=0$ in the second. An alternative procedure begins by suitably raising the determinant to the sixth order, so as to make the non-diagonal elements of the original replaceable by zeros.

MUIR, T. (1881).

(See under this heading in Chap. VII.)

MUIR, T. (1881).

[On new and recently discovered properties of certain symmetric determinants. *Quart. Journ. of Math.*, xviii. pp. 166–177.]

All the results save one concern symmetric determinants of special forms elsewhere dealt with. The identity excepted is, in the case of the third order,

$$\begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} \cdot \begin{vmatrix} x & y & z \\ y & u & v \\ z & v & w \end{vmatrix} = \frac{1}{2^6} \begin{vmatrix} a+x & b+y & c+z & c-z & b-y & a-x \\ b+y & d+u & e+v & e-v & d-u & b-y \\ c+z & e+v & f+w & f-w & e-v & c-z \\ c-z & e-v & f-w & f+w & e+v & c+z \\ b-y & d-u & e-v & e+v & d+u & b+y \\ a-x & b-y & c-z & c+z & b+y & a+x \end{vmatrix},$$

the determinant on the right being bi-axisymmetric.

BURNSIDE, W. S. AND PANTON, A. W. (1881).

[The Theory of Equations, ... xvi+387 pp. Dublin.]

By bordering

$$\begin{vmatrix} \lambda x^2+cy^2+bz^2-1 & (\lambda-c)\,xy & (\lambda-b)\,xz \\ (\lambda-c)\,xy & \lambda y^2+az^2+cx^2-1 & (\lambda-a)\,yz \\ (\lambda-b)\,xz & (\lambda-a)\,yz & \lambda z^2+bx^2+ay^2-1 \end{vmatrix}$$

by the column 1, λx, λy, λz and the row 1, 0, 0, 0, it is shown to contain the factor $1-\lambda(x^2+y^2+z^2)$, the cofactor being what the given determinant becomes on putting $\lambda=0$.

MUIR, T. (1882).

[On a symmetric determinant connected with Lagrange's interpolation-problem. *Proceed. London Math. Soc.*, xiii. pp. 156–161.]

The subject is the determinant

$$\left| \sin\frac{1\cdot 1\pi}{n+1} \cdot \sin\frac{2\cdot 2\pi}{n+1} \ldots \sin\frac{n\cdot n\pi}{n+1} \right|, \quad \text{or } \mathrm{L}_n \text{ say,}$$

and quite a number of results in connection with it are given. Besides being symmetric with respect to the main diagonal, it is alternately symmetric and skew with respect to the secondary diagonal and also with respect to the bisecting line drawn parallel to the columns. Its most important property is that *its elements are proportional to their cofactors.** This is established by multiplying the s^{th} column by the element in the $(r, s)^{\text{th}}$ place, the $(s-1)^{\text{th}}$ column by the element in the $(r, s-1)^{\text{th}}$ place, the $(s-2)^{\text{th}}$ column by the element in the $(r, s-2)^{\text{th}}$ place, and so on, when it is found that the result of summation is a new s^{th} column with all its elements equal to 0 except that in the r^{th} row, so that

$$\mathrm{L}_n \times \sin\frac{rs\pi}{n+1} = \tfrac{1}{2}(n+1)\cdot \text{cofactor of } \sin\frac{rs\pi}{n+1}.$$

By multiplying L_n by itself and using the identity

$$\sin\frac{r\pi}{n+1}\cdot\sin\frac{s\pi}{n+1} + \sin\frac{2r\pi}{n+1}\cdot\sin\frac{2s\pi}{n+1} + \ldots + \sin\frac{nr\pi}{n+1}\cdot\sin\frac{ns\pi}{n+1} = \begin{cases} 0 & (s \neq r) \\ \frac{1}{2}(n+1) & (s = r) \end{cases},$$

there is obtained

$$\mathrm{L}_n^{\,2} = \left\{\tfrac{1}{2}(n+1)\right\}^n.$$

In the case of L_{2m} addition of rows suffices to make the elements in m^2 places vanish, and after some transformations there results

$$\mathrm{L}_{2m} = (-1)^m 2^m \left| \sin\frac{1\cdot 1\pi}{2m+1}\cdot\sin\frac{2\cdot 3\pi}{2m+1}\cdot\ldots\cdot\sin\frac{m(2m-1)\pi}{2m+1}\right|^2,$$

from which and a previous result we have readily

$$\left|\sin\frac{1\cdot 1\pi}{2m+1}\sin\frac{2\cdot 3\pi}{2m+1}\cdot\ldots\cdot\sin\frac{m(2m-1)\pi}{2m+1}\right| = \left\{\tfrac{1}{4}(2m+1)\right\}^{\frac{1}{2}m}.$$

Another evaluation is thus effected, as, curiously enough, the new determinant on the left here is shown to be transformable into

$$\left|\sin\frac{1\cdot 1\cdot 2\pi}{2m+1}\sin\frac{2\cdot 2\cdot 2\pi}{2m+1}\cdot\ldots\cdot\sin\frac{m\cdot m\cdot 2\pi}{2m+1}\right|.$$

Lastly, the determinant which differs from L in having cosines in place of sines is referred to, and a few analogous results given, one agreeing with Hunyady's of 1872 so far as the latter goes.

* Note might have been taken here in passing that the determinant is thus an orthogonant.

KRONECKER, L. (1882).

[Die Subdeterminanten symmetrischer Systeme. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), pp. 821–824 ; or *Bull. des Sci. Math.*, (2) xi. (2), pp. 102–107 ; or *Werke*, ii. pp. 391–396.]

As already recorded, the basis of this paper professes to be the conception of ' reciprocal systems.' In the portion devoted to such systems that are axisymmetric, the author intimates the existence of linear relations between the minors of any axisymmetric determinant. These he states in the form

$$|a_{gh}| = \sum |a_{1k}|$$

$$(g = 1, 2, \ldots, m; \quad h = 1, 2, \ldots, m);$$

$$(i = 1, 2, \ldots, m-1, r; \quad k = m+1, \ldots, r-1, m, r+1, \ldots 2m);$$

$$r = m+1, m+2, \ldots, 2m,$$

which is somewhat lengthy and complicated, even although the requisite axisymmetry is not prescribed in it. The theorem simply is that *if* $\begin{vmatrix} 1\ 2 \ldots 2m \\ 1\ 2 \ldots 2m \end{vmatrix}$ *be axisymmetric, the minor*

$$\begin{vmatrix} 1 & 2 & \ldots & m \\ m+1 & m+2 & \ldots & 2m \end{vmatrix}$$

is equal to the sum of all the others got from it by interchanging an under index with the last of the upper ; for example, $\begin{vmatrix} 123456 \\ 123456 \end{vmatrix}$ being given axisymmetric, we have

$$\begin{vmatrix} 123 \\ 456 \end{vmatrix} = \begin{vmatrix} 124 \\ 356 \end{vmatrix} + \begin{vmatrix} 125 \\ 436 \end{vmatrix} + \begin{vmatrix} 126 \\ 453 \end{vmatrix},$$

or

$$\begin{vmatrix} 123 \\ 456 \end{vmatrix} - \begin{vmatrix} 124 \\ 356 \end{vmatrix} + \begin{vmatrix} 125 \\ 346 \end{vmatrix} - \begin{vmatrix} 126 \\ 345 \end{vmatrix} = 0.$$

It is pointed out that the identity is verifiable on expanding each determinant in terms of the elements of its last row and their cofactors. As a matter of fact, in these last rows the elements 34, 35, 36, 45, 46, 56 occur twice, and the cofactor in one position cancels its fellow in the other position.

RUNGE, C. (1882).

[Die lineare Relationen zwischen den verschiedenen Subdeterminanten symmetrischer Systeme. *Crelle's Journ.*, xciii. pp. 319–327.]

This paper is the record of a vigorous investigation of Kronecker's identity with two main purposes in view, a section being devoted to each.

In the first place, it having been made evident that linear relations with constant coefficients other than 1 may exist between the m-line minors of a $2m$-line axisymmetric determinant, the question arises whether such relations are self-dependent or derivative. With this inquiry § I. is occupied, and the result is to show (pp. 320–322) that they are all deducible from Kronecker's. At the outset attention is drawn to the fact that in every minor of a Kronecker identity the row-numbers and column-numbers involve between them all the integers 1, 2, . . . , $2m$, the number of minors of this kind being therefore $\frac{1}{2}(2m)_m$; but that an identity will still subsist if in its leading minor

$$\begin{vmatrix} 1\ , & 2\ , & \ldots\ , & m-1, & m \\ m+1, & m+2, & \ldots\ , & 2m-1, & 2m \end{vmatrix}$$

any upper index except m be made the same as one of the under indices. Thus, from the identity

$$\left|\begin{matrix}123\\456\end{matrix}\right| - \left|\begin{matrix}124\\356\end{matrix}\right| + \left|\begin{matrix}125\\346\end{matrix}\right| - \left|\begin{matrix}126\\345\end{matrix}\right| = 0,$$

by replacing 2 by 5 we find

$$\left|\begin{matrix}135\\456\end{matrix}\right| - \left|\begin{matrix}145\\356\end{matrix}\right| + \left|\begin{matrix}156\\345\end{matrix}\right| = 0,$$

and by replacing 1 in this by 4 we arrive back at the simplest consequence of axisymmetry

$$-\left|\begin{matrix}345\\456\end{matrix}\right| + \left|\begin{matrix}456\\345\end{matrix}\right| = 0.$$

The result of the second section (pp. 323–326) is more interesting and important, namely, that *a system of linearly-independent* m-*line minors can be found in terms of which each of the others can be linearly*

and integrally expressed: further, that the number of the former is $(2m)!/(m+1)!\,m!$, *being the* $\frac{1}{2}(m+1)^{th}$ *part of the total number that are admissible.*

The third section (pp. 326–327) establishes the absence of such linear relations in the case of a determinant that is skew, the mode of demonstration employed being gradational.

MUIR, T. (1884).

[Note on a unique property of an axisymmetric determinant of the fourth order. *Annals of Math.*, i. pp. 31–33.]

The property in question is that *if* H *and* F *stand for the complementary minors of* h *and* f *in*

$$\begin{vmatrix} . & a & b & c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & k \end{vmatrix},$$

then

$$\begin{vmatrix} . & H & F \\ H & d & e \\ F & e & g \end{vmatrix} \div \begin{vmatrix} . & a & b \\ a & d & e \\ b & e & g \end{vmatrix} = \begin{vmatrix} . & a & b & -c \\ a & d & e & f \\ b & e & g & h \\ c & f & h & . \end{vmatrix},$$

Incidentally it is noted and proved that *if a non-diagonal element in an axisymmetric determinant be changed in sign, the number of terms in the final expansion is diminished by*

$$2(n-2)\cdot(n-2)!$$

SYLVESTER, J. J. (1884).

[Question 7889. *Educ. Times*, xxxvii. p. 356; or *Math. from Educ. Times*, xlviii. p. 37.]

When simplified the result here established is that *the discriminant of*

$$\begin{vmatrix} a_1x+a_2y+a_3z+a_4w & b_1x+b_2y+b_3z+b_4w \\ c_1x+c_2y+c_3z+c_4w & d_1x+d_2y+d_3z+d_4w \end{vmatrix}$$

is $|\,a_1b_2c_3d_4\,|^2$. For proof all that is needed is to write the dis-

criminant in its definitional form, when it will be found to be the row-by-row product of

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} d_1 & -c_1 & -b_1 & a_1 \\ d_2 & -c_2 & -b_2 & a_2 \\ d_3 & -c_3 & -b_3 & a_3 \\ d_4 & -c_4 & -b_4 & a_4 \end{vmatrix}.$$

Sylvester, however, evidently had in mind a general theorem involving a determinant of order m^2, not merely 2^2.

BUCHHEIM, A. (1884).

[On a theorem relating to symmetrical determinants. *Messenger of Math.*, xiv. pp. 143–144.]

The theorem is that regarding the reality of the roots of Lagrange's determinantal equation, and the proof given rests on the consideration of a set of simultaneous linear equations.

MUIR, T. (1884).

[Note on the equation connecting the mutual distances of four points in a plane. *Proceed. Edinburgh Math. Soc.*, iii. pp. 34–38.]

The fact that one of the determinants in the two relations

$$\begin{vmatrix} . & 1 & 1 & 1 & 1 \\ 1 & . & a^2 & b^2 & c^2 \\ 1 & a^2 & . & d^2 & e^2 \\ 1 & b^2 & d^2 & . & f^2 \\ 1 & c^2 & e^2 & f^2 & . \end{vmatrix} = 0, \quad \begin{vmatrix} . & a^2 & b^2 & c^2 \\ a^2 & . & d^2 & e^2 \\ b^2 & d^2 & . & f^2 \\ c^2 & e^2 & f^2 & . \end{vmatrix} = 0$$

connecting the distances, a, b, c, d, e, f, of four points on a circumference is a coaxial primary minor of the other, is used to make the immediate deduction

$$-\left\{(d^2e^2f^2)^{\frac{1}{2}} + (b^2c^2f^2)^{\frac{1}{2}} + (a^2c^2e^2)^{\frac{1}{2}} + (a^2b^2d^2)^{\frac{1}{2}}\right\} = 0.$$

On account, however, of the ambiguities of sign it is thought better to alter the original equations * into

* The mode of transforming the four-line determinant, though not given in the paper, deserves note. Multiplication of the rows in order by

def, bcf, ace, abd,

$$d^2e^2f^2 + b^2c^2f^2 + a^2c^2e^2 + a^2b^2d^2 - (a^2+f^2)(-a^2f^2+b^2e^2+c^2d^2)$$
$$- (b^2+e^2)(a^2f^2-b^2e^2+c^2d^2)$$
$$- (c^2+d^2)(a^2f^2+b^2e^2-c^2d^2) = 0,$$
$$(af+be+cd)(af-be-cd)(af-be+cd)(af+be-cd) = 0,$$

when the deductions with the proper signs are readily obtained.

MEHMKE, R. (1885): SCHENDEL, L. (1885).

[Bemerkung über die Subdeterminanten symmetrischer Systeme. *Math. Annalen*, xxvi. pp. 209–210.]

[Das Kronecker'sche Subdeterminantensatz. *Zeitschrift f. Math. u. Phys.*, xxxii. pp. 119–120.]

The author of the first of these two brief notes quotes from p. 131 of Grassmann's *Ausdehnungslehre* of 1862 a theorem regarding a vanishing sum of products, and proceeds to show that when the usual transition is made from a vector to a scalar relation a case of the theorem is the same as Kronecker's of 1882.

The other author, with his own changes on Grassmann's notation, seeks to prove the latter theorem anew, and thereafter to obtain by means of it a fresh expression for a bilinear form, and another of similar type for a quadric.

SYLVESTER, J. J. (1885).

[Note on certain elementary geometrical notions and determinations.* *Proceed. London Math. Soc.*, xvi. pp. 201–215; or *Collected Math. Papers*, iv. pp. 259–271.]

The mathematical notions in question are those which lead up to

and then division of the columns by

$$abc,\ ade,\ bdf,\ cef,$$

gives

$$\begin{vmatrix} . & af & be & cd \\ af & . & cd & be \\ be & cd & . & af \\ cd & be & af & . \end{vmatrix},$$

whence come immediately the four factors.

* At the top of four pages the word is as here 'determinations,' at the top of four others it is 'determinants.'

the consideration of Cayley's 'mutual-distance' determinant and another connected therewith, namely,

$$\begin{vmatrix} . & 1 & 1 & 1 & \dots \\ 1 & . & 12 & 13 & \dots \\ 1 & 12 & . & 23 & \dots \\ 1 & 13 & 23 & . & \dots \\ . & . & . & . & \dots \end{vmatrix}_{n+1}, \qquad \begin{vmatrix} . & . & A & B & C & \dots \\ . & . & 1 & 1 & 1 & \dots \\ A & 1 & . & 12 & 13 & \dots \\ B & 1 & 12 & . & 23 & \dots \\ C & 1 & 13 & 23 & . & \dots \\ . & . & . & . & . & \dots \end{vmatrix}_{n+2}.$$

For the former he finds the development

$$\sum(-1)^{n+i+j} \cdot k_1 k_2 \dots k_i \cdot 2^j \chi_1 \chi_2 \dots \chi_j,$$

where the k's are cycles of two umbrae, and the χ's are cycles of more than two: for example, $12{\cdot}21$, $12{\cdot}23{\cdot}31$, where $21 \equiv 12$ and $31 \equiv 13$ (cf. *Hist.*, ii. pp. 469–470). The development of the other he arranges according to products whose one factor is a 2-line minor formed from the first two rows and whose other is a 2-line minor from the first two columns. To this he devotes considerable attention (pp. 211–215), separating the terms of the development into three types, and examining them in more or less detail for the first seven cases. Unfortunately, there are serious inaccuracies in the evaluations.

CESÀRO, E. (1885).

[Determinanti in aritmetica. *Giornale di Mat.*, xxiii. pp. 182–187.]

This is the first of a vigorous series of more or less overlapping papers * belonging to the year 1885 and occupied with determinants of the type brought to light by Smith in 1876 and already studied by Mansion and others. The series, however, throws little, if any, additional light on determinants, the interest attaching to them being more closely connected with the Theory of Integers. We

* The others are in
Nouv. Annales de Math. (3), v. pp. 44–47.
Rendic. . . . Accad. dei Lincei (1885), pp. 709–711, 711–715.
Annales de l'Éc. Normale (3), ii. pp. 425–435.
Mathesis, v. pp. 248–250.
Giornale di Mat., xxv. pp. 14–19.
Two 'questions' taken out of the first of the series appeared also in *Nouv. Annales de Math.* (3), iv. p. 56, p. 451. See also (3) iv. p. 560.

have, for example, the evaluation of $|\, a_{1,n-1} \,|$, whose $(r, s)^{\text{th}}$ element is the number of common divisors of $r+1$, $s+1$, excluding unity, and $|\, a_{1n} \,|$, whose $(r, s)^{\text{th}}$ element is 1 or 0, according as the highest common divisor of r, s is or is not a square.

LORIA, G. (1886).

[Intorno ad alcune relazioni fra distanze. *Periodico di Mat.*, i. pp. 33–43.]

Loria rests his investigation on the lemma that *if* α, β, γ *be angles satisfying one of the relations*

$$\alpha+\beta+\gamma = 2\pi, \quad -\alpha+\beta+\gamma = 0, \quad \alpha-\beta+\gamma = 0, \quad \alpha+\beta-\gamma = 0,$$

then

$$\begin{vmatrix} 1 & \cos\gamma & \cos\beta \\ \cos\gamma & 1 & \cos\alpha \\ \cos\beta & \cos\alpha & 1 \end{vmatrix} = 0,$$

considering first points on a sphere, and then five points in space. In dealing with the latter his initial step is to obtain the four-line determinant corresponding to the preceding, then to substitute for the cosines their equivalents in terms of lengths, and, lastly, to simplify the elements by raising the determinant first to the fifth order and then to Cayley's sixth order.

EDWARDES, D. (1886).

[Question 8767. *Educ. Times*, xxxix. p. 372; *Math. from Educ. Times*, li. p. 76.]

When stated a little more fully, Edwardes' interesting theorem is that if

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = 0,$$

then

$$\begin{vmatrix} ax^2+2bxy+cy^2 & bx^2+2cxy+dy^2 \\ bx^2+2cxy+dy^2 & cx^2+2dxy+ey^2 \end{vmatrix} = \begin{vmatrix} ax+by & bx+cy \\ bx+cy & cx+dy \end{vmatrix}^2 \div (ac-b^2)$$

$$= \begin{vmatrix} ax+by & cx+dy \\ bx+cy & dx+ey \end{vmatrix}^2 \div (ae-c^2) = \begin{vmatrix} bx+cy & cx+dy \\ cx+dy & dx+ey \end{vmatrix}^2 \div (ce-d^2).$$

Verification is readily effected by expressing the members of the desired identity in descending powers of x, namely,

$$\mathrm{E}x^4-2\mathrm{D}x^3y+(2\mathrm{C}+\Gamma)x^2y^2-2\mathrm{B}xy^3+\mathrm{A}y^4 = (\mathrm{E}x^2-\mathrm{D}xy+\mathrm{C}y^2)^2\div\mathrm{E},$$

and using the fact that the adjugate

$$\begin{vmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} \\ \mathrm{B} & \Gamma & \mathrm{D} \\ \mathrm{C} & \mathrm{D} & \mathrm{E} \end{vmatrix}$$

of the given vanishing determinant has all its primary minors equal to 0.

We may add that

$$-\begin{vmatrix} . & y^2 & xy & x^2 \\ y^2 & a & b & c \\ xy & b & c & d \\ x^2 & c & d & e \end{vmatrix}$$

is a fifth equivalent expression.

SCHRADER, W. (1887).

[Beiträge zur Theorie der Determinanten. vi+156 pp. Halle a. S.]

In the section on axisymmetric determinants (pp. 81–84) it is stated that *if* D *be any determinant and* X *be axisymmetric, then* DXD *is axisymmetric.* The cause of this being at variance with Muir's expression (1885) for the product of three determinants is that with Schrader row-by-row multiplication is meant. If the standard mode of multiplying in determinants be made that of matrices, it is DXD' that is axisymmetric, the superfixed comma indicating change of rows into columns.

SHARP, W. J. C. (1887).

[Questions 8940, 8970. *Math. from Educ. Times*, xlvi. pp. 131, 135; xlix. pp. 133–135, 136–138.]

The identities here dealt with belong to the same series as a simpler one formulated by us when describing a paper of Cauchy's of the year 1844 (*Hist.*, ii. p. 26) and referred to again in our notice of a different kind of generalization of it made by Cayley in 1846. What is now specially interesting is the mode of proof given by

Edwardes. For the case of the 4th order the result may be stated as follows : *The bilinear*

x_r	y_r	z_r	w_r	
a	n	m	p	x_s
n	b	l	q	y_s
m	l	c	r	z_s
p	q	r	d	w_s

being denoted by F_{rs},

then

$$|F_{11}F_{22}F_{33}| = \begin{array}{|cccc|c} |y_1z_2w_3| & |z_1w_2x_3| & |w_1x_2y_3| & |x_1y_2z_3| & \\ \hline A & N & M & P & |y_1z_2w_3| \\ N & B & L & Q & |z_1w_2x_3| \\ M & L & C & R & |w_1x_2y_3| \\ P & Q & R & D & |x_1y_2z_3| \end{array}.$$

Edwardes points out conclusively that the members of this asserted equality are the two forms of

$$\left\| \begin{matrix} \frac{\partial F_{11}}{\partial x_1} & \frac{\partial F_{11}}{\partial y_1} & \frac{\partial F_{11}}{\partial z_1} & \frac{\partial F_{11}}{\partial w_1} \\ \frac{\partial F_{22}}{\partial x_2} & \frac{\partial F_{22}}{\partial y_2} & \frac{\partial F_{22}}{\partial z_2} & \frac{\partial F_{22}}{\partial w_2} \\ \frac{\partial F_{33}}{\partial x_3} & \frac{\partial F_{33}}{\partial y_3} & \frac{\partial F_{33}}{\partial z_3} & \frac{\partial F_{33}}{\partial w_3} \end{matrix} \right\| \cdot \left\| \begin{matrix} x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{matrix} \right\|$$

known to us from the extended multiplication-theorem.

MUIR, T. (1888).

[On vanishing aggregates of determinants. *Proceed. R. Soc. Edinburgh*, xv. pp. 96–105.]

A fresh departure is made here, a vanishing aggregate being used which degenerates into Kronecker's when special values are given to a certain number of the elements involved. For the case of the 4th order the elements involved are

14	15	16	17	18
24	25	26	27	28
34	35	36	37	38
	45	46	47	48
		56	57	58
			67	68
				78,

and the identity is

$$\begin{vmatrix} 15 & 16 & 17 & 18 \\ 25 & 26 & 27 & 28 \\ 35 & 36 & 37 & 38 \\ 45 & 46 & 47 & 48 \end{vmatrix} - \begin{vmatrix} 14 & 16 & 17 & 18 \\ 24 & 26 & 27 & 28 \\ 34 & 36 & 37 & 38 \\ 45 & 56 & 57 & 58 \end{vmatrix} + \begin{vmatrix} 14 & 15 & 17 & 18 \\ 24 & 25 & 27 & 28 \\ 34 & 35 & 37 & 38 \\ 46 & 56 & 67 & 68 \end{vmatrix}$$

$$- \begin{vmatrix} 14 & 15 & 16 & 18 \\ 24 & 25 & 26 & 28 \\ 34 & 35 & 36 & 38 \\ 47 & 57 & 67 & 78 \end{vmatrix} + \begin{vmatrix} 14 & 15 & 16 & 17 \\ 24 & 25 & 26 & 27 \\ 34 & 35 & 36 & 37 \\ 48 & 58 & 68 & 78 \end{vmatrix} = 0.$$

But the five determinants here would be minors of any 8-line determinant in which

$$\begin{array}{rrrr} & & & 45 = 54, \\ & & 46 = 64, & 56 = 65, \\ & 47 = 74, & 57 = 75, & 67 = 76, \\ 48 = 84, & 58 = 85, & 68 = 86, & 78 = 87\,; \end{array}$$

and in the axisymmetric determinant

$$|\,11 \cdot 22 \cdot 33 \ldots 88\,|_{rs=sr}$$

all these conditions hold. Consequently, what we have reached is an identity connecting five of the minors of the said determinant, that is to say, Kronecker's identity.

The number of distinct elements of the axisymmetric determinant which do not appear in the identity is

$$m(2m+1) - \tfrac{1}{2}(m+1)(3m-2),$$

i.e.

$$\tfrac{1}{2}m^2+\tfrac{1}{2}m+1,$$

or, if we leave the diagonal elements out of account,

$$\tfrac{1}{2}m^2-\tfrac{3}{2}m+1, \quad i.e.\ \tfrac{1}{2}(m-1)(m-2).$$

These it is next sought to identify, and, as a consequence, the natural result is reached that they are those elements of the axisymmetric determinant which do not occur in the mixed array on which the theorem is here founded. Thus, in the example just used they are

$$\begin{array}{cc} 12 & 13 \\ & 23. \end{array}$$

From this it is suggested that the number of Kronecker identities connected with a given axisymmetric determinant can be found.

The curious result is also thus led up to that *if zeros be put in the places of the absent elements, the* 2m*-line Pfaffian associated with the axisymmetric determinant*—for example,

$$\begin{vmatrix} . & . & 14 & 15 & 16 & 17 & 18 \\ & . & 24 & 25 & 26 & 27 & 28 \\ & & 34 & 35 & 36 & 37 & 38 \\ & & & . & . & . & . \\ & & & & & 67 & 68 \\ & & & & & & 78 \end{vmatrix}$$

—represents exactly half the terms of the Kronecker identity, the other half being the same with opposite signs.

Lastly, the naturalness of the descent from a general to an axisymmetric determinant being accompanied by a descent from a quadratic to a linear relation between minors suggests that the axisymmetric is not the only special determinant with minors linearly related: and this is fully verified.

HERMES, J. (1888).

[Determinanten bei wiederholten Halbirung des ganzen Winkels. *Archiv d. Math. u. Phys.*, (2) vi. pp. 276–293.]

The main result of this lengthy paper is to the effect that *the* n*-line determinant whose* $(r, s)^{th}$ *element is*

$$\cos\frac{(2r-1)(2s-1)\pi}{4n}$$

is equal to $(\frac{1}{2}n)^{\frac{1}{2}n}$, n *being any power of* 2. Considerable space is devoted to the verification of the first five cases, each determinant being resolved into the product of two-line determinants. Thus, putting $c_1, c_3, \ldots$ for $\cos\frac{\pi}{16}, \cos\frac{3\pi}{16}, \ldots$, so that the four-line determinant is

$$\begin{vmatrix} c_1 & c_3 & c_5 & c_7 \\ c_3 & c_9 & c_{15} & c_{21} \\ c_5 & c_{15} & c_{25} & c_{35} \\ c_7 & c_{21} & c_{35} & c_{49} \end{vmatrix},$$

the writer increases the 4th column by the 1st and the 3rd by the 2nd, thus arriving at

$$2\cos\frac{\pi}{4}\cdot 2\cos\frac{\pi}{4}\begin{vmatrix} c_1 & c_3 & c_1 & c_3 \\ c_3 & -c_7 & -c_3 & c_7 \\ c_5 & -c_1 & -c_5 & c_1 \\ c_7 & -c_5 & c_7 & -c_5 \end{vmatrix}, \text{ thence } 2^3\begin{vmatrix} c_1 & c_3 & c_1 & c_3 \\ . & . & -c_3 & c_7 \\ . & . & -c_5 & c_1 \\ c_7 & -c_5 & c_7 & -c_5 \end{vmatrix},$$

and so forth. A generalization of the procedure is next attempted, and then the remaining half of the paper is devoted to remarks more or less of the nature of suggested thoughts and corollaries,—for example, the ineffectiveness of the substitution of sines for cosines, etc.

STIELTJES, T. J. (1888).

[Sur l'équation d'Euler. *Comptes Rendus . . . Acad. des Sci.* (Paris), cvii. pp. 617–618; *Bulletin des Sci. math.*, (2) xii. pp. 222–227.]

We have already seen Cayley using alternants (1877) in connection with the equation $\partial x/\sqrt{X} = \partial y/\sqrt{Y}$: the solution now given is

$$\begin{vmatrix} 0 & 1 & -\frac{1}{2}(x+y) & xy \\ 1 & a_0 & a_1 & a_2-2C \\ -\frac{1}{2}(x+y) & a_1 & a_2+C & a_3 \\ xy & a_2-2C & a_3 & a_4 \end{vmatrix} = 0,$$

where the a's are the coefficients of the originating quartic and C is an arbitrary constant.

SHARP, W. J. C. (1888).

[Question 9779. *Math. from Educ. Times*, xlix. pp. 141–142; (2) iv. pp. 83–86.]

The first of the three "new" determinants here evaluated is Ferrers' of 1861; the second is Scott's of 1880; and the third is a simple case of Sylvester's circulant of 1855. Interest attaches to the second only, the result now given being

$$\begin{vmatrix} . & a_1+a_2 & a_1+a_3 & \dots \\ a_1+a_2 & . & a_2+a_3 & \dots \\ a_1+a_3 & a_2+a_3 & . & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n$$

$$= \tfrac{1}{2}(-2)^{n-1}a_1a_2\dots a_n\left\{(a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)-(n-2)^2\right\}$$

or, say, for shortness' sake,

$$\tfrac{1}{2}(-2)^{n-1}\Pi(a)\left\{\sum(a)\cdot\sum\left(\frac{1}{a}\right)-(n-2)^2\right\}.$$

In both cases the form of the evaluation is doubtless influenced by the form of the evaluation of a determinant preceding, and, unfortunately, both forms conceal the important fact that there are not more than two kinds of terms in the final expansion. A better form is

$$(-2)^{n-2}\left\{(n-1)(n-4)a_1a_2\ldots a_n-\sum a_1^2a_2\ldots a_{n-1}\right\},$$

which also has the merit of attracting attention to the case where $n=4$, the result then being simply

$$-4\sum a_1^2a_2a_3.$$

CAYLEY, A. (1888): MUIR, T. (1889).

[Note on the relation between the distances of five points in space. *Messenger of Math.*, xviii. pp. 100–102.]

[Note on the relation between the mutual distances of five points in space. *Proceed. R. Soc. Edinburgh*, xvi. pp. 86–88.]

Cayley draws attention to the fact that in a memoir of Lagrange's * there is contained the relation

$$\begin{aligned}0 \;=\; &-4\Delta^2f+\alpha(a+f-g)^2+\alpha'(a'+f-g')^2+\alpha''(a''+f-g'')^2\\&+2\beta(a'+f-g')(a''+f-g'')+2\beta'(a+f-g)(a''+f-g'')\\&+2\beta''(a+f-g)(a'+f-g'),\end{aligned}$$

where, if the points be called 1, 2, 3, 4, 5, the letters

$$\left.\begin{array}{cccc} c'' & c' & a & g \\ & c & a' & g' \\ & & a'' & g'' \\ & & & f \end{array}\right\}$$ represent the squares of the distances $$\left\{\begin{array}{cccc} 12 & 13 & 14 & 15 \\ & 23 & 24 & 25 \\ & & 34 & 35 \\ & & & 45 \end{array}\right.$$

* Lagrange, J. L. Solutions analytiques de quelques problèmes sur les pyramides triangulaires. *Nouv. Mém. de l'Acad.* . . . (de Berlin), Ann. 1773, pp. 149–176; or *Œuvres*, iii. (pp. 659–692), p. 677.

and Δ, a, a', a'', β, β', β'' are certain functions of the said distances. Reason is given for believing that this cannot be essentially different from Cayley's own relation of 1841 (*Hist.*, ii. pp. 109–110), and the belief is justified by an actual verification.

Following on this Muir shows that the one vanishing expression can be directly transformed into the other. Beginning with Cayley's determinant

$$\begin{vmatrix} . & c'' & c' & a & g & 1 \\ c'' & . & c & a' & g' & 1 \\ c' & c & . & a'' & g'' & 1 \\ a & a' & a'' & . & f & 1 \\ g & g' & g'' & f & . & 1 \\ 1 & 1 & 1 & 1 & 1 & . \end{vmatrix}$$

he subtracts the 4th column from each of the others except the last, and thereafter the 4th row from each of the other rows except the last, thus arriving at

$$\begin{vmatrix} -2a & c''-a-a' & c'-a-a'' & g-a-f \\ c''-a'-a & -2a' & c-a'-a'' & g'-a'-f \\ c'-a''-a & c-a''-a & -2a'' & g''-a''-f \\ g-f-a & g'-f-a' & g''-f-a'' & -2f \end{vmatrix},$$

which, when expressed according to binary products of the last row and column, gives Lagrange's expression

$$-4\Delta^2 f + a(a+f-g)^2 + \dots .$$

He is thus led to suggest a fresh expression for the relationship in question, namely,

$$\begin{vmatrix} 15^2+15^2-11^2 & 25^2+15^2-21^2 & 35^2+15^2-31^2 & 45^2+15^2-41^2 \\ 15^2+25^2-12^2 & 25^2+25^2-22^2 & 35^2+25^2-32^2 & 45^2+25^2-42^2 \\ 15^2+35^2-13^2 & 25^2+35^2-23^2 & 35^2+35^2-33^2 & 45^2+35^2-43^2 \\ 15^2+45^2-14^2 & 25^2+45^2-24^2 & 35^2+45^2-34^2 & 45^2+45^2-44^2 \end{vmatrix} = 0.$$

Further, the determinant here proves to be a missing link giving connection with another already-known expression—that with cosines for elements,—all that is needful being to divide in every case the s^{th} row and s^{th} column by $s5$.

KRONECKER, L. (1889).

[Ueber symmetrische Systeme. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), pp. 349–362; or *Werke*, iii. pp. 295–314.]

The main contents of this paper do not concern us; but the foundational portion (pp. 350–354), which deals with the 'composition of systems,' or what Cayley at an early date had spoken of as 'multiplication of matrices,' craves suitable notice. It consists of a series of numbered results culminating in the theorem that *any non-zero axisymmetric determinant can be transformed so as to have all its elements zeros except those in the diagonal, and that the reverse changes can also be effected.* The noteworthy point about it is that every one of the necessary changes is brought about by multiplying by a unit determinant. For example, the determinant being three-line, the operation of increasing the 2nd row by t times the 1st, and thereafter the 2nd column by t times the 1st, takes the form

$$\begin{vmatrix} 1 & t & . \\ . & 1 & . \\ . & . & 1 \end{vmatrix} \cdot \begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} \cdot \begin{vmatrix} 1 & . & . \\ t & 1 & . \\ . & . & 1 \end{vmatrix};$$

and the interchange of the 1st and 2nd rows followed by interchange of the 1st and 2nd columns is effected by

$$\begin{vmatrix} -1 & . \\ 1 & . \\ . & 1 \end{vmatrix} \begin{vmatrix} 1 & . & . \\ 1 & 1 & . \\ . & . & 1 \end{vmatrix} \begin{vmatrix} 1 & -1 & . \\ . & 1 & . \\ . & . & 1 \end{vmatrix} \begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} \begin{vmatrix} 1 & . & . \\ -1 & 1 & . \\ . & . & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & . \\ . & 1 & . \\ . & . & 1 \end{vmatrix} \begin{vmatrix} 1 & . & . \\ -1 & 1 & . \\ . & . & 1 \end{vmatrix},$$

where it is to be noted that any determinant to the right of the given axisymmetric determinant is the conjugate of the corresponding determinant to the left, in accordance with the theorem that *if* $|\,z_{1n}\,|$ *be axisymmetric and* $|\,a_{1n}\,|$ *be any determinant whatever, then*

$$|\,a_{1n}\,| \cdot |\,z_{1n}\,| \cdot |\,a_{1n}\,|'$$

is also axisymmetric. Unfortunately, no reference is made to H. J. S. Smith's similar work of 1861. (See below in Chap. XXI.)

In the last section (pp. 358–362) Kronecker advances to the similar consideration of a *general* determinant, and he continues the subject at some length a month or two later.*

* Kronecker, L. Die Decomposition der Systeme von n^2 Grossen . . . *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), 1889, pp. 474–505, 603–614; or *Werke*, iii. pp. 315–368.

SYLVESTER, J. J. (1890).

[Question 10751. *Educ. Times*, xliii. p. 484; lxv. (1912), p. 263, pp. 337–338.]

[Question 10792. *Educ. Times*, xliii. p. 528.]

These questions were neglected for over twenty years, on account, most probably, of inaccuracy in the statement of the first of them. Ultimately Muir pointed out what the theorem intended to be enunciated really was, and extended it to determinants in general, his enunciation being *If* Δ *and* δ *be any* n*-line determinant and the complementary minor of its last element, and* S, s *be the sums of the primary coaxial minors of* Δ, δ *respectively, then*

$$\begin{vmatrix} \delta & \mathrm{s} \\ \Delta & \mathrm{S} \end{vmatrix}$$

is equal to the product of the first n−1 *rows of* Δ *by the first* n−1 *rows of the conjugate of* Δ. Thus, for the case of n equal to 4, it is

$$\begin{vmatrix} \mathrm{D}_4 & |a_1b_2|+|a_1c_3|+|b_2c_3| \\ |a_1b_2c_3d_4| & \mathrm{A}_1+\mathrm{B}_2+\mathrm{C}_3+\mathrm{D}_4 \end{vmatrix} = \begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{Vmatrix} \cdot \begin{Vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{Vmatrix},$$

which can be verified by noting that the right-hand member equals

$$\mathrm{D}_4{}^2+\mathrm{D}_3\mathrm{C}_4+\mathrm{D}_2\mathrm{B}_4+\mathrm{D}_1\mathrm{A}_4,$$

and that

$$|a_1b_2c_3d_4|\{|a_1b_2|+|a_1c_3|+|b_2c_3|\}_j = |\mathrm{C}_3\mathrm{D}_4|+|\mathrm{B}_2\mathrm{D}_4|+|\mathrm{A}_1\mathrm{D}_4|.$$

The theorem involved in the other 'question' is that *If the sum of the squares of the primary minors of an axisymmetric determinant vanish, and if in addition either the determinant itself vanish or the sum of its coaxial secondary minors, then the sum of the coaxial primary minors must also vanish.* Part of the basis of this is an identity just seen to be necessary for the establishment of the previous theorem, namely, that *in the case of an* n*-line axisymmetric determinant the sum of the coaxial* r*-line minors of the adjugate is equal to the product of the original by the sum of the* (n−r)*-line minors of the latter.* Thus, when r is 2 and the determinant is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \text{or } \Delta_3 \text{ say,}$$

we have

$$\begin{vmatrix} B & F \\ F & C \end{vmatrix} + \begin{vmatrix} A & G \\ G & C \end{vmatrix} + \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \Delta_3 (a+b+c).$$

Consequently, if either Δ_3 or $a+b+c$ vanish, there results

$$BC+CA+AB = F^2+G^2+H^2;$$

and the sum of the squares of the primary minors being

$$A^2+B^2+C^2+2(F^2+G^2+H^2)$$

is equal to

$$A^2+B^2+C^2+2BC+2CA+2AB,$$

i.e.

$$(A+B+C)^2.$$

It thus follows from the first datum that $A+B+C$ vanishes also, as asserted.

HORTA, F. DA P. (1890).

(See under this heading in Chap. I.)

LONGCHAMPS, G. DE (1890, 1891).

[Questions 298, 332. *Journ. de Math. Spéc.*, (3) iv. p. 263; (4) i. pp. 142–144; (3) v. p. 264.]

The theorem here is to the effect that *the roots of the equation resulting from putting equal to* 0 *Torelli's determinant of* 1864 *are either all real or all real except two.* The subject dates back to Garbieri's text-book of 1874 (*Hist.*, iii. pp. 100–101, 109–110). In view of the fact that the so-called proof which is given contains a serious error, it may be worth noting that the determinant is expressible as an axisymmetric continuant: for example, when n is 4 it is equal to

$$\begin{vmatrix} a_1+a_2-2x & x-a_2 & . & . \\ x-a_2 & a_2+a_3-2x & x-a_3 & . \\ . & x-a_3 & a_3+a_4-2x & x-a_4 \\ . & . & x-a_4 & a_4 \end{vmatrix}.$$

The author's other determinant results from bordering Torelli's with the row $\lambda, \alpha, \alpha, \alpha, \ldots$ and the column $\lambda, \beta, \beta, \beta, \ldots$.

MUIR, T. (1891).

[Note on a peculiar determinant of the sixth order. *Philos. Magazine*, (5) xxxi. pp. 429–430.]

This concerns the identity

$$\begin{vmatrix} . & c & b & -2a' & . & . \\ c & . & a & . & -2b' & . \\ b & a & . & . & . & -2c' \\ a' & . & . & a & -c' & -b' \\ . & b' & . & -c' & b & -a' \\ . & . & c' & -b' & -a' & c \end{vmatrix} = 2 \begin{vmatrix} a & c' & b' \\ c' & b & a' \\ b' & a' & c \end{vmatrix}^2,$$

of which Cayley desired to have a direct proof. Muir's procedure consists essentially in the multiplication of $-a^2b^2c^2$ in the form

$$\begin{vmatrix} -\frac{1}{2}ac' & -\frac{1}{2}bc' & \frac{1}{2}cc' & . & . & ab \\ -\frac{1}{2}ab' & \frac{1}{2}bb' & -\frac{1}{2}cb' & . & ca & . \\ \frac{1}{2}aa' & -\frac{1}{2}ba' & -\frac{1}{2}ca' & bc & . & . \\ . & . & 1 & . & . & . \\ . & 1 & . & . & . & . \\ 1 & . & . & . & . & . \end{vmatrix}$$

by the left-hand member, the product obtained being manifestly resolvable in a way suitable for giving the desired result (*Hist.*, i. p. 244 ; ii. pp. 143–144).

In the following year another proof was incidentally given by him (*Proceed. R. Soc. Edinburgh*, xx. p. 302).

LAURENT, H. (1892).

[Sur l'élimination. *Nouv. Annales de Math.*, (3) xi. pp. 5–7.]

The resultant here found for two binary quantics is essentially the same as Borchardt's of the year 1859, the determinant involved thus being the axisymmetric case of Sylvester's unisignant of 1855 (*Hist.*, ii. pp. 456–459). With Laurent, however, the quantics are no longer restricted to the same degree ; and the mode of arriving at the resultant is entirely different, being simple and clear and not at all lengthy.

MUIR, T. (1892): CAYLEY, A. (1894).

It deserves note under this head in Chap. I. that the study of Sylvester's well-known set of equations

$$0 = cy^2-2fyz+bz^2 = az^2-2gzx+cx^2 = bx^2-2hxy+ay^2$$

practically brought out the result that *if the cofactors of* a, b, c *in*

$$\begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & . \end{vmatrix}$$

vanish, then the other three-line minors vanish also.

SYLVESTER, J. J. (1892).

[Question 11580. *Educ. Times*, xlv. p. 306, lxii. pp. 329–330; *Math. from Educ. Times*, (2) xvii. p. 40.]

The theorem meant to be announced here is interesting in connection with the author's paper of 1853 (*Hist.*, ii. pp. 127-129), being that

$$\text{Norm}\left[1 + (f+\sqrt{f^2-1})(g+\sqrt{g^2-1})(h+\sqrt{h^2-1})\right] = 16 \begin{vmatrix} 1 & -h & -g \\ -h & 1 & -f \\ -g & -f & 1 \end{vmatrix}^2.$$

It was not established till 1909, when Muir made the requisite correction and gave a proof.

THE PASSING OF SYLVESTER.

It deserves note that after the year here reached there is no record of any direct contribution made by Sylvester to the pure theory of determinants, although his mind continued to be active on other subjects almost till the day of his death, 15th March, 1897. His first paper of all was published in 1838, and his first paper on determinants,—then known to him only as 'zeta-ic products of differences'—in 1839. Thus in a working life of almost sixty years, determinants figured for about fifty-four,—exactly the same stretch of time as in the case of his great contemporary and co-worker, Cayley. See above, p. 52.

PRIME, F. (1892).

[Sur un déterminant nul. *Journ. de Math. Spéc.*, xvi. pp. 177–179.]

The determinant in question is the n-line determinant whose $(r, s)^{\text{th}}$ element is $\sin(a_r+a_s)$, or, say for shortness' sake,

$$|\sin(a_r+a_s)|_n.$$

It manifestly equals

$$\begin{vmatrix} \sin a_1 & \cos a_1 & 0 & 0 & \dots \\ \sin a_2 & \cos a_2 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \sin a_n & \cos a_n & 0 & 0 & \dots \end{vmatrix} \times \begin{vmatrix} \cos a_1 & \sin a_1 & 0 & 0 & \dots \\ \cos a_2 & \sin a_2 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \cos a_n & \sin a_n & 0 & 0 & \dots \end{vmatrix};$$

and $\therefore$ vanishes when $n > 2$. So does $|\cos(a_r+a_s)|_n$; and we remark for ourselves, further, that endless results of this kind are included in the general proposition that *the product of two* n-*by*-k *arrays is an* n-*line determinant which necessarily vanishes when* $n > k$ (see *Hist.*, ii. pp. 465, 468; iii. pp. 122, 322; iv. p. 109).

LEUDESDORF, C. (1893): CESÀRO, E. (1894): TUCKER, R. (1894).

[Question 12015. *Educ. Times*, xlvi. p. 305; *Math. from Educ. Times*, lx. p. 89.]

[Corso di Analisi Algebrica, ... viii+500 pp. Torino.]

[Question 12532. *Educ. Times*, xlvii. p. 440; *Math. from Educ. Times*, lxiii. p. 116.]

The identity here given in regard to

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix},$$

namely,

$$\begin{vmatrix} hf'+h'f-bg'-b'g & gh'+g'h-af'-a'f & ab'+a'b-2hh' \\ \mathrm{G} & \mathrm{F} & \mathrm{C} \\ \mathrm{G}' & \mathrm{F}' & \mathrm{C}' \end{vmatrix}$$

$$= \begin{vmatrix} a'\mathrm{G}+h'\mathrm{F}+g'\mathrm{C} & h'\mathrm{G}+b'\mathrm{F}+f'\mathrm{C} \\ a\mathrm{G}'+h\mathrm{F}'+g\mathrm{C}' & h\mathrm{G}'+b\mathrm{F}'+f\mathrm{C}' \end{vmatrix}$$

is established by verifying the equality of the cofactors of C on the two sides, also the cofactors of F, and the cofactors of G.*

The identity involved in the other ' question ' is of little moment.

Cesàro's simple but interesting result is

$$\begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 & \dots & a_1b_n \\ a_1b_2 & a_2b_2 & a_2b_3 & \dots & a_2b_n \\ a_1b_3 & a_2b_3 & a_3b_3 & \dots & a_3b_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1b_n & a_2b_n & a_3b_n & \dots & a_nb_n \end{vmatrix} = a_1b_n \,|\, a_2b_1 \,|\cdot|\, a_3b_2 \,|\cdot \dots \cdot |\, a_nb_{n-1} \,|.$$

NEUBERG, J. (1893, 1896).

[Questions 820, 1080. *Mathesis*, (2) iii. p. 79, pp. 172–173 ; vi. p. 215 ; vii. pp. 232–233.]

[Question 242. *Revue de Math. Spéc.*, ii. p. 195.]

The result here given, which is said to be a generalization of one that appeared in 1892 in an Aberystwyth examination paper, is

$$\begin{vmatrix} 2 & b+\frac{1}{b} & c+\frac{1}{c} \\ b+\frac{1}{b} & 2 & a+\frac{1}{a} \\ c+\frac{1}{c} & a+\frac{1}{a} & 2 \end{vmatrix} = 2(bc-a)(ca-b)(ab-c)(abc-1)^2a^2b^2c^2.$$

---

* Here there might have been suggested a very interesting result in regard to the minors of a 3-by-4 array : namely, the array being

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix},$$

we have

$$\begin{vmatrix} |a_2d_3| - |b_2c_3| & |a_2b_3| & |c_2d_3| \\ |a_3d_1| - |b_3c_1| & |a_3b_1| & |c_3d_1| \\ |a_1d_2| - |b_1c_2| & |c_1b_2| & |c_1d_2| \end{vmatrix} = \begin{vmatrix} |a_1b_2c_3| & |a_1b_2d_3| \\ |a_1c_2d_3| & |b_1c_2d_3| \end{vmatrix}.$$

This is easily established by multiplying the left member columnwise by $|b_1c_2d_3|$. It involves all the minors except six,

$$|a_1c_2|,\ |a_1c_3|,\ |a_2c_3|,\ |b_1d_2|,\ |b_1d_3|,\ |b_2d_3|.$$

The special case referred to is where the factor $abc-1$ vanishes because of taking

$$a,\ b,\ c = \frac{l}{m},\quad \frac{m}{n},\quad \frac{n}{l}.$$

We may note that interest attaches to the factorization on account of the determinant being the duplicant of

$$\begin{vmatrix} 1 & b & c \\ \frac{1}{b} & 1 & a \\ \frac{1}{c} & \frac{1}{a} & 1 \end{vmatrix},$$

the value of which is $(ab-c)^2/abc$. In connection with the invariance of the determinants to the substitution

$$a,\ b,\ c = \frac{1}{a},\quad \frac{1}{b},\quad \frac{1}{c}$$

it is well also to note how the factors change, for example

$$bc-a \quad \text{into} \quad (bc-a) \times \frac{-1}{abc}.$$

SCHAPIRA, H. (1893).

[Ueber symmetrische quadratische Formen. *Jahresb. d. deutschen math. Verein.*, iii. pp. 99–102.]

Incidentally there is here considered the subject of determinants resolvable into rational linear factors, there being given as examples of a general theorem the circulant C $(x,\ a,\ a)$ and the interesting result

$$\begin{vmatrix} x & a & a & a & a & b & a & a & b & b \\ a & x & a & a & b & a & a & b & a & b \\ a & a & x & b & a & a & b & a & a & b \\ a & a & b & x & a & a & a & b & b & a \\ a & b & a & a & x & a & b & a & b & a \\ b & a & a & a & a & x & b & b & a & a \\ a & a & b & a & b & b & x & a & a & a \\ a & b & a & b & a & b & a & x & a & a \\ b & a & a & b & b & a & a & a & x & a \\ b & b & b & a & a & a & a & a & a & x \end{vmatrix} = \begin{cases} (x+6a+3b) \\ \cdot(x+a-2b)^4 \\ \cdot(x-2a+b)^5, \end{cases}$$

where the construction of the determinant is best indicated by denoting its rows in order by

11, 12, 13, 14, 15, 23, 24, 25, 34, 35, 45,

and its columns in the same way, and then filling the $(\alpha\beta, \gamma\delta)^{\text{th}}$ place with an x when $\alpha\beta$ and $\gamma\delta$ do not at all differ, with an a when they differ in one of the integers, and a b when they differ in both.

METZLER, W. H. (1893).

[Compound determinants. *American Journ. of Math.*, xvi. pp. 131–150.]

The first theorem here given (pp. 146–148) is Gravelaar's of 1875, and the other is that *All the minors of order* n−m *in an* n*-line axisymmetric determinant will vanish if the coaxial minors of this order vanish, and if for each of the higher orders the sum of the coaxial minors vanishes.* The latter follows from one of the main theorems (pp. 132–145) of the paper regarding sums of sets of k-line minors of the m^{th} compound of $|\, a_{1n} \,|$.

FROBENIUS, G. (1894).

[Ueber das Trägheitsgesetz der quadratischen Formen. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), pp. 241–256, 407–431 ; or *Crelle's Journ.*, cxiv. pp. 187–230.]

Frobenius here formulates (§ 2, pp. 245–246) a set of three theorems regarding the evanescence of minors, the third being that which he had arrived at in 1876, though at that date less readily. They are: (1) *If in an axisymmetric determinant* $|\, a_{1n} \,|$ *the* r*-line coaxial minor* $|\, a_{11}a_{22} \ldots a_{rr} \,|$ *does not vanish, but all minors of the forms*

$$|\, a_{11}a_{22} \ldots a_{rr}a_{\sigma\sigma} \,|, \quad |\, a_{11}a_{22} \ldots a_{rr}a_{\sigma\sigma}a_{\tau\tau} \,|$$

do vanish, then the non-coaxial minors of the $(r+1)^{th}$ *order vanish also.* (2) *If in an axisymmetric determinant the coaxial minors of the* r^{th} *order and of the* $(r+1)^{th}$ *order all vanish, then the non-coaxial minors of the* r^{th} *and higher orders vanish also.* (3) *If in an axisymmetric determinant the minors of order* r+1 *all vanish, but not all those of order* r*, then one of the latter is coaxial.* (See *Hist.*, iii. p. 275.)

He also at a later stage (§ 12) gives a proof of Kronecker's so-called linear relation of 1882.

CIAMBERLINI, C. (1895): LOVETT, E. O. (1897).

[Intorno alla relazione tra le distanze di 5 punti dello spazio. *Giornale di Mat.*, xxxiv. pp. 279–289.]

[Note on the invariants of n points. *Bull. American Math. Soc.*, iv. pp. 58–59.]

Both of these touch lightly on Cayley's determinants of 1841 (*Hist.*, ii. pp. 109–110). The second is interesting from its fresh standpoint.

WHITE, H. S. (1895).

[Kronecker's linear relation among the minors of a symmetric determinant. *Bull. American Math. Soc.*, ii. pp. 136–138.]

This note makes the fresh suggestion that Bezout's relation between the three-line minors of a 3-by-6 array, namely,

$$|a_1b_2c_3|\cdot|a_4b_5c_6| = |a_1b_2c_4|\cdot|a_3b_5c_6| + |a_1b_2c_5|\cdot|a_4b_3c_6| + |a_1b_2c_6|\cdot|a_4b_5c_3|,$$

if not the source of Kronecker's linear relation, can at least be viewed as a basis for establishing the latter. For, performing columnwise the multiplications indicated in Bezout's relation, and denoting by $r\cdot s$ the product of the r^{th} and s^{th} columns, there is obtained

$$\begin{vmatrix} 1\cdot4 & 2\cdot4 & 3\cdot4 \\ 1\cdot5 & 2\cdot5 & 3\cdot5 \\ 1\cdot6 & 2\cdot6 & 3\cdot6 \end{vmatrix} = \begin{vmatrix} 1\cdot3 & 2\cdot3 & 4\cdot3 \\ 1\cdot5 & 2\cdot5 & 4\cdot5 \\ 1\cdot6 & 2\cdot6 & 4\cdot6 \end{vmatrix} + \begin{vmatrix} 1\cdot4 & 2\cdot4 & 5\cdot4 \\ 1\cdot3 & 2\cdot3 & 5\cdot3 \\ 1\cdot6 & 2\cdot6 & 5\cdot6 \end{vmatrix} + \begin{vmatrix} 1\cdot4 & 2\cdot4 & 6\cdot4 \\ 1\cdot5 & 2\cdot5 & 6\cdot5 \\ 1\cdot3 & 2\cdot3 & 6\cdot3 \end{vmatrix},$$

which is Kronecker's linear relation connected with the determinant

$$|1\cdot1 \quad 2\cdot2 \quad 3\cdot3 \quad 4\cdot4 \quad 5\cdot5 \quad 6\cdot6|,$$

the axisymmetry of this last being implicated in the definition of $r\cdot s$. The possible objection that a theorem in determinants might hold for elements of the type

$$a_ra_s+b_rb_s+c_rc_s,$$

and not hold for all others, is met by pointing out that as the linear relation involves no diagonal elements of the six-line determinant, there are available for determining the 15 non-diagonal elements

the 18 parameters of the 3-by-6 array. It is added that this reasoning is always applicable since, just as $15 < 18$, so generally

$$\frac{2m(2m-1)}{2} < 2m^2.$$

MUIR, T. (1897).

[The automorphic linear transformation of a quadric. *Transac. R. Soc. Edinburgh*, xxxix. pp. 209–230.]

Incidentally (§ 11, pp. 225–226) attention is drawn to the fact that *in the axisymmetric determinant of the fifth order there exist three secondary minors whose aggregate vanishes*, namely,

$$\begin{vmatrix} 135 \\ 124 \end{vmatrix} - \begin{vmatrix} 125 \\ 134 \end{vmatrix} - \begin{vmatrix} 145 \\ 123 \end{vmatrix} = 0.$$

Further, the identity is viewed as the ' extensional ' of

$$\begin{vmatrix} 23 \\ 45 \end{vmatrix} - \begin{vmatrix} 24 \\ 35 \end{vmatrix} + \begin{vmatrix} 25 \\ 34 \end{vmatrix} = 0,$$

which, we know from Kronecker, holds for the axisymmetric determinant

$$\left| \begin{matrix} 2345 \\ 2345 \end{matrix} \right|_{rs=sr};$$

not only so, but the general applicability of the Law of Extensible Minors to Kronecker's identities is thus suggested; for example, knowing that

$$\begin{vmatrix} 123 \\ 456 \end{vmatrix} - \begin{vmatrix} 124 \\ 356 \end{vmatrix} + \begin{vmatrix} 125 \\ 346 \end{vmatrix} - \begin{vmatrix} 126 \\ 345 \end{vmatrix} = 0$$

if the determinants be minors of the axisymmetric determinant

$$\left| \begin{matrix} 123456 \\ 123456 \end{matrix} \right|_{rs=sr},$$

we can at once vouch for the identity

$$\begin{vmatrix} 12378\ldots \\ 45678\ldots \end{vmatrix} - \begin{vmatrix} 12478\ldots \\ 35678\ldots \end{vmatrix} + \begin{vmatrix} 12578\ldots \\ 34678\ldots \end{vmatrix} - \begin{vmatrix} 12678\ldots \\ 34578\ldots \end{vmatrix} = 0$$

in connection with the axisymmetric determinant

$$\left| \begin{matrix} 12345678\ldots \\ 12345678\ldots \end{matrix} \right|_{rs=sr}.$$

CHAPTER V.

SYMMETRIC DETERMINANTS THAT ARE NOT AXISYMMETRIC, FROM 1884 TO 1897.

LEUDESDORF, C. (1884).

[Question 7718. *Educ. Times*, xxxvii. pp. 162, 211; *Math. from Educ. Times*, xlii. p. 26.]

The determinant, D_n, evaluated here is one in which each odd-numbered diagonal element is a, each even-numbered diagonal element 0, and every other element b. It is found that

$$D_{2m+2} = (a-2b)D_{2m+1} - (a-b)^2 D_{2m},$$
$$D_{2m+1} = (a-2b)D_{2m} - b^2 D_{2m-1};$$

and thence that

$$D_{2m+2} = (-1)^m b^{m+1}(a-b)^m\{ma-(2m+1)b\},$$
$$D_{2m+1} = (-1)^{m+1}b^m(a-b)^m\{(m-1)a-2mb\}.$$

TUCKER, R. (1886).

[Question 8609. *Educ. Times*, xxxix. p. 227; *Math. from Educ. Times*, xlvi. pp. 69–70.]

The identity here noted is much better put in the form

$$\begin{vmatrix} -\alpha & \gamma+\alpha & \alpha+\beta \\ \beta+\gamma & -\beta & \alpha+\beta \\ \beta+\gamma & \gamma+\alpha & -\gamma \end{vmatrix} = (\alpha+\beta+\gamma)^3.$$

This also has the merit of suggesting a 'next case,' in which the first column is

$$-\alpha, \quad \beta+\gamma+\delta, \quad \beta+\gamma+\delta, \quad \beta+\gamma+\delta,$$

and the right-hand member

$$-2(\alpha+\beta+\gamma+\delta)^4.$$

EMMERICH, . (1888).

[Question 9906. *Educ. Times*, xli. p. 489 ; *Math. from Educ. Times*, li. p. 96.]

The equation here set for solution has for its non-zero member the case of Ferrers' determinant, in which the diagonal elements are

$$\frac{\sin(x-\alpha_1)}{\sin x}, \quad \frac{\sin(x-\alpha_2)}{\sin x}, \quad \ldots, \quad \frac{\sin(x-\alpha_n)}{\sin x},$$

and the remaining elements of the r^{th} column each equal to $\cos \alpha_r$; and Wolstenholme shows that the determinant is equal to

$$(-1)^n \sin\alpha_1 \sin\alpha_2 \ldots \sin\alpha_n \cdot (\cot x)^n \cdot \{1 - \tan x(\cot\alpha_1 + \ldots + \cot\alpha_n)\},$$

and that therefore the non-zero value of $\cot x$ is

$$\cot\alpha_1 + \cot\alpha_2 + \ldots + \cot\alpha_n.$$

HOWSE, G. F. (1889, 1890) : SHARP, W. J. C. (1891, 1894).

[Question 10354. *Educ. Times*, xlii. p. 483, xliii. p. 38 ; *Math. from Educ. Times*, liii. p. 38.]

[Questions 10934, 10977, 11184, 12316. *Educ. Times*, xliv. p. 157, p. 199, p. 345, xlvii. p. 199 ; *Math. from Educ. Times*, lv. p. 123, p. 116, lxiii. p. 36.]

The first result is already known, the others already familiar.

GILLET, J. (1894).

[Question 395. *L'Intermédiaire des Math.*, i. p. 235, ii. p. 127 ; solution by A. Hurwitz, ii. pp. 367–368.]

What seems desired here is a development of

$$\begin{vmatrix} (s-\alpha_1)^p & \alpha_1^p & \alpha_1^p & \ldots & \alpha_1^p \\ \alpha_2^p & (s-\alpha_2)^p & \alpha_2^p & \ldots & \alpha_2^p \\ \alpha_3^p & \alpha_3^p & (s-\alpha_3)^p & \ldots & \alpha_3^p \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \alpha_n^p & \alpha_n^p & \alpha_n^p & \ldots & (s-\alpha_n)^p \end{vmatrix},$$

where $s = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, similar to the developments in the cases $n, p = 3, 2$ (Salmon, 1866), $n, p = 4, 2$ (Wolstenholme, 1870). Unfortunately the solvers consider it sufficient to make a mere substitution in Ferrers' result of 1855.

MUIR, T. (1888).

[On vanishing aggregates of determinants. *Proceed. R. Soc. Edinburgh*, xv. pp. 96–105.]

One of the main results of the paper is to the effect that by combining in different ways the rows and columns of two given n-line determinants there may be formed $2n$ n-line determinants connected by a linear relation with unit coefficients. For example, the originating determinants being

$$\begin{vmatrix} 41 & 42 & 43 \\ 51 & 52 & 53 \\ 61 & 62 & 63 \end{vmatrix}, \quad \begin{vmatrix} 46 & 45 & 44 \\ 56 & 55 & 54 \\ 66 & 65 & 64 \end{vmatrix},$$

we have the identity

$$\begin{vmatrix} 41 & 45 & 44 \\ 51 & 55 & 54 \\ 61 & 65 & 64 \end{vmatrix} + \begin{vmatrix} 46 & 42 & 44 \\ 56 & 52 & 54 \\ 66 & 62 & 64 \end{vmatrix} + \begin{vmatrix} 46 & 45 & 43 \\ 56 & 55 & 53 \\ 66 & 65 & 63 \end{vmatrix}$$

$$= \begin{vmatrix} 41 & 42 & 43 \\ 56 & 55 & 54 \\ 66 & 65 & 64 \end{vmatrix} + \begin{vmatrix} 46 & 45 & 44 \\ 51 & 52 & 53 \\ 66 & 65 & 64 \end{vmatrix} + \begin{vmatrix} 46 & 45 & 44 \\ 56 & 55 & 54 \\ 61 & 62 & 63 \end{vmatrix}.$$

Now the first three determinants here are minors of any six-line determinant, and the second three *would be* minors of such a determinant if its elements were so related that

$$\begin{aligned} &41,\ 42,\ 43,\ 51,\ 52,\ 53,\ 61,\ 62,\ 63 \\ &= 36,\ 35,\ 34,\ 26,\ 25,\ 24,\ 16,\ 15,\ 14\ ; \end{aligned}$$

—in other words, so that $r,\ s = 7-r,\ 7-s$. But this is exactly the definition of a centro-symmetric determinant. It thus follows that *if* $|\ 11,\ 22,\ \ldots,\ 66\ |$ *be centro-symmetric, then a linear relation connects six of its three-line minors, namely,*

$$\begin{vmatrix} 456 \\ 451 \end{vmatrix} + \begin{vmatrix} 456 \\ 426 \end{vmatrix} + \begin{vmatrix} 456 \\ 356 \end{vmatrix}$$

$$= \begin{vmatrix} 451 \\ 456 \end{vmatrix} + \begin{vmatrix} 426 \\ 456 \end{vmatrix} + \begin{vmatrix} 356 \\ 456 \end{vmatrix}.$$

NEUBERG, J. (1897).

[Question 1120. *Mathesis*, (2) vii. pp. 126, 260–261.]

The result here is

$$\begin{vmatrix} \sin\alpha & \sin\beta & \sin\gamma & \sin\delta \\ \cos\alpha & \cos\beta & \cos\gamma & \cos\delta \\ \sin\beta & \sin\alpha & \sin\delta & \sin\gamma \\ \cos\beta & \cos\alpha & \cos\delta & \cos\gamma \end{vmatrix} = -4\sin(\alpha-\beta)\sin(\gamma-\delta) \cdot \sin^2\tfrac{1}{2}(\alpha+\beta-\gamma-\delta).$$

Note might have been taken that the determinant merely differs in sign from the centro-symmetric determinant whose eight elements are the sines and cosines of α, γ, δ, β.

CHAPTER VI.

ALTERNANTS, FROM 1880 TO 1899.

THE rate of increase in the number of writings on Alternants continues as great as ever. In the preceding volume we stated that the number for 1860–1880 was almost double the number for 1841–1860: now we have to report that the doubling is not merely approximate but complete,—that is to say, almost exactly the same number of writings as were dealt with in the whole of our first volume covering the period 1693–1841. Notwithstanding the changed circumstances—the generally increased facilities for publication, the tolerant reception of short communications on mathematical subjects, and the constantly growing demand for school and university text-books—these are striking and significant facts for the historian of science.

BAEHR, G. F. W. (1880).

[Extraits d'une lettre (à M. Catalan). *Nouv. Corresp. Math.*, vi. pp. 225–227.]

Baehr expresses the sine of the sum of n angles as the quotient of two alternants. The expression when n is 2 is

$$-\begin{vmatrix} 1 & \sin^2\alpha \\ 1 & \sin^2\beta \end{vmatrix} \div \begin{vmatrix} \sin\alpha & \cos\alpha \\ \sin\beta & \cos\beta \end{vmatrix},$$

and, when n is 4,

$$-\begin{vmatrix} 1 & \sin^2\alpha & \sin^4\alpha & \sin\alpha\cos\alpha \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \div \begin{vmatrix} \sin\alpha & \sin^3\alpha & \cos\alpha & \sin^2\alpha\cos\alpha \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix},$$

his sign, $(-1)^2$, in the latter instance being incorrect. For an odd

number of angles the elements follow a curiously different law; for example, when n is 3, the expression is

$$-\begin{vmatrix} \sin\alpha & \sin^3\alpha & \cos\alpha \\ \sin\beta & \sin^3\beta & \cos\beta \\ \sin\gamma & \sin^3\gamma & \cos\gamma \end{vmatrix} \div \begin{vmatrix} 1 & \sin^2\alpha & \sin\alpha\cos\alpha \\ 1 & \sin^2\beta & \sin\beta\cos\beta \\ 1 & \sin^2\gamma & \sin\gamma\cos\gamma \end{vmatrix},$$

and, when n is 5,

$$\frac{\begin{vmatrix} \sin\alpha & \sin^3\alpha & \sin^5\alpha & \cos\alpha & \sin^2\alpha\cos\alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}}{\begin{vmatrix} 1 & \sin^2\alpha & \sin^4\alpha & \sin\alpha\cos\alpha & \sin^3\alpha\cos\alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}}.$$

No indication is given of the nature of Baehr's mode of investigation.

We note for ourselves that the elements of the first rows of the two alternants are the powers of $\sin\alpha$, whose indices are $0, 1, 2, \ldots, n$, and the multiples of the first $n-1$ of these powers by $\cos\alpha$; that the members of the former group are assigned to the dividend and divisor alternately, beginning with the dividend when n is even and with the divisor when n is odd; and that the other group is then apportioned in like manner, beginning this time, however, in a different determinant. For example, when n is 7, the group of elements used is

$$1, \quad \sin\alpha, \quad \sin^2\alpha, \quad \ldots., \quad \sin^5\alpha, \quad \sin^6\alpha, \quad \sin^7\alpha,$$
$$\cos\alpha, \quad \sin\alpha\cos\alpha, \quad \sin^2\alpha\cos\alpha, \quad \ldots., \quad \sin^5\alpha\cos\alpha,$$

and the expression for $\sin(\alpha+\beta+\ldots+\eta)$

$$\frac{\begin{vmatrix} \sin\alpha & \sin^3\alpha & \sin^5\alpha & \sin^7\alpha & \cos\alpha & \sin^2\alpha\cos\alpha & \sin^4\alpha\cos\alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}}{\begin{vmatrix} 1 & \sin^2\alpha & \sin^4\alpha & \sin^6\alpha & \sin\alpha\cos\alpha & \sin^3\alpha\cos\alpha & \sin^5\alpha\cos\alpha \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}}.$$

SCHWARZ, H. A. (1880).

[Verallgemeinerung eines analytischen Fundamentalsatzes. *Annali di Mat.*, (2) x. pp. 129–136.]

The generalization is: *If the values of* t, $f_1(t)$, $f_2(t)$, ..., $f_n(t)$ *be real, and the functions and their derivates up to and including the* $(n-1)^{th}$ *be finite and continuous, then*

$$\left| f_1(t_1) \cdot f_2(t_2) \cdot \ldots \cdot f_n(t_n) \right| \div \left| t_1^{\,0}\, t_2^{\,1}\, t_3^{\,2} \ldots t_n^{\,n-1} \right|$$

is not greater than the greatest value of

$$\begin{vmatrix} f_1(\tau_1) & f_2(\tau_1) & \dots & f_n(\tau_1) \\ f_1'(\tau_2) & f_2'(\tau_2) & \dots & f_n'(\tau_2) \\ f_1''(\tau_3) & f_2''(\tau_3) & \dots & f_n''(\tau_3) \\ \dots & \dots & \dots & \dots \\ f_1^{(n-1)}(\tau_n) & f_2^{(n-1)}(\tau_n) & \dots & f_n^{(n-1)}(\tau_n) \end{vmatrix}$$

and not less than its least value, where $t_1, t_2, \dots, t_n$ *are values, all different, belonging to the interval* a . . . b, *and*

$$a \leqq \tau_1 \leqq b, \quad \tau_1 \leqq \tau_2 \leqq b, \quad \dots, \quad \tau_{n-1} \leqq \tau_n \leqq b.$$

Schwarz shows how he was led to the theorem in the case of the 3rd order by geometry, and it is this case, so illustrated, which he also proves.

IGEL, B. (1880).

[Zur Theorie der Determinanten. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), lxxxii. pp. 1288–1294.]

The second part of Igel's paper concerns a case of the difference-product, namely, the case in which the variables are divided into s sets and those in each set are in equidifferent progression. The expressions are unwieldy, even after being divided by certain powers of the common differences and the limit taken of the quotient when the said common differences become indefinitely small. In the case where s is 1 the already known limit

$$1^{n-1} \cdot 2^{n-2} \dots (n-2)^2 \cdot (n-1)^1$$

is obtained.

SCOTT, R. F. (1881).

[Mathematical notes. *Messenger of Math.*, x. pp. 142–149.]

In continuation of a paper of 1879, Scott now shows in his first note (pp. 142–143) how to evaluate alternants of the types

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \dots & \sin(2m-1)\alpha & \cos(2m-1)\alpha & \sin\{(2m+1)\alpha+\xi\} \\ 1 & \sin 2\alpha & \cos 2\alpha & \dots & \sin 2m\alpha & \cos 2m\alpha & \cos\{(2m+2)\alpha+\xi\} \end{vmatrix},$$

the important point being that in the results of evaluation ξ occurs only in the factor $\sin(\Sigma\alpha+\xi)$.

RE, A. DEL (1881).

[Relazione tra due determinanti. *Giornale di Mat.*, xix. pp. 116–117.]

An unimportant reply to Eugenio's 'question' of 1871.

KOSTKA, C. (1881).

[Ueber den Zusammenhung zwischen einigen Formen von symmetrischen Functionen. *Crelle's Journ.*, xciii. pp. 89–123.]

This may be viewed as a continuation of the author's papers of the years 1875, 1876. So far as determinants are concerned, the interest of it lies in the fact that one of the symmetric functions in question is

$$\begin{vmatrix} c_p & c_{p+1} & c_{p+2} & \dots \\ c_{q-1} & c_q & c_{q+1} & \dots \\ c_{r-2} & c_{r-1} & c_r & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}, \quad \text{or N}(p, q, r, \dots) \text{ say,}$$

where $c_1, c_2, c_3, \dots$ stand for $\Sigma\alpha$, $\Sigma\alpha\beta$, $\Sigma\alpha\beta\gamma$, ... and $p+q+r+\dots$ is equal to the number of the letters α, β, γ, ... The paper is practically summarized towards its close in a set of tables, the fourth of which,

	$\alpha\beta\gamma\delta$	$\Sigma\alpha^2\beta\gamma$	$\Sigma\alpha^2\beta^2$	$\Sigma\alpha^3\beta$	$\Sigma\alpha^4$	
N(4)	1	−3	1	2	−1	N(1, 1, 1, 1)
N(3, 1)	3	1	−1	−1	1	N(2, 1, 1)
N(2, 2)	2	1	1	−1	0	N(2, 2)
N(2, 1, 1)	3	2	1	1	−1	N(3, 1)
N(1, 1, 1, 1)	1	1	1	1	1	N(4),
	c_1^4	$c_1^2c_2$	c_2^2	c_1c_3	c_4	

will suffice as an example. From the 4th row beginning on the left and using the facients at the top, we obtain

$$\text{N}(2, 1, 1) = 3\alpha\beta\gamma\delta + 2\Sigma\alpha^2\beta\gamma + \Sigma\alpha^2\beta^2 + \Sigma\alpha^3\beta,$$

and from the 2nd row, beginning on the right and using the facients at the bottom, we have

$$\text{N}(2, 1, 1) = c_4 - c_1c_3 - c_2^2 + c_1^2c_2.$$

If the use of the table ended here, there would be nothing in it out of the common,—simply the combining of two triangular tables into one rectangular. There is, however, the curious additional fact that from the columns we can obtain an expression, in terms of the N's, for any one of the symmetric functions at the bottom, and also for any one of those at the top; for example, from the second column

$$c_1^2 c_2 = \mathrm{N}(4) + 2\mathrm{N}(3,1) + \mathrm{N}(2,2) + \mathrm{N}(2,1,1),$$

and from the fourth column

$$\Sigma\alpha^3\beta = 2\mathrm{N}(4) - \mathrm{N}(3,1) - \mathrm{N}(2,2) + \mathrm{N}(2,1,1).$$

A point worth noting in regard to the formation of the table is that the partitions of 4, namely,

$$4;\ \ 3,\ 1;\ \ 2,\ 2;\ \ 2,\ 1,\ 1;\ \ 1,\ 1,\ 1,\ 1,$$

appear in the notation of each set of functions, namely, in the top outside row going leftwards, in the left outside column going downwards, in the bottom outside row going leftwards, and in the right outside column going upwards. The tables extend to the 8th order inclusive.

The user of them needs to bear in mind that, although by Naegelsbach's theorem of 1871 we have

$$\frac{|\alpha^0\beta^1\gamma^2\delta^4\epsilon^7|}{|\alpha^0\beta^1\gamma^2\delta^3\epsilon^4|} = \mathrm{N}(2,1,1),$$

an error might be made in taking the first of the above expansions of N(2, 1, 1) as the equivalent of this quotient. The fact is that with Kostka N(2, 1, 1) stands only for $|\alpha^0\beta^1\gamma^3\delta^6| \div |\alpha^0\beta^1\gamma^2\delta^3|$, that is to say, for what the quotient of the two five-line alternants becomes when the number of variables is reduced to *four* by putting $\epsilon = 0$.

Of course the possibility of making the above double use of each half-table implies two theorems expressible without any reference to tables; namely, (1) *the coefficient of* $c_u c_v c_w \ldots$ *in the expansion of* N(l, m, n, . . .) *is equal to the coefficient of* N(λ, μ, ν, . . .) *in the expansion of* $\Sigma\alpha^u\beta^v\gamma^w \ldots$, l, m, n, . . . *and* λ, μ, ν, . . . , *being conjugate partitions*, and the wording of (2) differing from this merely in having $c_u c_v c_w \ldots$ and $\Sigma\alpha^u\beta^v\gamma^w \ldots$ interchanged.

BURNSIDE, W. S. AND PANTON, A. W. (1881).

[The Theory of Equations, . . . xvi+387 pp. Dublin.]

Among the results deserving record here are (p. 258)

$$\begin{vmatrix} a_1a_2 & a_1b_2+a_2b_1 & b_1b_2 \\ a_2a_3 & a_2b_3+a_3b_2 & b_2b_3 \\ a_3a_1 & a_3b_1+a_1b_3 & b_3b_1 \end{vmatrix} = |a_1b_2| \cdot |a_2b_3| \cdot |a_3b_1|,$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ \frac{a_1^2}{b_1} & \frac{a_2^2}{b_2} & \frac{a_3^2}{b_3} & \frac{a_4^2}{b_4} \\ \frac{b_1^2}{a_1} & \frac{b_2^2}{a_2} & \frac{b_3^2}{a_3} & \frac{b_4^2}{a_4} \end{vmatrix} = \frac{|a_1b_2| \cdot |a_1c_2| \cdot |a_1d_2| \cdot |b_1c_2| \cdot |b_1d_2| \cdot |c_1d_2|}{a_1a_2a_3a_4 \cdot b_1b_2b_3b_4},$$

the former being comparable with one already used (*Hist.*, iii. p. 132), and the second being, for generalization's sake, better written in the form $|a_1^3 \cdot a_2^2b_2 \cdot a_3b_3^2 \cdot b_4^3| = |a_1b_2| \cdot \ldots$.

A more interesting pair (p. 258) is viewable as giving solutions of the equation

$$\xi^{\frac{1}{2}}(x, y, z) = -\xi^{\frac{1}{2}}(a, b, c, d),$$

namely,

$$x, y, z = ab+cd, \quad ac+bd, \quad ad+bc,$$

$$x, y, z = \tfrac{1}{4}(a+b-c-d)^2, \quad \tfrac{1}{4}(a+c-b-d)^2, \quad \tfrac{1}{4}(a+d-b-c)^2;$$

and alongside may be placed temporarily the equality

$$-\begin{vmatrix} 1 & a & a' & aa' \\ 1 & b & b' & bb' \\ 1 & c & c' & cc' \\ 1 & d & d' & dd' \end{vmatrix} = \begin{vmatrix} 1 & ab+cd & a'b'+c'd' \\ 1 & ac+bd & a'c'+b'd' \\ 1 & ad+bc & a'd'+b'c' \end{vmatrix},$$

where the determinant on the left (*Hist.*, ii. p. 451) is manifestly an alternant when $a', b', c', d' = a^2, b^2, c^2, d^2$.

There is also given in effect (p. 262) the equality

$$3\xi^{\frac{1}{2}}(a, b, c, d) = \begin{vmatrix} (a-b)^3 & (a-c)^3 & (a-d)^3 \\ & (b-c)^3 & (b-d)^3 \\ & & (c-d)^3 \end{vmatrix};$$

and lastly we note (pp. 250, 261) one regarding double alternants, namely,

$$\left|(a_1-b_1)^3\cdot(a_2-b_2)^3\cdot(a_3-b_3)^3\right|$$

$$=\xi^{\frac{1}{2}}(a_1,a_2,a_3)\cdot\xi^{\frac{1}{2}}(b_1,b_2,b_3)\cdot 3\Big\{3a_1a_2a_3-\Sigma a_1a_2\cdot\Sigma b_1+\Sigma a_1\cdot\Sigma b_1b_2-3b_1b_2b_3\Big\},$$

the last factor being given equal to

$$3\Big\{(a_1-b_1)(a_2-b_2)(a_3-b_3)+(a_1-b_2)(a_2-b_3)(a_3-b_1)$$
$$+(a_1-b_3)(a_2-b_1)(a_3-b_2)\Big\}.$$

The fresh point about the alternant here is that its order is the same as the power of its elements.

SCOTT, R. F. (1881).

[On some alternating functions of n variables. *Messenger of Math.*, xi. pp. 98–103.]

The real subject here is Jacobians of sets of similar symmetric functions, the paper following effectively on that of Crocchi of 1878, and the main functions being the c's, the $\aleph$'s, and the s's. In the first place are established anew the simple cases *

$$\mathrm{J}(c_1, c_2, \ldots, c_n) = (-1)^{\frac{1}{2}n(n-1)}\xi^{\frac{1}{2}},$$

$$\mathrm{J}(\aleph_1, \aleph_2, \ldots, \aleph_n) = \xi^{\frac{1}{2}},$$

$$\mathrm{J}(s_1, s_2, \ldots, s_n) = n!\,\xi^{\frac{1}{2}},$$

where $\xi^{\frac{1}{2}}$ stands for the difference-product of the independent variables $x_1, x_2, \ldots, x_n$. An advance is then made to the Jacobians in which the degrees of the functions (the c's necessarily excluded) are no longer 1, 2, ..., n, but any n integers in ascending order, say $p, q, \ldots, t$. Thus for the case of the s's :

* It is worth noting that, after the required differentiations have been performed, the first two of these Jacobians are readily seen to be resolvable into

$$\begin{vmatrix} 1 & . & . & . \\ c_1 & 1 & . & . \\ c_2 & c_1 & 1 & . \\ c_3 & c_2 & c_1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & -x & x^2 & -x^3 \\ 1 & -y & y^2 & -y^3 \\ 1 & -z & z^2 & -z^3 \\ 1 & -w & w^2 & -w^3 \end{vmatrix}, \quad \begin{vmatrix} 1 & . & . & . \\ \aleph_1 & 1 & . & . \\ \aleph_2 & \aleph_1 & 1 & . \\ \aleph_3 & \aleph_2 & \aleph_1 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & y & y^2 & y^3 \\ 1 & z & z^2 & z^3 \\ 1 & w & w^2 & w^3 \end{vmatrix},$$

where, merely for shortness, the order used is only the 4th, and x, y, z, w are used instead of x_1, x_2, x_3, x_4. In the case of the third Jacobian, the existence of the factor $|x^0y^1z^2w^3|$ is apparent without this kind of resolution.

$$\frac{\partial(s_p, s_q, \ldots, s_t)}{\partial(x_1, x_2, \ldots, x_n)} = pq \ldots t \cdot \begin{vmatrix} x_1^{p-1} & x_1^{q-1} & \ldots & x_1^{t-1} \\ x_2^{p-1} & x_2^{q-1} & \ldots & x_2^{t-1} \\ \ldots & \ldots & \ldots & \ldots \\ x_n^{p-1} & x_n^{q-1} & \ldots & x_n^{t-1} \end{vmatrix}$$

$$= pq \ldots t \cdot \begin{vmatrix} \aleph_{p-1} & \aleph_{q-1} & \ldots & \aleph_{t-1} \\ \aleph_{p-2} & \aleph_{q-2} & \ldots & \aleph_{t-2} \\ \ldots & \ldots & \ldots & \ldots \\ \aleph_{p-n} & \aleph_{q-n} & \ldots & \aleph_{t-n} \end{vmatrix} \cdot \xi^{\frac{1}{2}},$$

the latter step being Jacobi's (*Hist.*, i. pp. 341–342)*; and in the case of the alephs the result is the same divided by $n!$, because

$$\frac{\partial(\aleph_p, \aleph_q, \ldots, \aleph_t)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial(\aleph_p, \aleph_q, \ldots, \aleph_t)}{\partial(s_p, s_q, \ldots, s_t)} \cdot \frac{\partial(s_p, s_q, \ldots, s_t)}{\partial(x_1, x_2, \ldots, x_n)}$$

and the first Jacobian on the right is equal to $1/n!$ This last point, which is important, is made clear by noting that, since we have

$$\aleph_m = \frac{1}{m!} \begin{vmatrix} s_1 & -1 & . & . & \ldots. \\ s_2 & -s_1 & -2 & . & \ldots. \\ s_3 & s_2 & s_1 & -3 & \ldots. \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_m,$$

it follows that $\dfrac{\partial \aleph_m}{\partial s_m} = \dfrac{1}{m}$ and $\dfrac{\partial \aleph_m}{\partial s_{m+}} = 0.$

The obtaining of the Jacobians of $s_p, s_q, \ldots, s_t$ and of $\aleph_p, \aleph_q, \ldots, \aleph_t$ with respect to other sets of variables, namely,

$$c_1, c_2, \ldots, c_n \qquad \aleph_1, \aleph_2, \ldots, \aleph_n \qquad s_1, s_2, \ldots, s_n,$$

need only be mentioned. It is more interesting to note that a fresh function W being defined by the identity

$$1 + \mathrm{W}_1 x + \mathrm{W}_2 x^2 + \ldots \equiv (1 + \aleph_1 x + \aleph_2 x^2 + \ldots)^\nu,$$

and it being ascertained that

$$\frac{\partial \mathrm{W}_m}{\partial \aleph_m} = \nu \quad \text{and} \quad \frac{\partial \mathrm{W}_m}{\partial \aleph_{m+}} = 0,$$

repetition of the procedure above gives

$$\frac{\partial(\mathrm{W}_p, \mathrm{W}_q, \ldots, \mathrm{W}_t)}{\partial(x_1, x_2, \ldots, x_n)} = \nu^n \cdot \frac{\partial(\aleph_p, \aleph_q, \ldots, \aleph_t)}{\partial(x_1, x_2, \ldots, x_n)},$$

in which the Jacobian on the right has already been found.

* See also *Hist.* iii., under Trudi (1864).

MUIR, T. (1882).

[A Treatise on the Theory of Determinants. . . . viii+240 pp. London.]

We have here (pp. 161–182) the fullest account of alternants yet given in a text-book; and for the first time there is made known a general process for finding the quotient of a simple alternant by the corresponding difference-product. The process is quite similar to the ordinary division-process in Arithmetic, being dependent on the multiplication-theorem originally foreshadowed by Schweins. Thus, denoting the simple alternant

$$|\, a^p b^q c^r \,| \quad \text{by} \quad \mathrm{A}\,(p, q, r),$$

we have as an example of it

$$\begin{array}{l|ll} \underline{\mathrm{A}(012)} & \mathrm{A}(034) & \underline{|\,(022)+(112)} \\ & \underline{\mathrm{A}(034)-\mathrm{A}(124)} & \\ & \mathrm{A}(124) & \\ & \underline{\mathrm{A}(124)} & \end{array}$$

giving as result

$$|\, a^0 b^3 c^4 \,| \div |\, a^0 b^1 c^2 \,| = \Sigma a^2 b^2 + \Sigma a^2 bc.$$

Worthy of note among the exercises are the general equalities which include

$$\frac{|\, a^0 b^1 c^4 \,|\ |\, a^0 b^1 c^5 \,|}{|\, a^0 b^1 c^2 \,|} = |\, a^0 b^1 c^7 \,| + |\, a^0 b^2 c^6 \,| + |\, a^0 b^3 c^5 \,|,$$

$$\frac{|\, a^0 b^1 c^2 d^3 e^5 \,|\ |\, a^0 b^1 c^3 d^4 e^6 \,|}{|\, a^0 b^1 c^2 d^3 e^4 \,|} = |\, a^0 b^1 c^3 d^4 e^7 \,| + |\, a^0 b^1 c^3 d^5 e^6 \,| + |a^0 b^2 c^3 d^4 e^6 \,|.$$

MUIR, T. (1882).

[On a class of permanent symmetric functions. *Proceed. R. Soc. Edinburgh*, xi. pp. 409–418.]

The symmetric functions dealt with here, called *permanents*, being symmetric with respect to the rows of a square array, are not primarily of the simple type met with in the study of alternants, where the symmetry, as in $\Sigma a\beta\gamma$, $\Sigma a^2 bcd$, . . . , is between the individual variables. If, however, we specialize permanents as determinants are specialized into alternants, we pass over to the

simpler type of symmetric function at a single step; for to say that

$$\overset{+}{\begin{vmatrix} \alpha^m & \alpha^n & \alpha^p \\ \beta^m & \beta^n & \beta^p \\ \gamma^m & \gamma^n & \gamma^p \end{vmatrix}}$$

is symmetric with respect to its first and second row is the same as to say that it is symmetric with respect to α and β; in other words,

$$\overset{+}{|\, \alpha^m \beta^n \gamma^p \,|} \equiv \Sigma \alpha^m \beta^n \gamma^p.$$

An equality like

$$\cdot\, |\, a^1 b^2 c^5 \,| = |\, a^0 b^1 c^2 \,| \cdot \Sigma abc^3$$

is thus viewable as the result of multiplying a permanent by a determinant, namely,

$$\tfrac{1}{2} \overset{+}{\begin{vmatrix} a & a & a^3 \\ b & b & b^3 \\ c & c & c^3 \end{vmatrix}} \text{ by } \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Now in this paper, as we have already seen, a general multiplication-theorem of this kind is obtained, and what we have here to note is that specialization of it leads to the conclusion that *The product of a simple alternant and a single symmetric function of its variables is expressible by a sum of simple alternants, whose indices are got by arranging the variables in every term of the symmetric function in the same order, and adding the indices of each term to the indices of the original alternant, the first to the first, the second to the second, and so on* : * for example,

$$|\, a^0 b^1 c^2 \,| \cdot \Sigma a^3 b = |\, a^0 b^2 c^5 \,| - |\, a^0 b^3 c^4 \,| - |\, a b^2 c^4 \,|.$$

Another direct consequence of the said multiplication-theorem is Borchardt's equality

$$\overset{+}{|\, (a_1-\beta_1)^{-1}(a_2-\beta_2)^{-1} \ldots (a_n-\beta_n)^{-1} \,|} \cdot \left| (a_1-\beta_1)^{-1}(a_2-\beta_2)^{-1} \ldots (a_n-\beta_n)^{-1} \right|$$
$$= \left| (a_1-\beta_1)^{-2}(a_2-\beta_2)^{-2} \ldots (a_n-\beta_n)^{-2} \right|.$$

* Schweins' case of this (*Hist.*, i. pp. 311–318) should have been referred to in this paper, but Schweins' work was not then known to any one, having not been brought to light until 1884.

SCOTT, R. F. (1882).

[On some determinants whose elements are rational fractions. *Messenger of Math.*, xi. pp. 165–172.]

The determinants referred to are Cauchy's double alternant (*Hist.*, i. p. 345) and others formable from it by 'bordering.' With an eye on the known result,

$$\left|(x_1-a_1)^{-1}(x_2-a_2)^{-1}\ \ldots\ (x_n-a_n)^{-1}\right| = \frac{(-1)^{\frac{1}{2}n(n-1)}\xi^{\frac{1}{2}}(x_1,\ \ldots,\ x_n)\cdot\xi^{\frac{1}{2}}(a_1,\ \ldots,\ a_n)}{f(x_1)\cdot f(x_2)\cdot\ \ldots\ \cdot f(x_n)},$$

where $f(x)\equiv(x-a_1)(x-a_2)\ \ldots\ (x-a_n)$, Scott begins by multiplying the given alternant by the product of the f's, and then the determinant so obtained,

$$\begin{vmatrix} \frac{f(x_1)}{x_1-a_1} & \frac{f(x_1)}{x_1-a_2} & \ldots & \frac{f(x_1)}{x_1-a_n} \\ \frac{f(x_2)}{x_2-a_1} & \frac{f(x_2)}{x_2-a_2} & \ldots & \frac{f(x_2)}{x_2-a_n} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{f(x_n)}{x_n-a_1} & \frac{f(x_n)}{x_n-a_2} & \ldots & \frac{f(x_n)}{x_n-a_n} \end{vmatrix},$$

has only to be resolved into the two difference-products. Now, if in their expanded forms

$$f(x) \equiv x^n - c_1x^{n-1} + c_2x^{n-2} - \ldots.$$

and

$$\frac{f(x)}{x-a_k} \equiv x^{n-1} - c_{1k}x^{n-2} + c_{2k}x^{n-3} - \ldots,$$

$c_{h,k}$ thus becoming c_h on putting $a_k = 0$, it is clear that the said determinant is the product of

$$\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \ldots & 1 \\ x_2^{n-1} & x_2^{n-2} & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ x_n^{n-1} & x_n^{n-2} & \ldots & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & -c_{11} & c_{21} & -c_{31} & \ldots \\ 1 & -c_{12} & c_{22} & -c_{32} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & -c_{1n} & c_{2n} & -c_{3n} & \ldots \end{vmatrix},$$

and consequently

$$= (-1)^{\frac{1}{2}n(n-1)}\xi^{\frac{1}{2}}(x_1,\ \ldots,\ x_n)\cdot(-1)^{\frac{1}{2}n(n-1)}\ \frac{\partial(c_1,\ \ldots,\ c_n)}{\partial(a_1,\ \ldots,\ a_n)}$$

$$= (-1)^{\frac{1}{2}n(n-1)}\xi^{\frac{1}{2}}(x_1,\ \ldots,\ x_n)\cdot\xi^{\frac{1}{2}}(a_1,\ \ldots,\ a_n)$$

as desired. The procedure is worthy of attention because of the fresh results reached by it. With comparative ease there is evaluated Joachimsthal's determinant formed by annexing

$$\begin{array}{rcccc} \text{as a last row} & (x_{n+1}-a_1)^{-1} & \dots & (x_{n+1}-a_n)^{-1} & 1 \\ \text{and as a last column} & 1 & \dots & 1 & 1; \end{array}$$

the determinant formed by annexing

$$\begin{array}{rcccc} \text{as a last row} & (x_{n+1}-a_1)^{-1} & \dots & (x_{n+1}-a_n)^{-1} & \dfrac{x_{n+1}^r}{f(x_{n+1})} \\ \text{and as a last column} & \dfrac{x_1^r}{f(x_1)} & \dots & \dfrac{x_n^r}{f(x_n)} & \dfrac{x_{n+1}^r}{f(x_{n+1})}; \end{array}$$

the determinant formed by annexing as a last row and as a last column

$$1 \quad \dots \quad 1 \quad 0;$$

and the determinant formed by annexing

$$\begin{array}{rcccc} \text{as a last row} & 1 & \dots & 1 & 0 \\ \text{and as a last column} & \dfrac{x_1^r}{f(x_1)} & \dots & \dfrac{x_n^r}{f(x_n)} & 0. \end{array}$$

Perhaps the most interesting of these is the third, where the effect of the bordering is to multiply the original by

$$\Sigma x - \Sigma a;$$

in the other cases the like connecting factors are

$$(x_{n+1}-x_n)(x_{n+1}-x_{n-1}) \dots (x_{n+1}-x_1),$$
$$(x_{n+1}-x_n)(x_{n+1}-x_{n-1}) \dots (x_{n+1}-x_1) \div f(x_{n+1}),$$
$$-\aleph_{r-n+1}(x_1, x_2, \dots, x_n),$$

respectively. As a sort of appendix Scott establishes Brioschi's improved result of 1857 (*Hist.*, ii. p. 187). Here his procedure is to take Cauchy's result for the double alternant of the $(2n)^{\text{th}}$ order

$$\begin{vmatrix} (x_1-a_1)^{-1} & (x_1-a_2)^{-1} & \dots & (x_1-a_{2n})^{-1} \\ (x_1-a_1)^{-2} & (x_1-a_2)^{-2} & \dots & (x_1-a_{2n})^{-2} \\ (x_2-a_1)^{-1} & (x_2-a_2)^{-1} & \dots & (x_2-a_{2n})^{-1} \\ (x_2-a_1)^{-2} & (x_2-a_2)^{-2} & \dots & (x_2-a_{2n})^{-2} \\ \dots & \dots & \dots & \dots \\ (x_n-a_1)^{-1} & (x_n-a_2)^{-1} & \dots & (x_n-a_{2n})^{-1} \\ (x_n-a_1)^{-2} & (x_n-a_2)^{-2} & \dots & (x_n-a_{2n})^{-2} \end{vmatrix},$$

diminish each $(r+2)^{\text{th}}$ row by the $(r+1)^{\text{th}}$, remove the factor $x_r - y_r$ thus disclosed, and then put $y_r = x_r$. He also extends the equality from the $(2n)^{\text{th}}$ order to the $(mn)^{\text{th}}$ order.

MITCHELL, O. H. (1882).

[Note on determinants of powers. *American Journ. of Math.*, iv. pp. 341–344 ; *Johns Hopkins Univ. Circ.*, 1882, p. 242.]

The contribution here is that *the number of terms in the quotient of* $|\, a^\alpha b^\beta c^\gamma d^\delta \,|$ *by* $|\, a^0 b^1 c^2 d^3 \,|$ *is* $\xi^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta) \div \xi^{\frac{1}{2}}(0, 1, 2, 3)$. The author first shows at considerable length that all the terms in the quotient are positive, and then in Jacobi's expression for it, namely,

$$\begin{vmatrix} \aleph_\alpha & \aleph_\beta & \aleph_\gamma & \aleph_\delta \\ \aleph_{\alpha-1} & \aleph_{\beta-1} & \aleph_{\gamma-1} & \aleph_{\delta-1} \\ \aleph_{\alpha-2} & \aleph_{\beta-2} & \aleph_{\gamma-2} & \aleph_{\delta-2} \\ \aleph_{\alpha-3} & \aleph_{\beta-3} & \aleph_{\gamma-3} & \aleph_{\delta-3} \end{vmatrix},$$

he naturally puts $a = b = c = d = 1$. Since the number of terms in the m^{th} aleph function of n letters is $(m+n-1)_{n-1}$, he thus obtains

$$\begin{vmatrix} (\alpha+3)_3 & (\beta+3)_3 & (\gamma+3)_3 & (\delta+3)_3 \\ (\alpha+2)_3 & (\beta+2)_3 & (\gamma+2)_3 & (\delta+2)_3 \\ (\alpha+1)_3 & (\beta+1)_3 & (\gamma+1)_3 & (\delta+1)_3 \\ (\alpha)_3 & (\beta)_3 & (\gamma)_3 & (\delta)_3 \end{vmatrix},$$

which by subtraction of rows is seen to be

$$= \begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ (\alpha)_2 & (\beta)_2 & (\gamma)_2 & (\delta)_2 \\ (\alpha)_3 & (\beta)_3 & (\gamma)_3 & (\delta)_3 \end{vmatrix} = \xi^{\frac{1}{2}}(\alpha, \beta, \gamma, \delta) \div 1!\,2!\,3!,$$

as we know from Stern (1865) and Zeipel (1865).

NOVARESE, E. (1882).

[Intorno ad alcune formole di Hermite per l'addizione delle funzioni ellitiche. *Atti . . . Accad. . . . di Torino*, xvii. pp. 607–621, 723–739.]

One point of interest here is the fact that the expressions obtained are in the form of the quotient of two determinants. Of greater

note, however, is a theorem communicated to the author by Siacci and utilized as the basis of the result of the second paper, namely, *When* $x_1, x_2, \ldots, x_n$ *have the common value* x, *then*

$$\frac{|\phi_1(x_1)\cdot\phi_2(x_2)\ \ldots\ \phi_n(x_n)|}{|\psi_1(x_1)\cdot\psi_2(x_2)\ \ldots\ \psi_n(x_n)|} \text{ reduces to } \frac{|\phi_1(x)\cdot\phi'_2(x)\ \ldots\ \phi_n^{(n-1)}(x)|}{|\psi_1(x)\cdot\psi'_2(x)\ \ldots\ \psi_n^{(n-1)}(x)|}.$$

By means of this the passage is made from addition to multiplication.

BESSO, D. (1882).

[Di alcune proprietà dell' equazione differenziale lineare omogenea del second' ordine. *Atti . . . Accad. dei Lincei, Mem.* (Roma), xiv. pp. 14–29.]

Incidentally it is shown here that the determinant of the $2n^{\text{th}}$ order, whose h^{th} row $(h \leqq n)$ is

$$1 \quad x_h \quad x_h^2 \quad \ldots \quad x_h^{2n-1},$$

and whose $(n+h)^{\text{th}}$ row is

$$0 \quad 1 \quad 2x_h \quad \ldots. \quad (2n-1)x_h^{2n-2},$$

is equal to

$$(-1)^{\frac{1}{2}n(n-1)}\cdot\xi^{\frac{1}{2}}(x_1, x_2, \ldots, x_n).$$

In this connection Franke's paper of 1876 (*Hist.*, iii. p. 159) may be recalled.

GARBIERI, G. (1882).

[Sopra alcune classi di funzioni simmetriche. *Atti . . . Ist. Veneto di Sci.*, (6) i. pp. 33–74.]

This long paper is avowedly a continuation of that of 1879 (*Hist.*, iii. pp. 163–166). In its first section (pp. 34–53) it still concerns the quotient

$$|f_1(a_1)\cdot f_2(a_2)\ \ldots\ f_n(a_n)| \div \xi^{\frac{1}{2}}(a_1, a_2, \ldots, a_n);$$

but the result obtained, although interesting, is a little far-fetched and less likely to be of value. Thus, when n is 3 and the three functions are

$$g_0 + g_1x + \ldots + g_5x^5,$$
$$h_0 + h_1x + \ldots + h_5x^5,$$
$$k_0 + k_1x + \ldots + k_5x^5,$$

the equivalent of 1879 is

$$\begin{vmatrix} g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ h_0 & h_1 & h_2 & h_3 & h_4 & h_5 \\ k_0 & k_1 & k_2 & k_3 & k_4 & k_5 \\ c_0 & c_1 & c_2 & c_3 & . & . \\ . & c_0 & c_1 & c_2 & c_3 & . \\ . & . & c_0 & c_1 & c_2 & c_3 \end{vmatrix} \div c_3^3,$$

were the c's are defined by the identity

$$c_3x^3 + c_2x^2 + c_1x^1 + c_0 \equiv c_3(x-a_1)(x-a_2)(x-a_3);$$

whereas now the equivalent given us is

$$\begin{vmatrix} g_0 & g_1 & g_2 & g_3 & g_4 & g_5 \\ h_0 & h_1 & h_2 & h_3 & h_4 & h_5 \\ k_0 & k_1 & k_2 & k_3 & k_4 & k_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ c_0 & c_1 & c_2 & c_3 & c_4 & . \\ . & c_0 & c_1 & c_2 & c_3 & c_4 \end{vmatrix} \div \begin{vmatrix} b_3 & b_4 & b_5 \\ c_3 & c_4 & . \\ c_2 & c_3 & c_4 \end{vmatrix},$$

in which the b's and c's are defined by the identities

$$b_5x^5 + b_4x^4 + \ldots + b_0 \equiv b_5(x-a_1)(x-a_2)(x-a_3)(x-\beta_1)(x-\beta_2),$$
$$c_4x^4 + \ldots + c_0 \equiv c_4(x-a_1)(x-a_2)(x-a_3)(x-\gamma).$$

This means that a_1, a_2, a_3 are now roots common to two equations: and that the determinant-quotient just written, notwithstanding the presence in it of the other roots β_1, β_2, γ, is independent of them. The first and main step in the new proof is quite similar to that of the old, consisting in multiplying the dividend determinant by

$$\xi^{\frac{1}{2}}(a_1, a_2, a_3, \beta_1, \beta_2, \gamma), \quad i.e. \quad |\, a_1^{\,0}\, a_2^{\,1}\, a_3^{\,2}\, \beta_1^{\,3}\, \beta_2^{\,4}\, \gamma^5 \,|.$$

The quite general theorem is fully investigated, that is to say, the theorem in which the degree of the given functions is n, the number of functions and of common roots is m, the degree of the b-equation is s, and the degree of the c-equation t, s being taken for convenience $> t$. The dividend-determinant in the equivalent for $|f_1(a_1) \ldots f_n(a_n)| \div \xi^{\frac{1}{2}}(a_1, \ldots, a_n)$ has then $t-m$ rows of b's and is of the degree $n+1$ or $s+t-m$, according as the first or

second of these numbers is the greater : the divisor-determinant is the complementary of the first m-line minor of the dividend.

The second section (pp. 54–74) is devoted to a similar treatment of the quotient

$$\frac{\left|\mathrm{F}(\alpha_1,\beta_1)\cdot\mathrm{F}(\alpha_2,\beta_2)\ \ldots\ \mathrm{F}(\alpha_n,\beta_n)\right|}{\xi^{\frac{1}{2}}(\alpha_1,\ \ldots,\ \alpha_n)\cdot\xi^{\frac{1}{2}}(\beta_1,\ \ldots,\ \beta_n)},$$

as is of course natural in view of the author's previous work.

ŘEHOŘOVSKY, V. (1882).

[O vytvornjíci funkci Borchardt-ove. *Časopis pro pěstování math. a fys.*, xi. pp. 111–120.]

This is another examination of the equality

$$\left|(t_1-x_1)^{-2}\ldots(t_n-x_n)^{-2}\right|$$
$$=\left|(t_1-x_1)^{-1}\ldots(t_n-x_n)^{-1}\right|\cdot\overset{+}{\left|(t_1-x_1)^{-1}\ldots(t_n-x_n)^{-1}\right|}$$

and of the associated generating function, the author being moved thereto by Bruno's chapter on the subject.

POSSE, C. (1883).

[Sur le terme complémentaire de la formule de M. Tchebychef donnant l'expression approchée d'une intégrale définie par d'autres prises entre les mêmes limites. *Bull. des Sci. Math.*, (2) vii. pp. 214–224.]

Incidentally there is here (pp. 221–223) formulated and proved the theorem that *if* $x_1, x_2, \ldots, x_{n+1}$ *be independent quantities and* $f(x)$ *any function that is continuous for all values of* x *lying between the least and greatest of the said quantities, then*

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \ldots. & x_1^{n-1} & f(x_1) \\ 1 & x_2 & x_2^2 & \ldots. & x_2^{n-1} & f(x_2) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_{n+1} = \frac{1}{n!}\,\xi^{\frac{1}{2}}(x_1,\ldots.,x_{n+1})\cdot f^n(\xi),$$

where $f^n(x)$ *denotes the* n^{th} *derivate of* $f(x)$, *and* ξ *a quantity intermediate between the least and greatest of* $x_1, x_2, \ldots, x_{n+1}$. The proof is gradational, the known case where $n = 1$ leading to that where $n = 2$, and so on.

CRAIG, T. (1883).

[On a Θ-function formula. *American Journ. of Math.*, v. pp. 350–357.]

What is of value here properly concerns the Θ-functions: it is interesting, however, for us to note the peculiar type of determinant used, namely,

$$|\,\Theta(u)\cdot\Theta'(v)\cdot\Theta''(w)\cdot\Theta'''(t)\,|,$$

which is an alternant with respect to u, v, w, t, and in which the first element of each row gives rise to the others through being differentiated.

EDWARDES, D. (1884).

[Question 7952. *Educ. Times*, xxxvii. p. 388.]

The assertion here is that

$$D_2^3\begin{vmatrix} D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \\ D_4 & D_5 & D_6 \end{vmatrix} + \begin{vmatrix} . & D_2 & D_3 \\ D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \end{vmatrix} = 0 \quad \text{if} \quad D_n = \begin{vmatrix} 1 & a & a^n \\ 1 & b & b^n \\ 1 & c & c^n \end{vmatrix}.$$

In the absence of any proof we may note that the expression involving the D's is a two-line minor of the adjugate of

$$\begin{vmatrix} . & . & D_2 & D_3 \\ . & D_2 & D_3 & D_4 \\ D_2 & D_3 & D_4 & D_5 \\ D_3 & D_4 & D_5 & D_6 \end{vmatrix},$$

and is thus known to be equal to the product of this determinant by its central two-line minor. It will suffice, therefore, if we can prove that one of these factors vanishes when the D's are specialized as stated. Now, since the specialized D_n is divisible by D_2, the quotient being the aleph function $(a, b, c)^{n-2}$ or $\aleph_{n-2}$, the four-line determinant is seen to be equal to

$$D_2^4\begin{vmatrix} . & . & 1 & \aleph_1 \\ . & 1 & \aleph_1 & \aleph_2 \\ 1 & \aleph_1 & \aleph_2 & \aleph_3 \\ \aleph_1 & \aleph_2 & \aleph_3 & \aleph_4 \end{vmatrix},$$

and therefore equal to 0, as we know from Trudi (1864), or as the use of Wronski's relation

$$\mathrm{col}_4 - \Sigma a \cdot \mathrm{col}_3 + \Sigma ab \cdot \mathrm{col}_2 - abc \cdot \mathrm{col}_1 = 0$$

makes directly clear.

GORDAN, P. (1884).

[Ueber Gleichungen siebenten Grades, mit . . . . *Math. Annalen*, xxv. pp. 459–521.]

Three pages (§ 4) of this are devoted to the symmetric functions

$$| x^\lambda y^\mu z^\nu | \div \xi^{\frac{1}{2}}(x, y, z),$$

$$| x^\lambda \cdot y^\mu \cdot (z-x)(z-y) | \div \xi^{\frac{1}{2}}(x, y, z);$$

but the results obtained, though useful in connection with the main subject of the paper, are not otherwise noteworthy. They are based on what may be conveniently called the multiplication-theorem of alternants (see above, p. 151 and *Hist.*, iii. p. 155).

FOURET, G. (1884).

[Sur deux formules trigonométriques d'interpolation, applicables, l'une aux fonctions paires, l'autre aux fonctions impaires. *Comptes rendus . . . Acad. des Sci.* (Paris), xcix. pp. 963–966.]

[Sur une formule trigonométrique d'interpolation applicable à des valeurs quelconques de la valeur indépendante. *Comptes rendus . . . Acad. des Sci.* (Paris), xcix. pp. 1062–1064.]

So far as determinants are concerned Fouret's results are rediscoveries, those of the first paper having been given by Prouhet in 1857 and that of the second by Scott in 1879.

JACOBI, C. G. J. (1884).

[Additamentum ad Commentationem quae inscripta est: Disquisitiones analyticae de fractionibus simplicibus. *Gesammelte Werke*, iii. pp. 553–582.]

The theorems regarding alternants given in this posthumous memoir are essentially the same as those published by Jacobi in 1841 (*Hist.*, i. pp. 331–342).

JOHNSON, W. W. (1885).

[Reduction of alternating functions to alternants. *American Journ. of Math.*, vii. pp. 345–346.]

The function in question is

$$\begin{vmatrix} \phi_1(a_1;\ a_2, a_3, \dots, a_n) & \phi_2(a_1;\ a_2, a_3, \dots, a_n) & \dots & \phi_n(a_1;\ a_2, a_3, \dots, a_n) \\ \phi_1(a_2;\ a_1, a_3, \dots, a_n) & \phi_2(a_2;\ a_1, a_3, \dots, a_n) & \dots & \phi_n(a_2;\ a_1, a_3, \dots, a_n) \\ \dots & \dots & \dots & \dots \\ \phi_1(a_n;\ a_1, a_2, \dots, a_{n-1}) & \phi_2(a_n;\ a_1, a_2, \dots, a_{n-1}) & \dots & \phi_n(a_n;\ a_1, a_2, \dots, a_{n-1}) \end{vmatrix}$$ or, Φ sa

where $\phi(a_1;\ a_2, a_3, \dots, a_n)$ denotes a function of the a's which is symmetric with respect to those following the semicolon; and the theorem is that Φ is expressible as a sum of alternants the aggregate of whose principal terms is equal to the principal term of Φ. Thus the principal term of

$$\begin{vmatrix} 1 & b^2+c^2 & a^2+bc \\ 1 & c^2+a^2 & b^2+ca \\ 1 & a^2+b^2 & c^2+ab \end{vmatrix}$$

being equal to

$$c^4 + c^2a^2 + c^2ab + a^3b,$$

which can be written

$$a^0b^0c^4 + a^2b^0c^2 + a^1b^1c^2 + a^3b^1c^0,$$

the determinant itself

$$\begin{aligned} &= |\, a^0b^0c^4 \,| + |\, a^2b^0c^2 \,| + |\, a^1b^1c^2 \,| + |\, a^3b^1c^0 \,| \\ &= 0 + 0 + 0 - |\, a^0b^1c^3 \,| \\ &= -(a+b+c)\cdot \xi^{\frac{1}{2}}(a, b, c). \end{aligned}$$

The proof rests on the fact that the interchange of the h^{th} and k^{th} rows is equivalent to the interchange of a_h and a_k in the function.

JOHNSON, W. W. (1885).

[On a formula of reduction for alternants of the third order. *American Journ. of Math.*, vii. pp. 347–352.]

The formula in question furnishes a complete solution of the problem of finding the quotient of a three-line alternant by the

difference-product of the variables. It may be written, q being $> p$,

$$\frac{|\,a^0b^pc^q\,|}{|\,a^0b^1c^2\,|} = (\Sigma a^{q-2}b^{p-1} + \ldots) + abc(\Sigma a^{q-4}b^{p-2} + \ldots) + a^2b^2c^2(\Sigma a^{q-6}b^{p-3} + \ldots) + \ldots,$$

where $\Sigma a^{q-2}b^{p-1} + \ldots$ means $\Sigma a^{q-2}b^{p-1}$ and all the terms following it in the aleph function of the $(q+p-3)^{\text{th}}$ degree,—in other words, the said function as truncated by cutting off all terms that have an exponent greater than $q-2$. Thus

$$\frac{|\,a^0b^5c^7\,|}{|\,a^0b^1c^2\,|} = (\Sigma a^5b^4 + \ldots) + abc(\Sigma a^3b^3 + \ldots)$$
$$= \Sigma a^5b^4 + \Sigma a^5b^3c + \Sigma a^5b^2c^2 + 2\Sigma a^4b^4c + 2\Sigma a^4b^3c^2 + 2\Sigma a^3b^3c^3;$$

$$\frac{|\,a^0b^3c^8\,|}{|\,a^0b^1c^2\,|} = (\Sigma a^6b^2 + \ldots) + abc(\Sigma a^4b^1 + \ldots) + a^2b^2c^2(\Sigma a^2 + \ldots)$$
$$= \Sigma a^6b^2 + \Sigma a^6bc + \Sigma a^5b^3 + 2\Sigma a^5b^2c + \Sigma a^4b^4 + 2\Sigma a^4b^3c + 3\Sigma a^4b^2c^2 + 3\Sigma a^3b^3c^2.$$

All that requires to be proved is

$$\frac{|\,a^0b^pc^q\,|}{|\,a^0b^1c^2\,|} = (\Sigma a^{q-2}b^{p-1} + \ldots) + abc\frac{|\,a^0b^{p-1}c^{q-2}\,|}{|\,a^0b^1c^2\,|},$$

or
$$|\,a^0b^pc^q\,| = |\,a^0b^1c^2\,| \cdot (\Sigma a^{q-2}b^{p-1} + \ldots) + |\,a^1b^pc^{q-1}\,|,$$

which would be very simple if we were furnished with a theorem for the multiplication of an alternant by $\Sigma a^{q-2}b^{p-1} + \ldots$ similar to Schweins' for multiplication by a simple symmetric function or by an aleph function (*Hist.*, i. pp. 311–318). In default of this Johnson supplies the interesting lemma

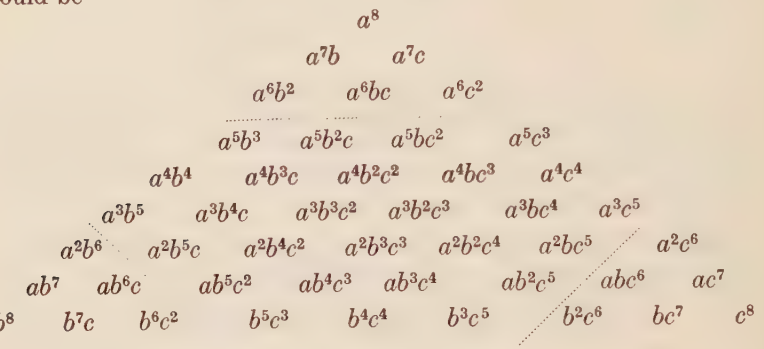
$$\Sigma a^{m-r-1}b^{r+1} + \ldots. = \aleph_m - \aleph_r \cdot \Sigma a^{m-r} \qquad (r < \tfrac{1}{2}m),$$

adducing by way of proof the fact that if the terms of $\aleph_m$ be symmetrically disposed in the form of an equilateral triangle, the terms of $\aleph_r \cdot \Sigma a^{m-r}$ occupy three equal-sized portions cut off by lines parallel to the sides, and the terms of $\Sigma a^{m-r-1}b^{r+1} + \ldots$ constitute

the remainder : * for example, the diagrammatic illustration of the equality

$$\aleph_8 = \aleph_2 \cdot \Sigma a^6 + (\Sigma a^5 b^3 + \ldots .)$$

would be


a^8
a^7b a^7c
a^6b^2 a^6bc a^6c^2
a^5b^3 a^5b^2c a^5bc^2 a^5c^3
a^4b^4 a^4b^3c a^4b^2c^2 a^4bc^3 a^4c^4
a^3b^5 a^3b^4c a^3b^3c^2 a^3b^2c^3 a^3bc^4 a^3c^5
a^2b^6 a^2b^5c a^2b^4c^2 a^2b^3c^3 a^2b^2c^4 a^2bc^5 a^2c^6
ab^7 ab^6c ab^5c^2 ab^4c^3 ab^3c^4 ab^2c^5 abc^6 ac^7
b^8 b^7c b^6c^2 b^5c^3 b^4c^4 b^3c^5 b^2c^6 bc^7 c^8

He is thus enabled to substitute

$$\aleph_{p+q-3} - \aleph_{p-2} \cdot \Sigma a^{q-1} \quad \text{for} \quad \Sigma a^{q-2} b^{p-1} + \ldots ,$$

so that the right-hand member of the desired result becomes

$$|\, a^0 b^1 c^{p+q-1} \,| - |\, a^0 b^1 c^p \,| \cdot \Sigma a^{q-1} + |\, a^1 b^p c^{q-1} \,|,$$

and therefore by Schweins' theorem

$$|\, a^0 b^1 c^{p+q-1} \,| - \left(|\, a^{q-1} b^1 c^p \,| + |\, a^0 b^q c^p \,| + |\, a^0 b^1 c^{p+q-1} \,| \right) + |\, a^1 b^p c^{q-1} \,|,$$

which reduces at once to the left-hand member.

Other proofs of the lemma are given, but the foregoing is the most instructive. The author also points out that it is true for alternants of any order, but adds that for the 4th and higher orders it does not in general provide a 'formula of reduction.' As an exceptional case where it does, he instances $|\, a^0 b^1 c^{p+2} d^{p+3} \,|$, the formula then being

$$\frac{|\, a^0 b^1 c^{p+2} d^{p+3} \,|}{|\, a^0 b^1 c^2 d^3 \,|} = (\Sigma a^p b^p + \ldots) + abcd \cdot \frac{|\, a^0 b^1 c^p d^{p+1} \,|}{|\, a^0 b^1 c^2 d^3 \,|},$$

which he illustrates by putting $p = 5$.

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* It should be noted that when m is odd there exists a unique equality of this kind, namely,

$$\aleph_{2\mu+1} = \aleph_\mu \cdot \Sigma a^{\mu+1} + (\Sigma a^\mu b^\mu c + \ldots).$$

JOHNSON, W. W. (1885).

[On the calculation of the cofactor of alternants of the fourth order. *American Journ. of Math.*, vii. pp. 380–388.]

The central fact of interest here is the discovery of a 'formula of reduction' for alternants of the 4th order similar to that for alternants of the 3rd order, namely,

$$\frac{|a^0b^pc^qd^r|}{|a^0b^1c^2d^3|} = Q_1 + abcd\cdot\frac{|a^0b^{p-1}c^{q-1}d^{r-2}|}{|a^0b^1c^2d^3|},$$

or, say,

$$(0, p, q, r) = Q_1 + abcd\cdot(0, p-1, q-1, r-2),$$

where Q_1, the first instalment of the quotient, begins as before with $\Sigma a^{r-3}b^{q-2}c^{p-1}$, but does not necessarily include all the simple symmetric functions following $\Sigma a^{r-3}b^{q-2}c^{p-1}$ in $\aleph_{p+q+r-6}$ nor have such functions confined to the coefficient 1. For example,

$$(0, 2, 3, 7) = \Sigma a^4bc + \Sigma a^3b^2c + 2\Sigma a^3bcd + \Sigma a^2b^2c^2 + 2\Sigma a^2b^2cd + abcd\cdot(0, 1, 2, 5),$$

where in Q_1 the term Σa^3b^3 does not occur. Much of the paper is occupied with explanation of the origin of Q_1 and with simplifications of the original mode of finding it. In the course of this

$$(0, 3, 5, 7) \quad \text{and} \quad (0, 4, 7, 9)$$

are calculated as illustrations, and at the close there is also given the calculation of the more serious example

$$(0, 5, 13, 17).$$

There being occasion to use as a check Mitchell's rule for giving the number of terms in the full quotient, a more natural proof of the said rule is appended.*

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* It would have been still more natural to say that the number of terms in $|a^pb^qc^r| \div |a^0b^1c^2|$ is the limiting value, when $h=k=l=0$, of

$$|(1+h)^p(1+k)^q(1+l)^r| \div |(1+h)^0(1+k)^1(1+l)^2|,$$

i.e. of

$$\left\|\begin{matrix}1 & (p)_1 & (p)_2 & (p)_3 & \dots\\ 1 & (q)_1 & (q)_2 & (q)_3 & \dots\\ 1 & (r)_1 & (r)_2 & (r)_3 & \dots\end{matrix}\right\| \cdot \left\|\begin{matrix}1 & h & h^2 & h^3 & \dots\\ 1 & k & k^2 & k^3 & \dots\\ 1 & l & l^2 & l^3 & \dots\end{matrix}\right\| \div \left|\begin{matrix}1 & \cdot & \cdot\\ 1 & 1 & \cdot\\ 1 & 2 & 1\end{matrix}\right| \cdot \left|\begin{matrix}1 & h & h^2\\ 1 & k & k^2\\ 1 & l & l^2\end{matrix}\right|,$$

i.e.

$$\left|\begin{matrix}1 & p & (p)_2\\ 1 & q & (q)_2\\ 1 & r & (r)_2\end{matrix}\right|$$

SALMON, G. (1885).

[. . . Modern Higher Algebra. 4th ed. xvi+360 pp. Dublin.]

We note the alternants

$$\begin{vmatrix} \sin(\alpha+\alpha') & \sin(\alpha+\beta') & \sin(\alpha+\gamma') \\ \sin(\beta+\alpha') & \sin(\beta+\beta') & \sin(\beta+\gamma') \\ \sin(\gamma+\alpha') & \sin(\gamma+\beta') & \sin(\gamma+\gamma') \end{vmatrix},$$

$$\begin{vmatrix} 1 & \sin\alpha & \sin(\alpha+\theta) & \sin\alpha\sin(\alpha+\theta) \\ 1 & \sin\beta & \sin(\beta+\theta) & \sin\beta\sin(\beta+\theta) \\ 1 & \sin\gamma & \sin(\gamma+\theta) & \sin\gamma\sin(\gamma+\theta) \\ 1 & \sin\delta & \sin(\delta+\theta) & \sin\delta\sin(\delta+\theta) \end{vmatrix},$$

the former as being the product of two zeros, and the other as being equal to

$$\Pi \sin\tfrac{1}{2}(\delta-\gamma)\cdot 16\sin\theta\sin(\alpha+\beta+\gamma+\delta+\theta).$$

The result of squaring zero in the form

$$\begin{vmatrix} \cos\alpha & \sin\alpha & . \\ \cos\beta & \sin\beta & . \\ \cos\gamma & \sin\gamma & . \end{vmatrix}$$

is also noted : likewise the producing of

$$|\,(a_1-b_1)^r(a_2-b_2)^r\ \ldots\ (a_n-b_n)^r\,|$$

by means of the multiplication-theorem.

JOHNSON, W. W. (1886).

[On a geometrical representation of alternants of the third order and of their quotients when divided by A (0, 1, 2). *Quart. Journ. of Math.*, xxi. pp. 217–224.]

This is an elaboration of the diagrammatic part of the author's second paper of 1885, culminating in the representation of

$$|\,a^0b^pc^q\,| \div |\,a^0b^1c^2\,|$$

by means of a series of concentric hexagons, the first, as we already know, corresponding to

$$\Sigma a^{q-2}b^{p-1} + \ldots\ldots,$$

the second to

$$abc\cdot(\Sigma a^{q-4}b^{p-2} + \ldots\ldots),$$

and so on.

ANGLIN, A. H. (1886).

[Sur le coefficient du terme général dans certains développements. *Journ.* (*de Liouville*) *de Math.*, (4) ii. pp. 139–150.]

The coefficient in question is

$$-\begin{vmatrix} 1 & a & a^{n+2}(b^m+c^m) \\ 1 & b & b^{n+2}(c^m+a^m) \\ 1 & c & c^{n+2}(a^m+b^m) \end{vmatrix},$$

but determinants are not at all used or mentioned.

ANGLIN, A. H. (1886).

[On certain theorems mainly connected with alternants. *Proceed. R. Soc. Edinburgh*, xiii. pp. 823–839.]

The first result here is that *if the* pth *aleph function of* a, b, c, . . . *be denoted by* (p), *then*

$$\begin{vmatrix} 1 & a & a^2 & a^3 & \dots \\ (p) & (p+1) & (p+2) & (p+3) & \dots \\ (q) & (q+1) & (q+2) & (q+3) & \dots \\ (r) & (r+1) & (r+2) & (r+3) & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

is independent of a. The reason of course is that any column is freed of a by being diminished by a times the immediately preceding column: and Anglin's equivalent determinant of lower order is at once got by putting $a = 0$. The next result is that *the product of the complementary minors of the elements of a column of an* n*-line alternant has the* (n−2)th *power of the difference-product as a factor.* Thus, in $|a^p b^q c^r d^s|$ such a set of complementary minors must be divisible by

$$\xi^{\frac{1}{2}}(b, c, d), \quad \xi^{\frac{1}{2}}(c, d, a), \quad \xi^{\frac{1}{2}}(d, a, b), \quad \xi^{\frac{1}{2}}(a, b, c)$$

respectively, and therefore their product by the $\left(\frac{4\times3}{6}\right)^{\text{th}}$ power of $\xi^{\frac{1}{2}}(a, b, c, d)$.

The expression of the product of the minors of an n-by-2 array is then taken up—a subject hitherto not explicitly considered—and some deductions are made. Thus, when the array is

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{matrix}$$

there is obtained

$$|a_1b_2|\,|a_1b_3|\,|a_1b_4|\,|a_2b_3|\,|a_2b_4|\,|a_3b_4|$$

$$= \begin{vmatrix} a_1^3 & a_1^2b_1 & a_1b_1^2 & b_1^3 \\ a_2^3 & a_2^2b_2 & a_2b_2^2 & b_2^3 \\ a_3^3 & a_3^2b_3 & a_3b_3^2 & b_3^3 \\ a_4^3 & a_4^2b_4 & a_4b_4^2 & b_4^3 \end{vmatrix} = \begin{vmatrix} a_1^3b_2b_3b_4 & a_1^2 & a_1b_1 & b_1^2 \\ a_2^3b_3b_4b_1 & a_2^2 & a_2b_2 & b_2^2 \\ a_3^3b_4b_1b_2 & a_3^2 & a_3b_3 & b_3^2 \\ a_4^3b_1b_2b_3 & a_4^2 & a_4b_4 & b_4^2 \end{vmatrix}$$

$$= a_1^3b_2b_3b_4\cdot|a_2b_3|\,|a_2b_4|\,|a_3b_4| - \ldots - a_4^3b_1b_2b_3\cdot|a_1b_2|\,|a_1b_3|\,|a_2b_3|;$$

and by taking the 'complementary' of this* with respect to $|a_1b_2c_3d_4|$ and dividing by

$$|b_1c_2|\,|b_1c_3|\,|b_1c_4|\,|b_2c_3|\,|b_2c_4|\,|b_3c_4|\cdot|a_2b_3c_4|\,|a_1b_3c_4|\,|a_1b_2c_4|\,|a_1b_2c_3|$$

there is deduced the equality

$$-\frac{|a_1b_2c_3d_4|^3}{|a_2b_3c_4|\,|a_1b_3c_4|\,|a_1b_2c_4|\,|a_1b_2c_3|}$$

$$= \frac{|b_2c_3d_4|^3}{|b_2c_3|\,|b_2c_4|\,|b_3c_4|\,|a_2b_3c_4|} - \frac{|b_1c_3d_4|^3}{|b_1c_3|\,|b_1c_4|\,|b_3c_4|\,|a_1b_3c_4|}$$

$$+ \frac{|b_1c_2d_4|^3}{|b_1c_2|\,|b_1c_4|\,|b_2c_4|\,|a_1b_2c_4|} - \frac{|b_1c_2d_3|^3}{|b_1c_2|\,|b_1c_3|\,|b_2c_3|\,|a_1b_2c_3|},$$

the right-hand member of which takes two other forms on account of the left-hand member being symmetric with respect to a, b, c.

The paper concludes with a geometrical application of the theorem of which this is a case.

ROGERS, L. J. (1886).

[Question 8765. *Educ. Times*, xxxix. p. 372 : *Math. from Educ. Times*, xlvii. pp. 88–89.]

The subject here is the double alternant

$$\begin{vmatrix} \tan(x_1+a_1) & \tan(x_1+a_2) & \tan(x_1+a_3) \\ \tan(x_2+a_1) & \tan(x_2+a_2) & \tan(x_2+a_3) \\ \tan(x_3+a_1) & \tan(x_3+a_2) & \tan(x_3+a_3) \end{vmatrix},$$

but all we learn is that it vanishes when

$$x_1 + x_2 + x_3 + a_1 + a_2 + a_3 = 0,$$

and that the same is true for the determinant which differs from it in having sines in place of tangents.

* Not exactly; it would be literally correct if the previous b's were d's.

SCHRADER, W. (1887): SPORER, B. (1887).

[Beiträge zur Theorie der Determinanten. vi+156 pp. Halle a. S.]

[Ein Satz über Determinanten. *Math.-naturw. Mitt.* (Tübingen), ii. pp. 103–106.]

There is here a very full section on alternants (pp. 98–112), but unfortunately the numerous special results had all already appeared, some in Muir's text-book of 1882 and many others in Johnson's papers of 1885. Other determinants to which a separate section is given, namely, those with so-called 'factorial' elements, are in reality alternants, being expressible as sums of multiples of simple alternants. For example, taking α, β, γ, δ instead of the d, $2d$, $3d$, $4d$ of $a^{5|d}$, we have

$$\begin{vmatrix} 1 & a & a(a+\alpha) & a(a+\alpha)(a+\beta)(a+\gamma)(a+\delta) \\ 1 & b & b(b+\alpha) & b(b+\alpha)(b+\beta)(b+\gamma)\ (b+\delta) \\ 1 & c & c(c+\alpha) & c(c+\alpha)(c+\beta)(c+\gamma)\ \ (c+\delta) \\ 1 & d & d(d+\alpha) & d(d+\alpha)(d+\beta)(d+\gamma)(d+\delta) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2+a\alpha & a^5+a^4\Sigma\alpha+a^3\Sigma\alpha\beta+a^2\Sigma\alpha\beta\gamma+a\cdot\alpha\beta\gamma\delta \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 & a^5 \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 & a^4 \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \cdot \Sigma\alpha + \begin{vmatrix} 1 & a & a^2 & a^3 \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \Sigma\alpha\beta.$$

The note of the other writer concerns the alternant already dealt with by Garbieri in 1878.

MUIR, T. (1887).

[On a class of alternating functions. *Transac. R. Soc. Edinburgh*, xxxiii. pp. 309–312.]

When the number of variables in each of the two sets involved is 4, and m, n, r are integers not greater than 4, the function is

$$\frac{|\,b^m c^n d^r\,|}{|\,b^0 c^1 d^2\,|} \cdot \frac{(a-\alpha)(a-\beta)(a-\gamma)(a-\delta)}{(a-b)(a-c)(a-d)} + \frac{|\,a^m c^n d^r\,|}{|\,a^0 c^1 d^2\,|} \cdot \frac{(b-\alpha)(b-\beta)(b-\gamma)(b-\delta)}{(b-a)(b-c)(b-d)}$$

$$+ \frac{|\,a^m b^n d^r\,|}{|\,a^0 b^1 d^2\,|} \cdot \frac{(c-\alpha)(c-\beta)(c-\gamma)(c-\delta)}{(c-a)(c-b)(c-d)} + \frac{|\,a^m b^n c^r\,|}{|\,a^0 b^1 c^2\,|} \cdot \frac{(d-\alpha)(d-\beta)(d-\gamma)(d-\delta)}{(d-a)(d-b)(d-c)}$$

or, say $F_{m,n,r}$,

and the object is to show that while it is symmetric with respect to a, b, c, d and symmetric with respect to $\alpha, \beta, \gamma, \delta$, it is alternating with respect to the interchange

$$\begin{pmatrix} a & b & c & d \\ \alpha & \beta & \gamma & \delta \end{pmatrix}.$$

This is accomplished so far as the symmetry is concerned by transforming it into

$$\frac{1}{|a^0b^1c^2d^3|}\Big[\begin{matrix} |a^mb^nc^rd^4| - |a^mb^nc^rd^3|\cdot\Sigma\alpha + |a^mb^nc^rd^2|\cdot\Sigma\alpha\beta \\ - |a^mb^nc^rd|\cdot\Sigma\alpha\beta\gamma + |a^mb^nc^rd^0|\cdot\Sigma\alpha\beta\gamma\delta \end{matrix}\Big].$$

Alternancy is then demonstrated by considering separately all the possible sets of values of m, n, r, the final result being *If* m, n, r *arranged in order of magnitude be any three of the values* 0, 1, 2, 3, 4 *and* u, v *arranged in order of magnitude be the remaining two, then*

$$\mathrm{F}_{\mathrm{m,\,n,\,r}} = |\,\mathrm{s}_{4-\mathrm{u}}\sigma_{4-\mathrm{v}}\,|,$$

where s *and* σ *are explained by the examples*

$$\mathrm{s}_2 = \Sigma\mathrm{ab}, \quad \sigma_3 = \Sigma\alpha\beta\gamma.$$

RUSSELL, A. (1887).

[Question 9017. *Educ. Times*, xl. p. 135 : *Math. from Educ. Times*, xlvii. p. 124.]

This and Sylvester's unsolved question 2810 of the year 1869 are both included in Muir's first paper of this year.

MUIR, T. (1887).

[On the quotient of a simple alternant by the difference-product of the variables. *Proceed. R. Soc. Edinburgh*, xiv. pp. 433–445.]

Three pages at the beginning of this are devoted to a resuscitation of the author's method of 1879, his action being due to the fresh interest created in the subject by Johnson's papers. The peculiarity of the method lies in that it determines the arithmetical coefficient of any one of the simple symmetric functions without regard to any of the rest. It is stated in the form of a 'rule' and leads in the case of the 3rd order to the conclusion that *the coefficient of* $\Sigma a^x b^y c^z$ *in the expansion of* $|\,a^0b^pc^q\,| \div |\,a^0b^1c^2\,|$ *is the number of*

ways in which by taking a number from

$$0,\ 1,\ 2,\ \ldots,\ z-1,\ z$$

and a number from

$$p,\ p+1,\ \ldots,\ q-2,\ q-1,$$

the sum $y+z+1$ *may be obtained.* To illustrate it in the case of the 4th order Johnson's example (0. 5, 13, 17) is taken and the coefficients of $\Sigma a^{10}b^9c^7d^3$, $\Sigma a^{10}b^9c^6d^4$, $\Sigma a^{10}b^9c^5d^5$ found to be 35, 46, 50. The following results connected with this order are then established at length.

(1) *The expansion of* $|\,a^0b^1c^3d^{3+s}\,| \div |\,a^0b^1c^2d^3\,|$ *consists of all the single symmetric functions which have no index greater than* s, *and the sum of whose indices is* s+1, *the coefficient of each function being less by* 1 *than the number of different letters appearing in any of its terms*; for example,

$$(0,\ 1,\ 3,\ 7) \;=\; \Sigma a^4b + \Sigma a^3b^2 + 2\Sigma a^3bc + 2\Sigma a^2b^2c + 3\Sigma a^2bcd.$$

(2) *The expansion of* $|\,a^0b^2c^3d^{3+s}\,| \div |\,a^0b^1c^2d^3\,|$ *consists of the single symmetric function* $\Sigma a^s bc$ *and all the like functions succeeding it, the coefficient of every term involving three letters being* 1, *and the coefficient of every term involving four letters being* 3; for example,

$$(0,\ 2,\ 3,\ 6) \;=\; \Sigma a^3bc + \Sigma a^2b^2c + 3\Sigma a^2bcd.$$

(3) *The expansion of*

$$(0,\ 1,\ q,\ r) - (0,\ 2,\ q,\ r-1) + (1,\ 2,\ q,\ r-2)$$

consists of the symmetric function $\Sigma a^{r-3}b^{q-2}$ *and the like functions succeeding it, the coefficient in every case being* 1, *i.e.* $(\Sigma a^{r-3}b^{q-2} + \ldots)$.

(4) *The expansion of* (0, 1, 2+s, 3+s) *is*

$$(\Sigma a^s b^s + \ldots) + (\Sigma a^{s-1}b^{s-1}cd + \ldots) + (\Sigma a^{s-2}b^{s-2}c^2d^2 + \ldots) + \ldots.$$

(5) *The expansion of*

$$(0,\ 2,\ 2+s,\ 3+s) - (1,\ 2,\ 1+s,\ 3+s)$$

is $(\Sigma a^s b^s c + \ldots)$.

(6) *The expansion of* (0, 1, 2+s, 4+s) *is*

$$(\Sigma a^{s-1}b^s + \ldots) + (\Sigma a^s b^s c + \ldots) + abcd(0,\ 1,\ s,\ s+2).$$

The paper concludes with tables giving the quotients from (0, 1, 2, 4) to (2, 3, 4, 6).

ANGLIN, A. H. (1887).

[Théorèmes sur les déterminants. *Bull. Soc. Math. de France*, xv. pp. 120–129.]

[On the summation of a certain series of alternants. *Proceed. R. Soc. Edinburgh*, xix. pp. 194–203.]

The theorems in the first paper are rediscoveries, having indeed already found their way into the ordinary text-books.

In the case of the second it is different, only the lemma which is brought forward and established being a rediscovery. In the case of the 4th order the new theorem is that *the sums*

$$|\,a^{p+1}b^qc^rd^s\,| + |\,a^pb^{q+1}c^rd^s\,| + |\,a^pb^qc^{r+1}d^s\,| + |\,a^pb^qc^rd^{s+1}\,|,$$
$$|\,a^{p+1}b^{q+1}c^rd^s\,| + |\,a^{p+1}b^qc^{r+1}d^s\,| + \ldots + |\,a^pb^qc^{r+1}d^{s+1}\,|,$$
$$|\,a^{p+1}b^{q+1}c^{r+1}d^s\,| + \ldots\ldots + |\,a^pb^{q+1}c^{r+1}d^{s+1}\,|$$

can each be expressed as a single determinant multiplied by the difference-product. By way of proof it has only to be noted that the quotient of the first sum by $\xi^{\frac{1}{2}}(a, b, c, d)$ is known from Jacobi to be

$$\begin{vmatrix} (p+1) & (q) & (r) & (s) \\ (p) & (q-1) & (r-1) & (s-1) \\ (p-1) & (q-2) & (r-2) & (s-2) \\ (p-2) & (q-3) & (r-3) & (s-3) \end{vmatrix} + \begin{vmatrix} (p) & (q+1) & (r) & (s) \\ (p-1) & (q) & (r-1) & (s-1) \\ (p-2) & (q-1) & (r-2) & (s-2) \\ (p-3) & (q-2) & (r-3) & (s-3) \end{vmatrix} + \ldots.$$

where (p) denotes the p^{th} aleph function, and this by Le Paige's theorem of 1880 is equal to

$$\begin{vmatrix} (p+1) & (q+1) & (r+1) & (s+1) \\ (p-1) & (q-1) & (r-1) & (s-1) \\ (p-2) & (q-2) & (r-2) & (s-2) \\ (p-3) & (q-3) & (r-3) & (s-3) \end{vmatrix}.$$

In the same way it follows that the two other sums are equal to

$$\begin{vmatrix} (p+1) & (q+1) & (r+1) & (s+1) \\ (p) & (q) & (r) & (s) \\ (p-2) & (q-2) & (r-2) & (s-2) \\ (p-3) & (q-3) & (r-3) & (s-3) \end{vmatrix} \xi^{\frac{1}{2}}, \quad \begin{vmatrix} (p+1) & (q+1) & (r+1) & (s+1) \\ (p) & (q) & (r) & (s) \\ (p-1) & (q-1) & (r-1) & (s-1) \\ (p-3) & (q-3) & (r-3) & (s-3) \end{vmatrix} \xi^{\frac{1}{2}}$$

respectively.

MARCOLONGO, R. (1887).

[Generalizzazione di un teorema sui determinanti. *Giornale di Mat.*, xxv. pp. 298–302.]

The portion of this which is new is readily suggested by Besso's theorem of 1882, being in effect that *the determinant of the* $(2n)^{th}$ *order whose* h^{th} *row* is

$$\phi_h(x_1),\ \phi_h(x_2),\ \ldots,\ \phi_h(x_n),\ \phi_h'(x_1),\ \phi_h'(x_2),\ \ldots,\ \phi_h'(x_n),$$

where $\phi_h(x) = x^{h-1} + \text{terms in } x^{h-2},\ x^{h-3},\ \ldots,$

is equal to $(-1)^{\frac{1}{2}n(n-1)} \cdot \xi^{\frac{4}{2}}(x_1,\ x_2,\ \ldots,\ x_n).$

WEIHRAUCH, K. (1887).

[Ueber gewisse Determinanten. *Zeitschrift f. Math. u. Phys.*, xxxiii. pp. 126–128.]

The determinants in question, P and Q say, closely resemble one dealt with by Scott in 1879, and Q is actually included in another of Scott's of 1881. They are

$$\begin{vmatrix} 1 & \cos a_1 & \sin a_1 & \cos 2a_1 & \sin 2a_1 & \ldots. & \cos(m-1)a_1 & \sin(m-1)a_1 & \cos ma_1 \\ 1 & \cos a_2 & \sin a_2 & \cos 2a_2 & \sin 2a_2 & \ldots. & \cos(m-1)a_2 & \sin(m-1)a_2 & \cos ma_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

and the one formable from it by putting sines for cosines in the last column. Instead of evaluating them separately with the help of the exponential values of the sine and cosine, Weihrauch evaluates with the same help

$$P + Qi,$$

which is also a determinant, and so finds readily

$$P = (-1)^m \cdot 2^{2m(m-1)+1} \cdot \Pi \sin \tfrac{1}{2}(a_{2m} - a_{2m-1}) \cdot \sin \tfrac{1}{2}\Sigma a,$$

$$Q = (-1)^{m-1} \cdot 2^{2m(m-1)+1} \cdot \Pi \sin \tfrac{1}{2}(a_{2m} - a_{2m-1}) \cdot \cos \tfrac{1}{2}\Sigma a.$$

It is interesting to learn from the writer that P and Q are not a product of fancy, but actually arose in the course of a meteorological investigation led up to by his professional duties.

We note for ourselves that if we increase each of the angles of the last column of P by ξ we obtain a determinant equal to

$$P \cos \xi - Q \sin \xi,$$

and therefore equal to

$$(-1)^m 2^{2m(m-1)+1} \cdot \Pi \sin \tfrac{1}{2}(a_{2m} - a_{2m-1}) \cdot \sin(\tfrac{1}{2}\Sigma a + \xi).$$

Further, if we treat the last column of Q in like fashion, the value of the resulting determinant differs from that of Q merely by having ξ added to $\frac{1}{2}\Sigma a$, which is Scott's generalization above referred to.

STIELTJES, T. J. (1887).

[Sur une généralisation de la formule des accroissements finis. *Bull. Soc. Math. de France*, xvi. pp. 100–113 : or *Nouv. Annales de Math.*, (3) vii. pp. 26–31.]

The theorem here reached is a variant of Schwarz's of 1880, namely, that if the functions

$$\begin{array}{cccc} f(u), & g(u), & h(u), & k(u), \\ f'(u), & g'(u), & h'(u), & k'(u), \\ f''(u), & g''(u), & h''(u), & k''(u), \end{array}$$

be finite and continuous, and x, y, z, t be values of u taken in ascending order of magnitude, then

$$\frac{|f(x)\,g(y)\,h(z)\,k(t)|}{|x^0 y^1 z^2 t^3|} = \frac{1}{1!\,2!\,3!} \begin{vmatrix} f(x) & g(x) & h(x) & k(x) \\ f'(\eta) & g'(\eta) & h'(\eta) & k'(\eta) \\ f''(\xi) & g''(\xi) & h''(\xi) & k''(\xi) \\ f'''(\tau) & g'''(\tau) & h'''(\tau) & k'''(\tau) \end{vmatrix},$$

where η lies between x and y, ξ between x and z, and τ between x and t. A simple proof is obtained by repeated use of the familiar theorem sought to be generalized ; and then on the further supposition that

$$f'''(u), \quad g'''(u), \quad h'''(u), \quad k'''(u)$$

are continuous, it is shown that if x, y, z, t tend toward a common limit a, the equality will still hold if η, ξ, τ are also put equal to a.

MUIR, T. (1888).

[On a class of alternants expressible in terms of simple alternants. *Proceed. R. Soc. Edinburgh*, xv. pp. 298–308.]

The alternants referred to are those dealt with by Johnson in his first paper of 1885, and the last of the three methods explained does

not really differ from Johnson's. The subject, however, is treated in greater detail, and pains are taken to make clear the possible simplifications of the procedure, application being made to show that

$$\begin{vmatrix} 1 & bc+bd+cd & b^3c+b^3d+c^3b+\ldots & b^4c^4d+b^4cd^4+bc^4d^4 \\ 1 & cd+ca+da & c^3d+c^3a+d^3c+\ldots & c^4d^4a+c^4da^4+cd^4a^4 \\ 1 & da+db+ab & d^3a+d^3b+a^3d+\ldots & d^4a^4b+d^4ab^4+da^4b^4 \\ 1 & ab+ac+bc & a^3b+a^3c+b^3a+\ldots & a^4b^4c+a^4bc^4+ab^4c^4 \end{vmatrix}$$

$$= |a^1b^2c^5d^7| + |a^1b^3c^4d^7| + |a^0b^3c^5d^7| - |a^0b^4c^5d^6| - |a^0b^2c^6d^7| - |a^2b^3c^4d^6| + |a^1b^3c^5d^6|.$$

A table is added giving the equivalents of

$$\begin{vmatrix} 1 & a & a^2 & a^r \cdot \mathrm{F}(b,c,d) \\ 1 & b & b^2 & b^r \cdot \mathrm{F}(c,d,a) \\ 1 & c & c^2 & c^r \cdot \mathrm{F}(d,a,b) \\ 1 & d & d^2 & d^r \cdot \mathrm{F}(a,b,c) \end{vmatrix}$$

for all cases where $a^r \cdot \mathrm{F}(b,c,d)$ is of the 9th degree, and where of course F is symmetric.

ANGLIN, A. H. (1888).

[On certain theorems mainly connected with alternants (II.). *Proceed. R. Soc. Edinburgh,* xv. pp. 381–396.]

[Alternants which are constant multiples of the difference-product of the variables. *Proceed. R. Soc. Edinburgh,* xv. pp. 468–476.]

The specified subject of the second paper here is also the actual subject of the algebraical part of the first, and both are mainly occupied with a special class of alternants belonging to the type

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \ldots. & a_1^{n-2} & \mathrm{F}(a_1;\ a_2, \ldots, a_n) \\ 1 & a_2 & a_2^2 & \ldots. & a_2^{n-2} & \mathrm{F}(a_2;\ a_1, \ldots, a_n) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n,$$

where F is a symmetric function of all the a's that follow the semicolon. Unlike his predecessors Anglin confines himself to the case where F is of the $(n-1)^{\text{th}}$ degree, and where therefore the cofactor of the difference-product is independent of the a's. The first result to be noted is that *if* F *be a* simple *symmetric function, and in the*

Σ-specification of it a of the letters used are alike in their exponent, β others have a similar mutual resemblance, γ others a similar, and so on, then the cofactor of $\xi^{\frac{1}{2}}$ is

$$(-1)^{\alpha+\beta+\gamma+\ldots} \cdot \frac{(\alpha+\beta+\gamma+\ldots)!}{\alpha!\,\beta!\,\gamma!\ldots}.$$

For example, if the variables be a, b, c, d, e, and if the simple symmetric function free of a be Σb^3c, the desired cofactor is

$$(-1)^2 \cdot \frac{2!}{1!\,1!}, \quad i.e.\ 2,$$

and we have

$$\begin{vmatrix} 1 & a & a^2 & a^3 & \Sigma b^3c \\ 1 & b & b^2 & b^3 & \Sigma c^3a \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 2 \cdot \xi^{\frac{1}{2}}(a, b, c, d, e).$$

Similarly, if the variables be a, b, c, d, e, f, g, h,

$$\begin{vmatrix} 1 & a & a^2 & a^3 & a^4 & a^5 & a^6 & a^3 \cdot \Sigma b^3c \\ 1 & b & b^2 & b^3 & b^4 & b^5 & b^6 & b^3 \cdot \Sigma c^3d \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 2 \cdot \xi^{\frac{1}{2}}(a, \ldots, h),$$

and so on. He then passes on to the case where F is the most general integral function of a, b, c, d, e, which is symmetric with respect to b, c, d, e, namely,

$$m \cdot a^4 + n \cdot a^3\Sigma b + p \cdot a^2\Sigma b^2 + q \cdot a^2\Sigma bc + \ldots + t \cdot a\Sigma bcd$$
$$+ u \cdot \Sigma b^4 + v \cdot \Sigma b^3c + w \cdot \Sigma b^2c^2 + x \cdot \Sigma b^2cd + y \cdot bcde,$$

giving as the desired cofactor

$$m - n - \ldots - t - u + 2v + w - 3x + y.$$

Lastly, he takes up for a moment such an alternant as

$$\begin{vmatrix} 1 & \Sigma b & \Sigma bc & \Sigma b^2c & \Sigma b^3c \\ 1 & \Sigma c & \Sigma cd & \Sigma c^2d & \Sigma c^3d \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \Sigma a & \Sigma ab & \Sigma a^2b & \Sigma a^3b \end{vmatrix},$$

where the Σ's of the first row refer to b, c, d, e, the Σ's of the second to c, d, e, a, and so on; and noting that the values of the cofactors corresponding to

$$\Sigma b, \quad \Sigma bc, \quad \Sigma b^2c, \quad \Sigma b^3c$$

in the previous case are

$$-1, \quad 1, \quad 2, \quad 2$$

shows that here the value is the product of these values, namely -4.

His method, like Muir's first method mentioned above, depends on the transformation of the last column by subtracting multiples of the others.

WEILL, G. (1888).

[Sur une forme du déterminant de Vandermonde. *Nouv. Annales de Math.*, (3) vii. pp. 427–429.]

The author formulates the set of equations for determining the A's so that

$$\frac{1}{(x-a_1)(x-a_2)\ldots(x-a_n)} \equiv \frac{A_1}{x-a_1}+\frac{A_2}{x-a_2}+\ldots+\frac{A_n}{x-a_n},$$

and notes (1) Tarleton's result of 1867, namely, that the determinant of the set is equal to the difference-product of the a's (*Hist.*, iii. pp. 141–2), although its r^{th} row, the conjugate having been taken, is free of a_r, whereas the r^{th} row of $|\,a_1{}^0a_2{}^1\ldots a_n{}^{n-1}\,|$ contains a_r alone; and (2) that the cofactor of the last element of each row, although involving all the a's, is a function of only $n-1$ of them.*

LORIA, G. (1888).

[Nota su una classe di determinanti. *Giornale di Mat.*, xxvi. pp. 329–333.]

A rediscovery of Naegelsbach's extension of Bellavitis' procedure of 1857.

STARKOFF, A. (1889).

[Théorie des équations générales. *Mém. . . . soc. des naturalistes . . .* (Odessa), x. pp. 143–200.]

Any freshness connected with the determinants used here, alternants chiefly and wronskians, is in the main a matter of notation.

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* The cofactors of other elements besides the last are worth noting; for example, in the case of the first row of

$$\begin{vmatrix} 1 & b+c+d & bc+cd+db & bcd \\ 1 & c+d+a & cd+da+ab & cda \\ 1 & d+a+b & da+ab+bd & dab \\ 1 & a+b+c & ab+bc+ca & abc \end{vmatrix}$$

the four are

$$-a^3\,|\,b^0c^1d^2\,|,\quad a^2\,|\,b^0c^1d^2\,|,\quad -a\,|\,b^0c^1d^2\,|,\quad |\,b^0c^1d^2\,|,$$

the last being independent of a.

For example, the general alternant $|f_0(x_0)\cdot f_1(x_1)\cdot \ldots \cdot f_n(x_n)|$ is denoted by $|x^0 x_1^{(1)} \ldots x_n^{(n)}|$, and the operation of changing $f_r(x)$ into $f_{r+1}(x)$ by $\mathrm{K}f_r(x)$, this and similar uses of K being frequent.

WEIHRAUCH, K. (1888, 1889).

[Neue Untersuchungen über die Bessel'sche Formel, und deren Verwendung in der Meteorologie. *Schriften . . . Naturf-Ges. bei der Univ. Dorpat*, iv.]

[Ueber gewisse goniometrische Determinanten, und damit . . . *Zeitschrift f. Math. u. Phys.*, xxxvi. pp. 71–77.]

These are continuations of the author's work of 1887, showing the same skill in the treatment of such determinants, and involving a considerable number of fresh evaluations. The opening result is a rediscovery, being that given by Scott in 1879 and already rediscovered by Fouret in 1884. As it has a column of 1's, columns of cosines and columns of sines, there are three types of primary minors, and all are dealt with, Scott, on the other hand, having only considered the first type. Similarly, Weihrauch's P and Q of 1887 lead each to three evaluations. More interesting are a companion-theorem of Scott's original, namely,

$$\begin{vmatrix} \cos a_1 & \sin a_1 & \cos 3a_1 & \sin 3a_1 & \ldots & \cos(2m-1)a_1 & \sin(2m-1)a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= 4^{m(m-1)} \cdot \Pi \sin(a_{2m} - a_{2m-1}),$$

and the like companions in the case of P and Q, namely,

$$\begin{vmatrix} \cos a_1 & \sin a_1 & \ldots & \cos(2m-1)a_1 & \sin(2m-1)a_1 & \cos(2m+1)a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= (-1)^m \cdot 4^{mm} \cdot \cos \Sigma a \cdot \Pi \sin(a_{2m+1} - a_{2m}),$$

$$\begin{vmatrix} \cos a_1 & \sin a_1 & \ldots & \cos(2m-1)a_1 & \sin(2m-1)a_1 & \sin(2m+1)a_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= (-1)^m \cdot 4^{mm} \cdot \sin \Sigma a \cdot \Pi \sin(a_{2m+1} - a_{2m}),$$

the latter, however, being included in a result indicated by Scott in 1881. Consideration of the primary minors of these leads to three pairs of evaluations, and with this the first section of the paper ends.

The second section concerns the sets of linear equations to which in the Theory of Interpolation the evaluated determinants pertain.

As before, we note for ourselves that if each angle in the last columns of the pair of determinants above be increased by ξ and $\Sigma\alpha$ in the evaluations receive the same increment, the identities still hold.

MERTENS, F. (1889).

[O wyznaczniku, którego elementami są wartości $n!$ funkcyj całkowitych n zmiennych $x_1, x_2, \ldots, x_n$ pochodzące z wszelkich możliwych przemian tychże zmiennych. *Pamiętnik Akad. . . . Wydz. mat.-przyr.*, xvi. pp. 60–69.]

The determinant here has for its $(r,s)^{\text{th}}$ element an integral function ϕ_r of the s^{th} permutation of the variables $x_1, x_2, \ldots, x_n$, and is therefore of course of the order $n!$ It is readily seen in the usual way to have $|\, x_1^{\,0}\, x_2^{\,1} \ldots x_n^{\,n-1} \,|$ for a factor, but the author's theorem is that $|\, x_1^{\,0} x_2^{\,1} \ldots x_n^{\,n-1} \,|^{\frac{1}{2}(n)!}$ is a factor.

He also generalizes the related result

$$\begin{vmatrix} 1 & x_1 & x_2 & x_1x_2 & x_2^{\,2} & x_1x_2^{\,2} \\ 1 & x_1 & x_3 & x_1x_3 & x_3^{\,2} & x_1x_3^{\,2} \\ 1 & x_2 & x_3 & x_2x_3 & x_3^{\,2} & x_2x_3^{\,2} \\ 1 & x_2 & x_1 & x_2x_1 & x_1^{\,2} & x_2x_1^{\,2} \\ 1 & x_3 & x_1 & x_3x_1 & x_1^{\,2} & x_3x_1^{\,2} \\ 1 & x_3 & x_2 & x_3x_2 & x_2^{\,2} & x_3x_2^{\,2} \end{vmatrix} = |\, x_1^{\,0} x_2^{\,1} x_3^{\,2} \,|^3,$$

where only two of the three variables occur in each row, and the sum of the elements of each row is of the form $(1+u)(1+v+v^2)$.

WEIHRAUCH, K. (1889).

[Ueber eine algebraische Determinante mit eigenthümlichem Bildungsgesetz der Elemente. *Zeitschrift f. Math. u. Phys.*, xxxvi. pp. 34–40.]

The first determinant here considered, $A_{n(m+1)}$ say, is of the order $n(m+1)$, consisting of the n rows

$$\begin{array}{cccccc} 1 & a_1 & a_1^{\,2} & \ldots. & a_1^{\,n(m+1)-1} \\ 1 & a_2 & a_2^{\,2} & \ldots. & a_2^{\,n(m+1)-1} \\ . & . & . & . & . & . \end{array}$$

and nm rows obtained by successive differentiation of each of these m times; and the result reached, naturally with some toil, is

$$A_{n(m+1)} = \{m!\,(m-1)!\,\ldots\,2!\,1!\,0!\}^n \cdot \{\zeta^{\frac{1}{2}}(a_1, a_2, \ldots, a_n)\}^{(m+1)^2}.$$

For example, when n is 2 and m is 2,

$$A_{2(2+1)} = \begin{vmatrix} 1 & a & a^2 & a^3 & a^4 & a^5 \\ . & 1 & 2a & 3a^2 & 4a^3 & 5a^4 \\ . & . & 2 & 6a & 12a^2 & 20a^3 \\ 1 & b & b^2 & b^3 & b^4 & b^5 \\ . & 1 & 2b & 3b^2 & 4b^3 & 5b^4 \\ . & . & 2 & 6b & 12b^2 & 20b^3 \end{vmatrix} = 4(b-a)^9.$$

The root-idea appears in Besso's paper of 1882, where $A_{n(1+1)}$ is dealt with.

The next type of determinant considered differs from this in having for its fundamental rows n rows like those of P in the author's paper of 1887, namely,

$$1 \quad \cos a_1 \quad \sin a_1 \quad \ldots\ldots \quad \cos(n-1)a_1 \quad \sin(n-1)a_1 \quad \cos na_1,$$

and in having $m = 1$. The two results obtained may be written

$$P_{n(1+1)} = (-1)^n \cdot 2^{(2n-1)(n-1)} \cdot \{\Pi \sin \tfrac{1}{2}(a_n - a_{n-1})\}^4 \cdot \sin \Sigma a,$$

$$Q_{n(1+1)} = (-1)^{n-1} \cdot 2^{(2n-1)(n-1)} \cdot \{\Pi \sin \tfrac{1}{2}(a_n - a_{n-1})\}^4 \cdot \cos \Sigma a.$$

The third type is a generalization, like Marcolongo's, of the first, each power of a_r being replaced by an integral function of a_r.

SEGAR, H. W. (1890).

[Some inequalities. *Messenger of Math.*, (2) xix. pp. 189–192; xx. pp. 54–59.]

So far as alternants are concerned, the two papers culminate in the result that *if* a, b, c, . . . , p, q, r, . . . *be positive quantities, then* $|a^p b^q c^r \ldots|$ *is of the same sign as* $\zeta^{\frac{1}{2}}(a, b, c, \ldots) \cdot \zeta^{\frac{1}{2}}(p, q, r, \ldots)$.

SCHENDEL, L. (1891).

[Mathematische Miscellen. III. Das alternierende Exponential-differenzenproduct. *Zeitschrift f. Math. u. Phys.*, xxxviii. pp. 84–87.]

What is fresh here is a theorem including Weihrauch's of 1889, given, however, in an unlike form and without any reference to

preceding workers. The relation between the two in the case where the number of variables in the difference-product is three, lies in the fact that whereas Weihrauch's determinant is equal to a multiple of

$$\{(x-y)(x-z)(y-z)\}^{p^2}$$

and is of the $2p^{\text{th}}$ order, Schendel's is equal to

$$(x-y)^{pq}(x-z)^{pr}(y-z)^{qr}$$

and is of the order $p+q+r$. If we write n for $p+q+r$, the first p columns of the latter determinant are

$$\begin{array}{lll} (n-1)_{p-1}x^{n-p} & \ldots. & (n-1)_0 x^{n-1} \\ (n-2)_{p-1}x^{n-p-1} & \ldots. & (n-2)_0 x^{n-2} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & , \end{array}$$

the next q columns are

$$\begin{array}{lll} (n-1)_{q-1}y^{n-q} & \ldots. & (n-1)_0 y^{n-1} \\ (n-2)_{q-1}y^{n-q-1} & \ldots. & (n-2)_0 y^{n-2} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & , \end{array}$$

and so on. For example, when $p, q, r = 1, 2, 3$, we have

$$\begin{vmatrix} x^5 & 5y^4 & y^5 & 10z^3 & 5z^4 & z^5 \\ x^4 & 4y^3 & y^4 & 6z^2 & 4z^3 & z^4 \\ x^3 & 3y^2 & y^3 & 3z & 3z^2 & z^3 \\ x^2 & 2y & y^2 & 1 & 2z & z^2 \\ x & 1 & y & . & 1 & z \\ 1 & . & 1 & . & . & 1 \end{vmatrix} = (x-y)^2(x-z)^3(y-z)^6.$$

Of the other results we need only note as an example

$$\begin{vmatrix} x^4 & (\alpha+4)_3 & (\beta+4)_2 & (\gamma+4)_1 & 1 \\ x^3y & (\alpha+3)_3 & (\beta+3)_2 & (\gamma+3)_1 & 1 \\ x^2y^2 & (\alpha+2)_3 & (\beta+2)_2 & (\gamma+2)_1 & 1 \\ xy^3 & (\alpha+1)_3 & (\beta+1)_2 & (\gamma+1)_1 & 1 \\ y^4 & (\alpha)_3 & (\beta)_2 & (\gamma)_1 & 1 \end{vmatrix} = (x-y)^4.$$

In no case is there any hint of a proof. Fortunately most of them are such as Zeipel had established in 1865.

For ourselves we must note that the specification of the main

determinant can be greatly simplified. Thus, in the case where it equals $(x-y)^{pq}(x-z)^{pr}(y-z)^{qr}$, we begin with the row

$$1 \quad x \quad x^2 \quad \ldots \quad x^{n-1},$$

obtain the next row by the operation $\partial/\partial x$, the 3rd row by the operation $\frac{1}{2}\partial/\partial x$ on the 2nd, the 4th row by the operation $\frac{1}{3}\partial/\partial x$ on the 3rd, and so on until p rows in all are got: then in similar fashion follow q rows involving y, and r rows involving z. Our six-line example above would thus take the form

$$\begin{vmatrix} 1 & x & x^2 & x^3 & x^4 & x^5 \\ 1 & y & y^2 & y^3 & y^4 & y^5 \\ . & 1 & 2y & 3y^2 & 4y^3 & 5y^4 \\ 1 & z & z^2 & z^3 & z^4 & z^5 \\ . & 1 & 2z & 3z^2 & 4z^3 & 5z^4 \\ . & . & 1 & 3z & 6z^2 & 10z^3 \end{vmatrix} = (y-x)^2(z-x)^3(z-y)^6.$$

For proof we might be content with the usual tedious verification obtained by removing the factors as methodically as possible one or more at a time; but a more interesting mode depends on the multiplication of the determinant by a certain multiple of itself. Thus, in the case just mentioned, we obtain the multiple in question by reversing the order of the columns and then multiplying them by 1, −5, 10, −10, 5, −1 respectively. The form of the product is zero-axial and skew, so that root-extraction is possible: thereafter little is needed.

SEGAR, H. W. (1892).

[On a determinantal theorem due to Jacobi. *Messenger of Math.*, (2) xxi. pp. 148–157.]

The theorem of Jacobi's referred to, and of which use is repeatedly made in the paper, is that regarding the quotient of $|a^\alpha b^\beta c^\gamma \ldots|$ by the difference-product $\zeta^{\frac{1}{2}}(a, b, c, \ldots)$. Of the results obtained, the most important is the first. In effect this is that

$$\frac{\begin{vmatrix} (\ )^{\alpha+\alpha'} & (\ )^{\alpha+\beta'} & (\ )^{\alpha+\gamma'} & \ldots \\ (\ )^{\beta+\alpha'} & (\ )^{\beta+\beta'} & (\ )^{\beta+\gamma'} & \ldots \\ (\ )^{\gamma+\alpha'} & (\ )^{\gamma+\beta'} & (\ )^{\gamma+\gamma'} & \ldots \\ . & . & . & . \end{vmatrix}}{(abc\ldots)^{n-1}} = (-1)^{\frac{1}{2}n(n-1)} \frac{\begin{vmatrix} s_{\alpha+\alpha'} & s_{\alpha+\beta'} & s_{\alpha+\gamma'} & \ldots \\ s_{\beta+\alpha'} & s_{\beta+\beta'} & s_{\beta+\gamma'} & \ldots \\ s_{\gamma+\alpha'} & s_{\gamma+\beta'} & s_{\gamma+\gamma'} & \ldots \\ . & . & . & . \end{vmatrix}}{\zeta(a, b, c, \ldots)}$$

where $(\ )^\theta \equiv (a, b, c, \ldots)^\theta \equiv \theta^{\text{th}}$ aleph function of $a, b, c, \ldots$,

$$s_\theta \equiv a^\theta + b^\theta + c^\theta + \ldots,$$

and the number of the variables $a, b, c, \ldots$ is n. The proof is indirect and somewhat lengthy, and is followed by the analogous theorem where the elements instead of being coefficients in the expansion of $1/(1 - x\Sigma a + x^2\Sigma ab - \ldots)$ are coefficients in the expansion of a multiple of this by a polynomial in x of the m^{th} degree. The third result is of a different type, being, when altered,*

$$\left| \mathrm{D}^\alpha \frac{1}{f(x)} \cdot \mathrm{D}^\beta \frac{x}{f(x)} \cdot \mathrm{D}^\gamma \frac{x^2}{f(x)} \cdot \ldots \right|$$

$$= \frac{(-1)^{\alpha+\beta+\gamma+\cdots} \cdot \alpha!\,\beta!\,\gamma! \ldots}{f(x) \cdot \xi^{\frac{1}{2}}(a_1, a_2, a_3, \ldots)} \cdot \left| (x+a_1)^{-\alpha}(x+a_2)^{-\beta}(x+a_3)^{-\gamma} \ldots \right|_n,$$

where $$f(x) \equiv (x+a_1)(x+a_2)(x+a_3) \ldots.$$

The fourth theorem is connected with the first, and the fifth with the second; the sixth deals with the unisignancy of

$$|\, a^\alpha b^\beta c^\gamma \ldots |\ \div\ |\, a^0 b^1 c^2 \ldots |$$

in the case of the lower orders, and the others with

$$(a+y, b+y, c+y, \ldots)^n, \quad \left(\frac{1}{a+y}, \frac{1}{b+y}, \frac{1}{c+y}, \ldots\right)^n$$

and related matters not requiring our attention. It is the first theorem that gives the paper its attraction, and that manifestly calls for further study.

ECHOLS, W. H. (1892).

[Exercise 336. *Annals of Math.*, vi. p. 136.]

The determinant form of $\xi^{\frac{1}{2}}(1^2, 2^2, \ldots, n^2)$ being denoted by Δ, and its $(r, s)^{\text{th}}$ primary minor by $\Delta_{(r,s)}$, the problem set is to find the limiting value of $\Delta_{(r,s)} \div \Delta$ when n approaches infinity.

* On the right Segar has no $f(x)$ in the denominator, but has it with the exponent $\frac{1}{2}n(n-3)$ in the numerator. As a check on the sign-factor the case where $\alpha, \beta, \gamma, \ldots = 1, 2, 3, \ldots$ may be compared with Segar's own result of 1891. Note is necessary, however, that with him $\xi^{\frac{1}{2}}(a_1, a_2, \ldots, a_n)$ stands for $(-1)^{\frac{1}{2}n(n-1)} |\, a_1{}^0 a_2{}^1 \ldots a_n{}^{n-1} |$.

ECHOLS, W. H. (1892).

[On certain determinant forms and their applications. *Annals of Math.*, vi. pp. 105–126; vii. pp. 11–59, 93–101, 109–142.]

The applications referred to concern the expansion of functions in series, and constitute the chief value of the paper. To the student of determinants it is mainly interesting as affording fresh evidence of the important uses to which his subject can be put.

As a first and fundamental example, we may give the following extension of the familiar theorem of Rolle, namely, *If* $f(x)$ *and its first* n *derivates be finite and continuous for all values of* x *from* a_0 *to* a_n, *and the quantities*

$$\begin{array}{ccccccccc} a_0 & a_1 & a_2 & \dots & a_{n-2} & & a_{n-1} & & a_n \\ & b_0 & b_1 & \dots & & b_{n-2} & & b_{n-1} & \\ & & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} & & \\ & & & \dots & \dots & \dots & \dots & & \\ & & & & w_0 & w_1 & & & \\ & & & & & u & & & \end{array}$$

be such that the a's *are in ascending order of magnitude, and*

$$a_r < b_r < a_{r+1},$$
$$b_r < c_r < b_{r+1},$$
$$\dots\dots\dots$$
$$w_0 < u < w_1,$$

then

$$f^n(u) = \frac{\begin{vmatrix} 1 & a_0 & a_0^2 & \dots & a_0^{n-1} & f(a_0) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} & f(a_n) \end{vmatrix}}{\begin{vmatrix} 1 & a_0 & a_0^2 & \dots & a_0^{n-1} & a_0^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^{n-1} & a_n^n \end{vmatrix}}.$$

It is very simply proved by noting that the function

$$\begin{vmatrix} 1 & x & x^2 & \dots & x^n & f(x) \\ 1 & a_0 & a_0^2 & \dots & a_0^n & f(a_0) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_n & a_n^2 & \dots & a_n^n & f(a_n) \end{vmatrix}$$

vanishes when x takes any one of the values $a_0, \dots, a_n$, and therefore its first derivate must vanish when x takes any one of the

values $b_0, \ldots, b_{n-1}$, its second derivate when x takes any one of the values $c_0, \ldots, c_{n-2}$, and so on, the final deduction being that the n^{th} derivate, namely,

$$\begin{vmatrix} . & . & . & \ldots & . & n! & f^n(x) \\ 1 & a_0 & {a_0}^2 & \ldots & {a_0}^{n-1} & {a_0}^n & f(a_0) \\ . & . & . & . & . & . & . \\ 1 & a_n & {a_n}^2 & \ldots & {a_n}^{n-1} & {a_n}^n & f(a_n) \end{vmatrix}$$

must vanish when x is put equal to u,—a result substantially the same as that desired.

In similar fashion it is shown that

$$\begin{vmatrix} f(x+h) & 1 & h & \frac{1}{2!}h^2 & \ldots. & \frac{1}{n!}h^n & \frac{1}{(n+1)!}h^{n+1} \\ f(x) & 1 & . & . & \ldots. & . & . \\ f'(x) & . & 1 & . & \ldots. & . & . \\ . & . & . & . & . & . & . \\ f^{(n)}(x) & . & . & . & \ldots. & 1 & . \\ f^{n+1}(x+\theta h) & . & . & . & \ldots. & . & 1 \end{vmatrix} = 0,$$

which is the familiar expansion of $f(x+h)$ in ascending powers of h. This in the next place is widely generalized; and in the following papers still further advances are made, the operation of Finite Differences, for one thing, taking the place of differentiation. It is not surprising therefore that the author finds himself ultimately brought into contact with Wronski and the 'loi suprême.'

SEGAR, H. W. (1892.)

[The deduction of certain determinants from others of indeterminate form. *Messenger of Math.*, xxii. pp. 57–67.]

In the examples dealt with here the indeterminate form is 0/0, and arises from fractions whose numerator and denominator are each a function of $x_1, x_2, \ldots, x_n$ and contain $\xi^{\frac{1}{2}}(x_1, x_2, \ldots, x_n)$ as a factor. The fundamental proposition used is that if there be no other disturbing factor in numerator and denominator, the limiting value when the suffixed x's are each made equal to x is got by first performing on both the operation

$$D_1^{n-1} D_2^{n-2} \ldots D_{n-1}^1,$$

where D_s^r denotes r successive differentiations with respect to x_s. After the three equalities

$$D_1^{n-1}D_2^{n-2}\ldots D_{n-1}^1\xi^{\frac{1}{2}}(x_1, x_2, \ldots, x_n) = (-1)^{\frac{1}{2}n(n-1)}\cdot(n-1)!!,$$

$$D_1^1\ D_2^2\ \ldots D_{n-1}^{n-1}\xi^{\frac{1}{2}}(x_1, x_2, \ldots, x_n) = (-1)^{n-1}\cdot(n-1)!!,$$

$$D_2^1\ D_3^2\ \ldots D_n^{n-1}\xi^{\frac{1}{2}}(x_1, x_2, \ldots, x_n) = (n-1)!!$$

are noted, the result of the first application of the proposition is the rediscovery of

$$\mathrm{L}\frac{|x_1^{\alpha}x_2^{\beta}x_3^{\gamma}\ldots|}{|x_1^{0}x_2^{1}x_3^{2}\ldots|} = \frac{\xi^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots)}{(n-1)!!}$$

as obtained by Johnson in 1885. Application is then made to Segar's own theorem regarding $|(\ )^{\alpha+\alpha'}(\ )^{\beta+\beta'}\ldots|_n$, the result being the equality

$$\begin{vmatrix} (\alpha+\alpha'+n-1)_{\alpha+\alpha'} & (\alpha+\beta'+n-1)_{\alpha+\beta'} & \ldots \\ (\beta+\alpha'+n-1)_{\beta+\alpha'} & (\beta+\beta'+n-1)_{\beta+\beta'} & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

$$= (-1)^{\frac{1}{2}n(n-1)}\cdot\frac{\xi^{\frac{1}{2}}(\alpha, \beta, \ldots)\cdot\xi^{\frac{1}{2}}(\alpha', \beta', \ldots)}{\{(n-1)!!\}^2}.$$

From the manifest identity

$$\begin{vmatrix} 1 & \frac{1}{f(x_1)} & \frac{x_1}{f(x_1)} & \ldots & \frac{x_1^{n-2}}{f(x_1)} \\ 1 & \frac{1}{f(x_2)} & \frac{x_2}{f(x_2)} & \ldots & \frac{x_2^{n-2}}{f(x_2)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f(x_1) & 1 & x_1 & \ldots & x_1^{n-2} \\ f(x_2) & 1 & x_2 & \ldots & x_2^{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix} = \frac{1}{f(x_1)}\cdot\frac{1}{f(x_2)}\cdot\ldots\cdot\frac{1}{f(x_n)}$$

there are obtained

$$\frac{\begin{vmatrix} D^{\circ}1 & D^{\circ}\frac{1}{f(x)} & D^{\circ}\frac{x}{f(x)} & \ldots & D^{\circ}\frac{x^{n-2}}{f(x)} \\ D'1 & D'\frac{1}{f(x)} & D'\frac{x}{f(x)} & \ldots & D'\frac{x^{n-2}}{f(x)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}}{\begin{vmatrix} f(x) & 1 & x & \ldots & x^{n-2} \\ f'(x) & . & 1 & \ldots & (n-2)x^{n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}} = \frac{1}{\{f(x)\}^n},$$

and

$$\left| \mathrm{D}'\frac{1}{f(x)} \quad \mathrm{D}^2\frac{x}{f(x)} \quad \ldots\ldots \quad \mathrm{D}^n\frac{x^{n-1}}{f(x)} \right| = (-1)^n \cdot (n-1)!! \, \frac{f^{(n)}(x)}{\{f(x)\}^{n+1}};$$

and from the equally simple equality

$$\frac{\left| \phi_1{}^1 \quad \phi_2{}^2 \quad \phi_3{}^3 \quad \ldots \quad \phi_n{}^n \right|}{\left| x_1{}^0 \quad x_2{}^1 \quad x_3{}^2 \quad \ldots \quad x_n{}^{n-1} \right|} = \phi_1\phi_2 \ldots \phi_n \cdot \frac{\xi^{\frac{1}{2}}(\phi_1, \phi_2, \ldots, \phi_n)}{\xi^{\frac{1}{2}}(x_1, x_2, \ldots, x_n)}$$

are got

$$\left| \mathrm{D}^\circ\phi \cdot \mathrm{D}^1\phi^2 \cdot \mathrm{D}^2\phi^3 \quad \ldots \quad \mathrm{D}^{n-1}\phi^n \right| = (n-1)!! \, \phi^n \cdot (\phi')^{\frac{1}{2}n(n-1)},$$

$$\left| \mathrm{D}f\phi \cdot \mathrm{D}^1f\phi^2 \cdot \mathrm{D}^2f\phi^3 \quad \ldots \quad \mathrm{D}^{n-1}f\phi^n \right| = (n-1)!! \, f^n\phi^n \cdot (\phi')^{\frac{1}{2}n(n-1)},$$

$$\left| \mathrm{D}'\phi \cdot \mathrm{D}^2\phi^2 \cdot \mathrm{D}^3\phi^3 \quad \ldots \quad \mathrm{D}^n\phi^n \right| = n!! \, (\phi')^{\frac{1}{2}n(n-1)}.$$

The last of these recalls to Segar a result of Wronski's (*Hist.*, i. p. 477), which he thereupon generalizes by applying his process to Jacobi's theorem regarding

$$\frac{\left| \phi_1{}^\alpha \quad \phi_2{}^\beta \quad \phi_3{}^\gamma \quad \ldots\ldots \right|_n}{\left| \phi_1{}^0 \quad \phi_2{}^1 \quad \phi_3{}^2 \quad \ldots\ldots \right|_n},$$

the result obtained being

$$\mathrm{D}'\phi^\alpha \cdot \mathrm{D}^2\phi^\beta \cdot \mathrm{D}^3\phi^\gamma \ldots \Big|_n = \alpha\beta\gamma \ldots \xi^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots) \cdot \left(\frac{\phi'}{\phi}\right)^{\frac{1}{2}n(n+1)} \cdot \phi^{\alpha+\beta+\gamma+\ldots}$$

Lastly, the process is applied to the supposed general equality

$$\frac{\left| \mathrm{F}_1(x_1) \cdot \mathrm{F}_2(x_2) \quad \ldots \quad \mathrm{F}_n(x_n) \right|}{\xi^{\frac{1}{2}}(x_1, x_2, \ldots, x_n)} = \Theta(x_1, x_2, \ldots, x_n),$$

and the outcome generalized by another process, with the result that there is found

$$\mathrm{L}\{\mathrm{D}_1{}^\alpha \mathrm{D}_2{}^\beta \mathrm{D}_3{}^\gamma \ldots\ldots (\Theta\xi^{\frac{1}{2}})\} = (n-1)!! \, \mathrm{L}\Delta\Theta,$$

where Δ is the operational determinant

$$\begin{vmatrix} \mathrm{D}_1{}^\alpha & \alpha\mathrm{D}_1{}^{\alpha-1} & (\alpha)_2\mathrm{D}_1{}^{\alpha-2} & \ldots \\ \mathrm{D}_2{}^\beta & \beta\mathrm{D}_2{}^{\beta-1} & (\beta)_2\mathrm{D}_2{}^{\beta-2} & \ldots \\ \mathrm{D}_3{}^\gamma & \gamma\mathrm{D}_3{}^{\gamma-1} & (\gamma)_2\mathrm{D}_3{}^{\gamma-2} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}.$$

As examples of this are given the values of three persymmetric determinants with arithmetical elements. A comparison of the two processes is also made, with interesting consequences.

VAUTRÉ, . (1893).

[Question 11858. *Educ. Times*, xlvi. p. 157 : or *Math. from Educ. Times*, lx. p. 108.]

A fresh proof of the already well-known result connecting the quotient

$$|\, a_1{}^0 a_2{}^1 \ldots a_{n-1}^{n-2}\ a_n^{n-1+h} \,| \div |\, a_1{}^0 a_2{}^1 \ldots a_{n-1}^{n-2}\ a_n^{n-1} \,|$$

with the h^{th} aleph function of the a's.

CAYLEY, A. (1893) : SEGAR, H. W. (1893).

[On the development of $(1+n^2x)^{\frac{m}{n}}$. *Messenger of Math.*, xxii. pp. 186–190 : or *Collected Math. Papers*, xiii. pp. 354–357.]

[Proof of a theorem in the theory of numbers. *Messenger of Math.*, xxiii. pp. 31–36.]

These both concern the theorem, spoken of by Cayley as Segar's, but implicitly expressed by Mitchell in 1882, that $\xi^{\frac{1}{2}}(p, q, r, \ldots)$ is divisible by $\xi^{\frac{1}{2}}(0, 1, 2, \ldots)$ when $p, q, r, \ldots$ are integers. Cayley's proof is essentially the same as that given above in a footnote to Johnson's third paper of 1885 : Segar's is non-determinant.

ECHOLS, W. H. (1893).

[Proof of a formula due to Cauchy. *Annals of Math.*, viii. p. 21.]

[On some forms of Lagrange's interpolation-formula. *Annals of Math.*, viii. pp. 22–24.]

In the first paper here the manifest vanishing of the function

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \ldots. & 1 \\ a^p x & 1 & a & a^2 & \ldots. & a^{n-1} \\ (a^p x)^2 & 1 & a^2 & a^4 & \ldots. & (a^{n-1})^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ (a^p x)^n & 1 & a^n & a^{2n} & \ldots. & (a^{n-1})^n \end{vmatrix} \div \xi^{\frac{1}{2}}(a, a^2, \ldots, a^n),$$

when $x = a^{-p}, a^{-p+1}, \ldots, a^{-p+n-1}$ is used to obtain for the function an n-factor product to be equated to the alternative expression got by expanding the determinant according to the elements of the first column and their complementaries.

In the second paper the kind of fresh forms brought forward will be understood from knowing that in the case of the fourth order

$$\begin{vmatrix} f(x) & (x-a_2)(x-a_3)(x-a_4) & (x-a_3)(x-a_4) & x-a_4 \\ f(a_1) & (a_1-a_2)(a_1-a_3)(a_1-a_4) & (a_1-a_3)(a_1-a_4) & a_1-a_4 \\ f(a_2) & . & (a_2-a_3)(a_2-a_4) & a_2-a_4 \\ f(a_3) & . & . & a_3-a_4 \end{vmatrix} \div \xi^{\frac{1}{2}}(a_1a_2a_3a_4)$$

takes the place of the known form

$$\begin{vmatrix} f(x) & 1 & x & x^2 \\ f(a_1) & 1 & a_1 & a_1^2 \\ f(a_2) & 1 & a_2 & a_2^2 \\ f(a_3) & 1 & a_3 & a_3^2 \end{vmatrix} \div \xi^{\frac{1}{2}}(a_1, a_2, a_3).$$

LERCH, M. (1893).

[Krátký důkaz Borchardtovy věty determinantní. *Časopis pro pěstování math. a fys.*, xxiii. pp. 76–78.]

Still another examination of the equality of 1855 (*Hist.*, ii. p. 174)

$$(t_1-x_1)^{-2} \dots (t_n-x_n)^{-2}\,| = |\,(t_1-x_1)^{-1} \dots (t_n-x_n)^{-1}\,|\cdot|\,(t_1-x_1)^{-1} \dots (t_n-x_n)^{-1}\,|$$

touched on above by Muir (p. 151) and Řehořovský (p. 157).

SCHLEGEL, V. (1894): NEUBERG, J. (1894): STUDNIČKA, F. J. (1896, 1896).

[Théorèmes relatifs aux déterminants. *El Progreso Mat.*, iv. pp. 176–177, 219.]

[Théorème sur les déterminants. *Mathesis*, (2) iv. p. 251.]

[Neuer Beitrag zur Theorie der Determinanten. *Sitzungsb. . . . Akad. d. Wiss.* (Prag), Jahrg. 1896, No. 6, 5 pp.]

[Přispěvek k nauce o determinantech. *Časopis pro pěstování math. a fyz.*, xxv. pp. 241–243.]

The first so-called theorem is that *if the elements of each of three rows of a determinant form an arithmetical series of order* 1, *the determinant must vanish*; and he extends it to determinants of higher order, for example,

$$|\,\alpha^2(\beta+1)^2(\gamma+2)^2(\delta+3)^2\,| \equiv 0.$$

Neuberg points out that of course

$$|\phi_1(x_1)\,\phi_2(x_2)\ldots\phi_n(x_n)| \equiv 0$$

so long as the ϕ's are integral functions of degree not higher than the $(n-2)^{\text{th}}$, the determinant being viewable as a function that vanishes for $n-1$ values of the independent variable, x_1 say.

Studnička's contribution is of less interest.

JONQUIERES, E. DE (1895).

[Sur les dépendances mutuelles des déterminants potentiels. *Comptes Rendus . . . Acad. des Sci.* (Paris), cxx. pp. 408–410, 580.]

[Démonstration d'un théorème sur les nombres entiers. *Comptes Rendus . . . Acad. des Sci.* (Paris), cxx. pp. 534, 537.]

The results here are essentially rediscoveries, the chief one being that proved in the second paper, namely, the divisibility of

$$|a_1{}^1 a_2{}^2 \ldots a_n{}^n| \quad \text{by} \quad |1^1 2^2 \ldots n^n|$$

when $a_1, a_2, \ldots$ are integers.

DELLAC, H. (1895).

[Note sur l'identité des polynomes. *Journ. de Math. spéc.* Année xix. pp. 169–177.]

A determinant which comes to the surface here is of considerable interest. It is a function of $2n$ variables, which may conveniently be looked on as the elements of a 2-by-n array, and it is of the order $\frac{1}{2}n(n+1)$. If the array be

$$\begin{array}{cccc} a_1, & a_2, & \ldots., & a_n, \\ b_1, & b_2, & \ldots., & b_n, \end{array}$$

the first row of the determinant is

$$a_1{}^{n-1},\ a_1{}^{n-2}b_1,\ \ldots,\ b_1{}^{n-1},\ a_1{}^{n-2},\ \ldots,\ b_1{}^{n-1},\ \ldots.,\ a_1, b_1, 1,$$

the second row is similarly formed from a_1, b_2, the third from a_1, b_3, and so on; then come $n-1$ rows similarly formed from

$$a_2 \text{ and } b_1,\ b_2,\ \ldots,\ b_{n-2},\ b_{n-1},$$

and following them $n-2$ rows formed from

$$a_3 \text{ and } b_1,\ b_2,\ \ldots,\ b_{n-2};$$

and so on, the two variables in the last row being thus a_n, b_1. For the cases where n is 3 and 4, evaluations are lengthily made, and it is thence concluded that the general result is

$$\xi^{\frac{1}{2}}(a_1, \ldots, a_n)\cdot\xi^{\frac{1}{2}}(a_1, \ldots, a_{n-1}) \ldots \xi^{\frac{1}{2}}(a_1, a_2)$$
$$\cdot\xi^{\frac{1}{2}}(b_1, \ldots, b_n)\cdot\xi^{\frac{1}{2}}(b_1, \ldots, b_{n-1}) \ldots \xi^{\frac{1}{2}}(b_1, b_2),$$

the determinant thus being the product of $\frac{1}{3}(n^3-n)$ differences. For example, when n is 3, we have

$$\begin{vmatrix} a_1^2 & a_1b_1 & b_1^2 & a_1 & b_1 & 1 \\ a_1^2 & a_1b_2 & b_2^2 & a_1 & b_2 & 1 \\ a_1^2 & a_1b_3 & b_3^2 & a_1 & b_3 & 1 \\ a_2^2 & a_2b_1 & b_1^2 & a_2 & b_1 & 1 \\ a_2^2 & a_2b_2 & b_2^2 & a_2 & b_2 & 1 \\ a_3^2 & a_3b_1 & b_1^2 & a_3 & b_1 & 1 \end{vmatrix} = (a_2-a_1)^2(a_3-a_2)(a_3-a_1)\cdot(b_2-b_1)^2(b_3-b_2)(b_3-b_1).$$

A good proof with attention to sign seems desirable, the determinant being worthy of further study.

LÉMERAY, E. M. (1896).

[Question 788. *L'Intermédiaire des Math.*, iii. pp. 58–59, 251–256, v. p. 151.]

This concerns the ratio of the complementary minors of the elements in the places $(k, 1)$, $(n, 1)$ in the case of each of the determinants

$$|\,1^0 2^1 3^2 \ldots n^{n-1}\,|, \quad |\,1^0 2^2 3^4 \ldots n^{2(n-1)}\,|$$

the value in the one case being $(n)_k$ and in the other $(2n)_{n-k}$. The starting point of the proof in the former case is the change of the first of the two minors into

$$\frac{n!}{r}\xi^{\frac{1}{2}}(1, 2, \ldots, r-1, r+1, \ldots, n);$$

and in the other case the procedure is similar.

JACOBI, C. G. J.: STÄCKEL, P. (1896).

[Ueber die alternierenden Functionen, ... *Ostwald's Klassiker .... Nr.* 77, pp. 50–65.]

This is the *De Functionibus Alternantibus* of 1841 in German. The editor appends a page of notes (pp. 72–73).

STUDNIČKA, F. J. (1896, 1897).

[Ueber Potenzdeterminanten und deren wichtigste Eigenschaften. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), No. 22, 8 pp.]

[Beitrag zur Theorie der Potenz- und Combinations-Determinanten. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), No. 1, 20 pp.]

[O základnich vlastnostech determinantů mocninných a jich upotřebení v theorii rovnic algebraických. *Časopis pro pěstováni math. a fys.*, xxvi. pp. 105–120.]

We have here the first three of eight papers written unfortunately under the belief that the subject had not been previously investigated. The first two are occupied with the gradual establishment of the theorem that the cofactor of $|a^0b^1c^2\ldots|$ in $|a^\alpha b^\beta c^\gamma\ldots|$ is expressible as a determinant whose elements are simple symmetric functions of $a, b, c, \ldots$. The third, if we leave aside the so-called application, is a re-exposition of the same theorem.

Studnička's oversight is the more noticeable because the theorem in the twenty-five years since its first publication by Nägelsbach had not only been more than once rediscovered, but had been generalized by Garbieri and actually in 1882 included in a text-book * to which Studnička himself for another purpose had occasion to refer. It is only fair to say, however, that others had made the same oversight, a school-programme published in a small provincial town being easily overlooked.

LONGCHAMPS, G. DE (1896).

[Exercices sur les déterminants. *Journ. de Math. Élém.*, xx. pp. 170–173, 198–201, 223–225, 244–245.]

Quite elementary, but not without interest.

LAISANT, C. A. (1896).

[Propriétés des coefficients du binôme. *Bulletin Soc. Math. de France*, xxiv. pp. 197–199.]

Laisant's result is such as would readily be suggested by the first of Léméray's of the same year, the minors now considered being those of the array got by deleting the last row of $|1^0 2^1 3^2 \ldots n^{n-1}|$

* Muir's, pp. 175-176.

instead of the first row. His mode of statement is: *If* Δ_r *denote the determinant of the array got by deleting the* rth *column of*

$$\begin{array}{ccccc} 1^0 & 2^0 & 3^0 & \ldots. & n^0 \\ 1^1 & 2^1 & 3^1 & \ldots. & n^1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1^{n-2} & 2^{n-2} & 3^{n-2} & \ldots. & n^{n-2} \end{array}$$

then

$$\frac{\Delta_1}{1} = \frac{\Delta_2}{(n-1)_1} = \frac{\Delta_3}{(n-1)_2} = \ldots = \frac{\Delta_n}{(n-1)_{n-1}}.$$

He reaches it by the solution of a set of $n-1$ quasi-equations, namely, the identities obtained by substituting -1 for x in the identity

$$(1+x)^{n-1} = 1 + (n-1)_1 x + (n-1)_2 x^2 + \ldots$$

and in the identities got therefrom by performing $n-2$ times in succession the operation $\frac{\partial}{\partial x} x$. The existence of a corresponding companion to Léméray's second theorem is not referred to.

We may note for ourselves (1) that the mode of proof used for Léméray's results is more readily applicable here; (2) that the said results may also be proved in Laisant's manner, the first, for example, by putting $x = -1$ in the equalities got by performing $x\frac{\partial}{\partial x}$ $n-1$ times in succession on $(1+x)^n = 1+(n)_1 x+\ldots$; (3) that *if the elements of the given array have for bases* 1, 2, 3, . . . , n *and for exponents* p, p+1, p+2, . . . , p+n−2, *then*

$$\frac{\Delta_r}{\Delta_n} = \left(\frac{n}{r}\right)^p \cdot (n-1)_{r-1};$$

(4) that *if on the other hand the bases be* $1^2, 2^2, \ldots, n^2$, *then*

$$\frac{\Delta_r}{\Delta_n} = \left(\frac{n}{r}\right)^{2p-2} (2n)_{n-r};$$

(5) that a fundamental fact in all the cases is *the possible expression of a binomial-coefficient as the quotient of two difference-products, namely,*

$$(n)_r = \frac{\zeta^{\frac{1}{2}}(1, 2, \ldots, r, r+2, \ldots, n+1)}{\zeta^{\frac{1}{2}}(1, 2, \ldots, n)}.$$

STUDNIČKA, F. J. (1897).

[Neuer Beitrag zur Theorie der Potenz- und Kombinations-Determinant. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), No. 16, 16 pp.]

The contents of this are a proof that

$$|\, a^0 b^1 c^2 \ldots |\; = \xi^{\frac{1}{2}}(a, b, c, \ldots),$$

a rediscovery of the divisibility of $|\, a^1 b^2 c^3 \ldots |$ by $|\, 1^0 2^1 3^2 \ldots |$ when $a, b, c, \ldots$ are positive integers, and the evaluation of a special persymmetric recurrent. The last result, namely,

$$\begin{vmatrix} m & 1 & . & . & \ldots \\ (m+1)_2 & m & 1 & . & \ldots \\ (m+2)_3 & (m+1)_2 & m & 1 & \ldots \\ (m+3)_4 & (m+2)_3 & (m+1)_2 & m & \ldots \\ . & . & . & . & . \end{vmatrix}_n = (m)_n$$

is interesting for comparison with Trudi's (*Hist.*, iii. p. 219). Like Trudi's it is obtainable from Zeipel's paper of 1865, whence also we might obtain Weihrauch's of 1874 (*Hist.*, iii. p. 231). In an appendix two trigonometrical identities are deduced by considering

$$\xi^{\frac{1}{2}}(\sin\alpha, \sin\beta, \sin\gamma) \quad \text{and} \quad \xi^{\frac{1}{2}}(\tan\alpha, \tan\beta, \tan\gamma).$$

The same matters are dealt with by Studnička a few months later in two papers written in Czech (*Věstník České Akad.*, vi. pp. 369–376, vii. pp. 165–167).

RUSSELL, J. W. (1897).

[Certain concomitant determinants. *Proceed. London Math. Soc.*, xxviii. pp. 430–439.]

The subject here is the invariancy of certain differential operators, namely, in the case of binary quantics, the operators

$$\begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial y_2} \end{vmatrix}, \quad \begin{vmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial y_1} & \frac{\partial^2}{\partial y_1^2} \\ \frac{\partial^2}{\partial x_2^2} & \frac{\partial^2}{\partial x_2 \partial y_2} & \frac{\partial^2}{\partial y_2^2} \\ \frac{\partial^2}{\partial x_3^2} & \frac{\partial^2}{\partial x_3 \partial y_3} & \frac{\partial^2}{\partial y_3^2} \end{vmatrix}, \ldots .$$

It belongs strictly to the algebra of quantics, but the determinants used have a more than superficial association with alternants. We give the first general theorem because there attaches to it the additional interest that a simple deduction from it is Schläfli's theorem of 1851 in regard to the $(C_{n-r+1,\ n})^{\text{th}}$ power of an n-line determinant. It is that *the determinant whose successive rows are made up of the several terms in the expansions*

$$\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\ldots+\frac{\partial}{\partial x_n}\right)^n u_1,$$

$$\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\ldots+\frac{\partial}{\partial x_n}\right)^n u_2,$$

$$\ldots\ldots\ldots$$

is a covariant of the q*-ary quantics* $u_1, u_2, \ldots, u_r$, *where* r *is the number of terms in each of these expansions.*

LORIA, G. (1897).

[Sopra certi determinanti i cui elementi sono funzioni trigonometriche. *Periodico di Mat.*, xii. pp. 33–34.]

[Sopra una classe notevoli di alternanti d'ordine qualsivoglia. *Věstnik Král. České Společnosti Náuk*, No. 57, 13 pp.; or *Periodico di Mat.*, xiii. pp. 129–138.]

The interest of these papers lies in the fact that they give a fresh mode of procedure for the evaluation of trigonometrical alternants, and especially of those in which the elements of the leading row are merely sines or cosines of multiples of the corresponding angle. The fundamental step in the procedure consists in expressing every function of each angle in terms of the tangent of half the angle, the means of doing this being furnished by the equality

$$\cos m\alpha+\sqrt{-1}\sin m\alpha = \frac{(1+A\sqrt{-1})^{2m}}{(1+A^2)^m}, \text{ where } A \equiv \tan\frac{1}{2}\alpha;$$

for example,

$$\begin{vmatrix} 1 & \sin\alpha & \cos 2\alpha \\ \ldots & \ldots & \ldots \end{vmatrix} \text{ is changed into } \begin{vmatrix} 1 & \frac{2A}{1+A^2} & \frac{1-6A^2+A^4}{(1+A^2)^2} \\ \ldots & \ldots & \ldots \end{vmatrix}.$$

The new form is then evaluated like an ordinary algebraical

alternant; for example, the factor $1/(1+A^2)^2(1+B^2)^2(1+C^2)^2$ is removed, and the cofactor

$$\begin{vmatrix} 1+2A^2+A^4 & 2A+2A^3 & 1-6A^2+A^4 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$$

partitioned into simpler alternants, so as to facilitate the removal of another factor, $\xi^{\frac{1}{2}}(A, B, C)$. This being done, all that remains is the work of resubstitution and simplification; for example,

$$1/(1+A^2)^2(1+B^2)^2(1+C^2)^2 = (\cos^2\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\beta\cos^2\tfrac{1}{2}\gamma)^2,$$

$$\xi^{\frac{1}{2}}(A, B, C) = (\tan\tfrac{1}{2}\gamma-\tan\tfrac{1}{2}\beta)(\tan\tfrac{1}{2}\gamma-\tan\tfrac{1}{2}\alpha)(\tan\tfrac{1}{2}\beta-\tan\tfrac{1}{2}\alpha),$$

$$= \frac{\sin\tfrac{1}{2}(\gamma-\beta)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\beta-\alpha)}{\cos^2\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\beta\cos^2\tfrac{1}{2}\gamma}$$

and the third factor—the only one that may require special treatment—

$$= 16\{-1+ABC(A+B+C)-(AB+BC+CA)+A^2B^2C^2\}$$

$$= 16(AB-1)(BC-1)(CA-1)$$

$$= (-1)^3 16\frac{\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha)}{\cos^2\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\beta\cos^2\tfrac{1}{2}\gamma};$$

so that

$$\begin{vmatrix} 1 & \sin\alpha & \cos 2\alpha \\ 1 & \sin\beta & \cos 2\beta \\ 1 & \sin\gamma & \cos 2\gamma \end{vmatrix} = -16\,\Pi\cdot\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha).$$

Other evaluations are:

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos 2\alpha \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = -4\Pi\cdot\Sigma\cos(\alpha+\beta),$$

$$\begin{vmatrix} 1 & \cos\alpha & \sin 2\alpha \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = 4\Pi\cdot(\Sigma\cos\alpha-\cos\Sigma\alpha),$$

$$\begin{vmatrix} 1 & \cos\alpha & \tan\alpha \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix} = 2\Pi\cdot\frac{1-\Sigma\cos(\alpha+\beta)}{\cos\alpha\cos\beta\cos\gamma};$$

but it is more important to note the general result, namely, that *when alternants of the type mentioned at the outset are divided by the alternating function* Π, *the quotient is a power of*

$$\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma\ldots.$$

multiplied by a symmetric function of $\tan\tfrac{1}{2}\alpha, \tan\tfrac{1}{2}\beta, \tan\tfrac{1}{2}\gamma, \ldots.$

STUDNIČKA, F. J. (1897).

[O determinantech mocninných a sestavných. *Král. Českе Společnosti Náuk* (Prag), 75 pp.]

The subject of this fifth paper is again the theorem dealt with in the first three. Now, however, the material is used with deductions and comments to form a little treatise or dissertation consisting of an Introduction and four Chapters, and extending to 75 pages. In the chapters the theorem is viewed in connection with alternants of the 2nd, 3rd, 4th, n^{th} orders respectively, and it consequently receives exceptionally full treatment. As a specimen deduction we may note from the first chapter the equality

$$\frac{|x^0y^n|}{|x^0y^1|} \equiv \frac{x^n-y^n}{x-y} = \begin{vmatrix} x+y & xy & . & . & \dots \\ 1 & x+y & xy & . & \dots \\ . & 1 & x+y & xy & \dots \\ . & . & . & . & . \end{vmatrix}_{n-1},$$

whence, on putting $\cos\alpha+i\sin\alpha$, $\cos\alpha-i\sin\alpha$ for x, y, there is obtained the known determinant for $\sin n\alpha/\sin\alpha$.

The memoir is published separately by the Bohemian Royal Society of Sciences, being the ninth memoir awarded the Society's Jubilee Prize for a work written in Czech.

BRAMBILLA, A. (1898).

[Intorno ad alcuni determinanti. *Periodico di Mat.*, xiii. pp. 169–175.]

The determinants here considered are those whose elements are of the form

$$x(x+1)\dots(x+m-1) \quad \text{or} \quad x^{[m]} \text{ say.}$$

The opening example is the alternant

$$\begin{vmatrix} 1 & a_1 & a_1^{[2]} & \dots & a_1^{[n-1]} \\ 1 & a_2 & a_2^{[2]} & \dots & a_2^{[n-1]} \\ . & . & . & . & . \end{vmatrix}_n$$

which is manifestly equal to

$$|a_1^0 \quad a_2^1 \quad a_3^2 \quad \dots \quad a_n^{n-1}|;$$

in other words, the simplest alternant is not altered by changing its every element from a_r^{s-1} into $a_r^{[s-1]}$. The next, also an alternant, is

$$\begin{vmatrix} a^{[m]} & (a+1)^{[m]} & (a+2)^{[m]} & \dots \\ b^{[m]} & (b+1)^{[m]} & (b+2)^{[m]} & \dots \\ c^{[m]} & (c+1)^{[m]} & (c+2)^{[m]} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

which is practically the same as to say that its $(r, s)^{\text{th}}$ element is $(a_r+s-1)^{[m]}$. By a process of repeated removal of factors it is found that

$$\begin{aligned} &|\, a_r+s-1)^{[m]}\,|_n \\ &= (-1)^{\frac{1}{2}n(n-1)} \\ &\quad \cdot \zeta^{\frac{1}{2}}(a_1, a_2, \dots, a_n) \\ &\quad \cdot m^{n-1}(m-1)^{n-2} \dots (m-n+2)^1 \\ &\quad \cdot (a_1+n-1)^{[m-n+1]}(a_2+n-1)^{[m-n+1]} \dots (a_n+n-1)^{[m-n+1]}; \end{aligned}$$

and this is the fundamental result of the paper. From it results already known regarding determinants whose elements are combinatorial numbers are obtained by using the fact that when x is a positive integer

$$x^{[m]} = m!\, C_{x+m-1,\, m};$$

but the procedure is not helpful.

We may add for ourselves that the corresponding evaluation of $|\,(a_n+s-1)^m|_n$ is not given, nor the fact brought out that when $n = m+1$ the two evaluations are identical.*

---

* Taking the case where m is 4, we have by the multiplication-theorem :

$$\begin{vmatrix} a^4 & (a+1)^4 & (a+2)^4 & (a+3)^4 & (a+4)^4 \\ b^4 & (b+1)^4 & (b+2)^4 & (b+3)^4 & (b+4)^4 \\ \dots & \dots & \dots & \dots & \dots \\ e^4 & (e+1)^4 & (e+2)^4 & (e+3)^4 & (e+4)^4 \end{vmatrix}$$

$$= \begin{vmatrix} a^4 & a^3 & a^2 & a & 1 \\ b^4 & b^3 & b^2 & b & 1 \\ \dots & \dots & \dots & \dots & \dots \\ e^4 & e^3 & e^2 & e & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & . & . & . & . \\ 1 & 4\cdot 1 & 6\cdot 1^2 & 4\cdot 1^3 & 1^4 \\ 1 & 4\cdot 2 & 6\cdot 2^2 & 4\cdot 2^3 & 2^4 \\ 1 & 4\cdot 3 & 6\cdot 3^2 & 4\cdot 3^3 & 3^4 \\ 1 & 4\cdot 4 & 6\cdot 4^2 & 4\cdot 4^3 & 4^4 \end{vmatrix}$$

$$= \zeta^{\frac{1}{2}}(a, b, c, d, e) \times 4^4 \cdot 3^3 \cdot 2^2 \cdot 1^1;$$

and this is exactly what is found above for the other determinant in which the index of the elements is not 4 but [4].

STUDNIČKA, F. J. (1898).

[Příspěvek k nauce o determinantech mocninných. *Věstník České Akad.* . . . , vii. pp. 350–358.]

The main results here are Stern's theorem of 1865 and Naegelsbach's of 1871 (*Hist.*, iii. pp. 138, 144–146), the latter being in particular applied to finding the product of the binomial sums of $a, b, c, d, \ldots$, that is to say, the quotient of $|\, a^0b^2c^4d^6 \ldots |_n$ by $|\, a^0b^1c^2d^3 \ldots |_n$.

BEKE, E. (1898).

[Ueber die Fundamentalgleichungen der homogenen linearen Differentialgleichungen. *Math. és Phys. Lapok*, vii. pp. 115–123 ; or *Math. u. naturw. Berichte aus Ungarn*, xv. pp. 273–281.]

As an auxiliary there is here established a result essentially the same as Schendel's (1891), but the factorization given is in more complicated form. No preceding writing on the subject is referred to save a problem by Rados (*Math. és Phys. Lapok*, i. p. 367).

STUDNIČKA, F. J. (1898, 1899).

(See under this heading in Chap. I.)

ROE, E. D. (1898, 1899).

[Note on a formula of symmetric functions. *American Math. Monthly*, v. pp. 161–164.]

[On symmetric functions. *American Math. Monthly*, vi. pp. 1–6, 25–30, 53–58, 103–107, 129–135, 161–165.]

The first paper here is devoted to establishing the accuracy of Brioschi's determinant of 1854 as a representation of Waring's expression for a symmetric function. The service rendered by it was seriously called for in view of the action of Bellavitis and Bruno. It quite forestalls Muir's paper of ten years later on the same subject (*Hist.*, iii. pp. 473–474).

Near the outset of the second paper proof is given of the theorem regarding the multiplication of $|\, a^0b^1c^2 \ldots |$ by a symmetric function of $a, b, c, \ldots$ (*Hist.*, iv. pp. 150-151). It occupies five pages.

LERCH, M. (1899).

[O některých vzorcích z theorie determinantů. *Rozpravy České Akad.*, viii., no. 12, 16 pp.]

The determinants here dealt with are almost entirely alternants, the memoir being merely a careful reconsideration of four well-known important theorems (*Hist.*, i. pp. 339–342, 343–345; ii. pp. 173–175; iii. pp. 144–146).

The last two pages concern the known expansion for the product of a general determinant by one of its minors (*Hist.*, iii. pp. 79–80).

MUIR, T. (1899).

[The multiplication of an alternant by a symmetric function of the variables. *Proceed. R. Soc. Edinburgh*, xxii. pp. 539–542.]

The theorem here established is that *If the alternant*

$$|\,a^0b^1c^2\ldots k^{n-2}l^{n-1}\,|$$

be multiplied by any symmetric function of a, b, c, . . . *of the* t^{th} *degree,* t *being not greater than* n, *one term of the product is got from the multiplicand by increasing each of its last* t *indices by* 1, *and prefixing as coefficient the same symmetric function of the roots of the equation*

$$x^n - x^{n-1} + x^{n-2} - \ldots + (-1)^n = 0.$$

The fundamental step in the proof consists in substituting for the symmetric function its equivalent in terms of Σa, Σab, Σabc, ... Thus in the example used for illustration,

$$|\,a^0b^1c^2d^3\,|\cdot\Sigma a^3b,$$

there is substituted for Σa^3b the expression

$$(\Sigma a)^2\Sigma ab - 2(\Sigma ab)^2 - \Sigma a\Sigma abc + 4\Sigma abcd,$$

the product obtained being

$$2|a^1b^2c^3d^4| - |a^0b^2c^3d^5| - |a^0b^1c^4d^5| + |a^0b^1c^3d^6|,$$

as is otherwise known.

The theorem has the additional interest of being the basis of Sylvester's previously unexplained theorem regarding "zeta-ic" multiplication (*Hist.*, i. pp. 233–235, 323–325).

LEWICKY, W. (1899).

[Einige Bemerkungen zur Lagrange'schen Interpolationsformel. *Archiv d. Math. u. Phys.*, (2) xvii. pp. 214–224 ; or, in Polish, *Wiadomosci mat.*, iii. pp. 264–274.]

This discussion of Lagrange's formula is only of interest to us in that the mode of stating the formula is determinantal, namely,

$$\begin{vmatrix} y & 1 & x & x^2 & \ldots. & x^{n-1} \\ y_1 & 1 & x_1 & x_1{}^2 & \ldots. & x_1{}^{n-1} \\ y_2 & 1 & x_2 & x_2{}^2 & \ldots. & x_2{}^{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ y_n & 1 & x_n & x_n{}^2 & \ldots. & x_n{}^{n-1} \end{vmatrix} = 0.$$

(See *Hist.*, ii. p. 155, and iv. chap. xxii. under Echols 1893.)

MUIR, T. (1899).

[On a development of a determinant of the mn^{th} order. *Transac. R. Soc. Edinburgh*, xxxix. pp. 623–628.]

When the determinant is a difference-product, the development takes a specially interesting form ; for example,

$$|\, a^0b^1c^2d^3e^4f^5 \,| = \sum \left\{ |\, a^0b^1c^2 \,|\, |\, d^0e^1f^2 \,| \cdot \left| (abc)^0(def)^3 \right| \right\}$$

$$= \sum \left\{ (c-b)(c-a)(b-a)(f-e)(f-d)(e-d) \cdot (d^3e^3f^3 - a^3b^3c^3) \right\},$$

$$|\, a^0b^1c^2d^3e^4f^5g^6h^7i^8 \,| = \sum \left\{ |\, a^0b^1c^2 \,|\, |\, d^0e^1f^2 \,|\, |\, g^0h^1i^2 \,| \cdot \left| (abc)^0(def)^3(ghi)^6 \right| \right\},$$

where the expression under the last Σ is the product of twelve binomial factors, namely,

$$(c-b)(c-a)(b-a) \ldots. (g^3h^3i^3 - d^3e^3f^3)(g^3h^3i^3 - a^3b^3c^3)(d^3e^3f^3 - a^3b^3c^3).$$

JUNG, V. (1899).

[Příspěvek k theorii determinantů mocninných. *Rozpravy České Akad.* viii., no. 38, 22 pp.]

Like Lerch's, this clearly written but lengthy paper shows the influence of Studnička. The first nine pages are taken up with the rediscovery of Garbieri's theorem of 1878 (*Hist.*, iii. p. 163). A notice of Cauchy's double alternant leads to a similar detailed

investigation of a double alternant with squared elements, a specimen result being

$$\begin{vmatrix} (t_1-x_2)^2(t_1-x_3)^2 & (t_1-x_3)^2(t_1-x_1)^2 & (t_1-x_1)^2(t_1-x_2)^2 \\ (t_2-x_2)^2(t_2-x_3)^2 & (t_2-x_3)^2(t_2-x_1)^2 & (t_2-x_1)^2(t_2-x_2)^2 \\ (t_3-x_2)^2(t_3-x_3)^2 & (t_3-x_3)^2(t_3-x_1)^2 & (t_3-x_1)^2(t_3-x_2)^2 \end{vmatrix}$$

$$= \xi_1^{\frac{1}{2}}\xi_2^{\frac{1}{2}} \begin{vmatrix} \mathrm{K}_2(x) & -2\mathrm{K}_1(x) & 3 & . & . \\ -\mathrm{K}_3(x) & . & \mathrm{K}_1(x) & -2 & . \\ . & 2\mathrm{K}_3(x) & -\mathrm{K}_2(x) & . & 1 \\ -\mathrm{K}_3(t) & \mathrm{K}_2(t) & -\mathrm{K}_1(t) & 1 & . \\ . & -\mathrm{K}_3(t) & \mathrm{K}_2(t) & -\mathrm{K}_1(t) & 1 \end{vmatrix}$$

where

$$\mathrm{K}_1(x) \equiv x_1+x_2+x_3, \quad \mathrm{K}_2(t) \equiv t_1t_2+t_2t_3+t_3t_1, \quad \ldots .$$

and the $\xi^{\frac{1}{2}}$'s are the difference-products of the two sets of variables. Borchardt's double alternant (*Hist.*, ii. pp. 173–175) is also again adverted to (see above, p. 187).

JUNG, V. (1899).

[Poznámka o jistém determinatu mocninném. *Časopis pro pěstování math. a fys.*, xxix. pp. 41–42.]

This concerns numbers of the type Σa, Σab, Σabc, . . . , wherein $a, b, c, \ldots = 1, 2, 3, \ldots$; it is quite unimportant in its bearing on determinants.

BURNSIDE, W. S. AND PANTON, A. W. (1899).

[An Introduction to Determinants . . . 84 pp. Dublin.]

From this (p. 73) we may extract for reference

$$|\,1 \quad \sin\alpha \quad \cos(\beta-\gamma)\,| \;=\; 4\Pi\cdot\Sigma\cos\alpha,$$

$$|\,\sin\alpha \quad \sin 2\alpha \quad \sin 4\alpha\,| \;=\; 16\sin\alpha\sin\beta\sin\gamma\cdot\Sigma\cos\alpha\cdot|\,1\ \cos\alpha\ \cos^2\alpha\,|,$$

$$|\,1 \quad \sin^4\alpha \quad \cos^4\alpha\,| \;=\; 2\Pi'\cdot\sin(\alpha+\beta)\sin(\beta+\gamma)\sin(\gamma+\alpha),$$

$$|\,1 \quad \tan\alpha \quad \tan 2\alpha\,| \;=\; 2\Pi'\cdot\frac{\Sigma\tan\alpha+\tan\alpha\tan\beta\tan\gamma}{\cos 2\alpha\cos 2\beta\cos 2\gamma},$$

where $\Pi = \sin\tfrac{1}{2}(\gamma-\beta)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\beta-\alpha)$,

and $\Pi' = \sin(\gamma-\beta)\sin(\gamma-\alpha)\sin(\beta-\alpha)$.

In this connection it is desirable to remark that writers of text-books might well point out to learners how from a very few such evaluations many others are readily deducible, including those got by merely substituting for the angles their complementaries: for example, from

$$|1 \quad \sin\alpha \quad \cos\alpha| \quad \text{and} \quad |1 \quad \cos\alpha \quad \cos^2\alpha|$$

which are at once seen to be equal to

$$\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta), \quad i.e.\ 4\Pi,$$

and

$$(\cos\gamma-\cos\beta)(\cos\gamma-\cos\alpha)(\cos\beta-\cos\alpha), \quad i.e.\ 2\Pi\cdot(\Sigma\sin\alpha-\sin\Sigma\alpha),$$

almost all those of the third order hitherto noted by us, as well as many others, can be obtained. This is not unimportant, for otherwise recourse must be had to the tedious process of removing separately the sines of the differences of the angles, or to substituting exponential values for the angle-functions.

MÉRAY, C. (1899).

[Sur un déterminant dont celui de Vandermonde n'est qu'un cas particulier. *Revue de math. spéc.*, ix. pp. 217–219.]

This might have been described as giving merely an improved enunciation and treatment of Schendel's theorem of 1891, if it were not that the author is able to show that long before Schendel, and indeed before Schendel's earliest predecessors Franke (1876) and Besso (1882), he had occupied himself with the subject and made good progress. The early contribution to which he refers appeared in 1865 in vol. iv. of the *Annales sci. de l'École norm. sup.*, the strictly relevant passage commencing on page 175.* The improved enunciation is similar to that given by us in the note appended to our account of Schendel's paper.

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* The full title of the paper is: Extension aux équations simultanées des formules de Newton pour le calcul de puissances semblables des racines des équations entières. iv. pp. 159–193.

CHAPTER VII.

COMPOUND DETERMINANTS, FROM 1880 TO 1900.

UNDER this heading there are a half more writings than there were in the preceding period. About a fourth of them originate in Hungary—a curious fact in view of the different state of matters in other chapters.

HUNYADY, J. (1880).

[A másodfokú görbék és felületek meghatározásáról. *Értekezések a math. tud. Köréből*, vii. 18 ; 39 pp.]

This professedly geometrical paper is also particularly interesting from the point of view of determinants, by reason of the fact that in each case where the equation of a quadric satisfying given conditions is determined, it takes the form of a vanishing determinant, and that in each case also an alternative form is found by transformation, the consequence being that half a score of determinant identities is presented to us. For example, the determinant reached in solving the first problem being

$$\begin{vmatrix} x_0^2 & y_0^2 & z_0^2 & 2y_0z_0 & 2z_0x_0 & 2x_0y_0 \\ x_4^2 & y_4^2 & z_4^2 & 2y_4z_4 & 2z_4x_4 & 2x_4y_4 \\ x_5^2 & y_5^2 & z_5^2 & 2y_5z_5 & 2z_5x_5 & 2x_5y_5 \\ x_2x_3 & y_2y_3 & z_2z_3 & y_2z_3+y_3z_2 & z_2x_3+z_3x_2 & x_2y_3+x_3y_2 \\ x_3x_1 & y_3y_1 & z_3z_1 & y_3z_1+y_1z_3 & z_3x_1+z_1x_3 & x_3y_1+x_1y_3 \\ x_1x_2 & y_1y_2 & z_1z_2 & y_1z_2+y_2z_1 & z_1x_2+z_2x_1 & x_1y_2+x_2y_1 \end{vmatrix}$$

Hunyady multiplies it by $|x_1y_2z_3|^8$ in the form

$$\begin{vmatrix} X_1^2 & Y_1^2 & Z_1^2 & Y_1Z_1 & Z_1X_1 & X_1Y_1 \\ X_2^2 & Y_2^2 & Z_2^2 & Y_2Z_2 & Z_2X_2 & X_2Y_2 \\ X_3^2 & Y_3^2 & Z_3^2 & Y_3Z_3 & Z_3X_3 & X_3Y_3 \\ 2X_2X_3 & 2Y_2Y_3 & 2Z_2Z_3 & Y_2Z_3+Y_3Z_2 & Z_2X_3+Z_3X_2 & X_2Y_3+X_3Y_2 \\ 2X_3X_1 & 2Y_3Y_1 & 2Z_3Z_1 & Y_3Z_1+Y_1Z_3 & Z_3X_1+Z_1X_3 & X_3Y_1+X_1Y_3 \\ 2X_1X_2 & 2Y_1Y_2 & 2Z_1Z_2 & Y_1Z_2+Y_2Z_1 & Z_1X_2+Z_2X_1 & X_1Y_2+X_2Y_1 \end{vmatrix}$$

and obtains

$$\begin{vmatrix} |x_0y_2z_3|^2 & |x_1y_0z_3|^2 & |x_1y_2z_0|^2 & r_1R_4 & r_1R_5 & r_1R_6 \\ |x_4y_2z_3|^2 & |x_1y_4z_3|^2 & |x_1y_2z_4|^2 & r_2R_4 & r_2R_5 & r_2R_6 \\ |x_5y_2z_3|^2 & |x_1y_5z_3|^2 & |x_1y_2z_5|^2 & r_3R_4 & r_3R_5 & r_3R_6 \\ . & . & . & |x_1y_2z_3|^2 & . & . \\ . & . & . & . & |x_1y_2z_3|^2 & . \\ . & . & . & . & . & |x_1y_2z_3|^2 \end{vmatrix},$$

whence it follows that the original six-line determinant is equal to

$$\begin{vmatrix} |x_0y_2z_3|^2 & |x_1y_0z_3|^2 & |x_1y_2z_0|^2 \\ |x_4y_2z_3|^2 & |x_1y_4z_3|^2 & |x_1y_2z_4|^2 \\ |x_5y_2z_3|^2 & |x_1y_5z_3|^2 & |x_1y_2z_5|^2 \end{vmatrix} \div |x_1y_2z_3|^2.$$

This interesting result is seen to depend—and so it is with all the others—on equally interesting identities in row-multiplication : and although Hunyady does not refer to the said identities, it is well to take note of them. Denoting by r_kR_k the product of the h^{th} row of the multiplicand by the k^{th} row of the multiplier, we have

$$\begin{aligned} r_1R_1 &= (x_0X_1+y_0Y_1+z_0Z_1)^2 = |x_0y_1z_2|^2, \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ r_1R_4 &= 2(x_0X_2+y_0Y_2+z_0Z_2)(x_0X_3+y_0Y_3+z_0Z_3) \\ &= 2\,|x_1y_0z_3|\cdot|x_1y_2z_0| \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ r_4R_1 &= (x_2X_1+y_2Y_1+z_2Z_1)(x_3X_1+y_3Y_1+z_3Z_1) \\ &= |x_2y_2z_3|\cdot|x_3y_2z_3| = 0, \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ r_4R_4 &= (x_2X_2+y_2Y_2+z_2Z_2)(x_3X_3+y_3Y_3+z_3Z_3) \\ &\qquad + (x_3X_2+y_3Y_2+z_3Z_2)(x_2X_3+y_2Y_3+z_2Z_3) \\ &= |x_1y_2z_3|\cdot|x_1y_2z_3| + 0\cdot 0 = |x_1y_2z_3|^2. \\ &\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

The next four results are similar in character, the determinants to start with being all of the sixth order and having rows confined to one or other of two types, namely, the types

$$x_r x_s \quad y_r y_s \quad z_r z_s \quad y_r z_s + y_s z_r \quad z_r x_s + z_s x_r \quad x_r y_s + x_s y_r,$$

and

$$x_r^2 \quad y_r^2 \quad z_r^2 \quad 2y_r z_r \quad 2z_r x_r \quad 2x_r y_r.$$

The remaining half of the paper is concerned with corresponding results in solid geometry, the determinants in their first form being now of the tenth order, and the elements of the evolved compound determinants being of the fourth order.

HUNYADY, J. (1880).

[Tételek azon determinánsokról melyek elemei adjungált rendszerek elemeiböl vannak componalva. *Értekezések a math. tud. Köréből*, vii. 19 ; 27 pp.]

The first half of this paper (§§ 1–5, pp. 1–13) concerns determinants whose rows are taken partly from a given determinant $|a_{1n}|$ and partly from the adjugate determinant ; and the second half is quite similar, the m^{th} compound and the $(n-m)^{\text{th}}$ compound taking the places of $|a_{1n}|$ and $|A_{1n}|$. The first half thus deals with the case of the second half where m is 1.

Most of the theorems are found very near the surface, being in general brought to light by a single application of the multiplication-theorem. For example, if n be 5, and the mixture of rows be the 1st, 2nd, 4th from $|A_{15}|$ and the 2nd, 5th from $|a_{15}|$, we have

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix} \cdot |a_{15}| = \begin{vmatrix} |a_{15}| & . & . & . & . \\ . & |a_{15}| & . & . & . \\ . & . & . & |a_{15}| & . \\ r_2r_1 & r_2r_2 & r_2r_3 & r_2r_4 & r_2r_5 \\ r_5r_1 & r_5r_2 & r_5r_3 & r_5r_4 & r_5r_5 \end{vmatrix},$$

where $r_h r_k$ denotes the product of the h^{th} row of $|a_{15}|$ by the k^{th}, and thence at once the final result is obtained. Writing the latter for the moment in the form

$$M_{25}^{124} = -|a_{15}|^2 \cdot \begin{vmatrix} r_2r_3 & r_2r_5 \\ r_5r_3 & r_5r_5 \end{vmatrix},$$

we readily see with Hunyady that

$$\begin{aligned} \mathrm{M}_{35}^{134} &= |\, a_{15} \,|^2 \cdot \begin{vmatrix} r_3 r_2 & r_3 r_5 \\ r_5 r_2 & r_5 r_5 \end{vmatrix} \\ &= -\mathrm{M}_{25}^{124}, \end{aligned}$$

and generally how the subject is proceeded with.

HUNYADY, J. (1880).

[Tételek a componált determinánsoknak egy különös neméről. *Értekezések a math. tud. Köréből*, vii. 21 ; 15 pp.]

This paper has considerable value, viewed as an orderly arrangement and exposition of known theorems, namely, Bazin's unextended theorem, the Reiss-Picquet theorem, and theorems on minors of the compound determinants involved in these. The reference to the authors of the said theorems is, however, strangely misleading, Sylvester in particular being wrongly favoured.

Since one of the authors, Bazin, has suffered much in this way, it seems worth while here to draw attention to the fact that almost simultaneously in the year 1851 two theorems in compound determinants were published, one by Bazin and one by Sylvester, and that outwardly there was very considerable resemblance between them, the statement of them in symbols taking in both cases the form

$$\| \quad \| = | \quad |^{n-1} \cdot | \quad |.$$

where the determinant on the left is compound. Nevertheless the two theorems are perfectly distinct. In Sylvester's case the elements of the compound are formed by bordering a given determinant, in Bazin's by supplanting a column of a given determinant by a column from another : in Sylvester's the determinant on the extreme right has the other determinants of the equality for minors, while in Bazin's the two determinants on the right are of the same order and are independent ; Sylvester's theorem is nothing if not an extensional, while Bazin's is as important without extension as with it.*

* On the general question of these relationships it may be helpful at this stage to recall Nanson's paper of 1900 (see above, p. 76).

PAIGE, C. LE (1880).

[Sur quelques propriétés des déterminants. *Nouv. Corresp. Math.*, vi. pp. 490–496.]

Of the two identities here given, namely,

$$\| a_1b_2 | \ | b_2c_3 | \ | c_3a_4 \| = | a_1b_2c_3 | \cdot | a_2b_3c_4 |,$$

$$\| a_1b_2c_3d_4 | \ | b_2c_3d_4e_5 | \ | c_3d_4e_5a_6 \| = | c_3d_4 | \cdot | a_1b_2c_3d_4e_5 | \cdot | a_2b_3c_4d_5e_6 |,$$

the second is an ' extensional ' of

$$\| a_1b_2 | \ | b_2e_5 | \ | e_5a_6 \| = | a_1b_2e_5 | \cdot | a_2b_5e_6 |,$$

which is equivalent to the first, and the first is a rediscovery. (See *Hist.*, iii. p. 196.)

CASPARY, F. (1880, 1881).

[Ueber die Umformung gewisser Determinanten, welche in der Lehre von den Kegelschitten vorkommen. *Crelle's Journ.*, xcii. pp. 123–144.]

[Ueber einige Determinanten-Identiteiten, welche. . . . *Crelle's Journ.*, xcv. pp. 36–43.]

Caspary's work is suggested by the papers of Hunyady, Mertens and Pasch on the same subject, and is interesting, not as giving any fresh result in determinants, but as providing Grassmannian verifications of known identities.

MUIR, T. (1881).

[Note on the multiplication of the $(n-1)^{\text{th}}$ power of a symmetric determinant of the n^{th} order by the second power of any determinant of the same order. *American Journ. of Math.*, iv. pp. 273–275.]

This is a generalization of an identity of Hesse's, and, as the title indicates, in a different direction from Hesse's own generalization of 1853 (*Hist.*, ii. pp. 132–133). It is, however, of comparatively little importance, being included in Sylvester's theorem of 1851. For example, from Sylvester we have

$$\| a_1b_2c_3d_4 | \ | a_1b_2c_3e_5 | \ | a_1b_2c_3f_6 \| = | a_1b_2c_3d_4e_5f_6 | \cdot | a_1b_2c_3 |^2;$$

and the degeneration referred to takes place when

$$\begin{vmatrix} a & b & c & x_1 & y_1 & z_1 \\ b & d & e & x_2 & y_2 & z_2 \\ c & e & f & x_3 & y_3 & z_3 \\ x_1 & x_2 & x_3 & . & . & . \\ y_1 & y_2 & y_3 & . & . & . \\ z_1 & z_2 & z_3 & . & . & . \end{vmatrix}$$

is substituted for $|\, a_1 b_2 c_3 d_4 e_5 f_6 \,|$.

Kronecker's theorem of 1869 (*Hist.*, iii. p. 19) is similarly included, and the same may be said of Scott's of 1881 as reported on next page.

HUNYADY, J. (1881).

[Egy negyedrendű felületről. *Értekezések a math. tud. Köréből*, viii. 12 ; 20 pp.]

[Ueber den geometrischen Ort der Kegelspitzen der durch sechs Punkte gehenden Kegelflächen zweiten Grades. *Crelle's Journ.*, xcii. pp. 304–306.]

[Zusatz zur Abhandlung : Ueber die verschiedenen Formen . . . *Crelle's Journ.*, xcii. pp. 307–310.]

Having ascertained that two forms of the equation of the locus in question had been found, Hunyady as before sets himself to prove the determinant identity arising therefrom, namely,

$$\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & u_1^2 & x_1y_1 & x_1z_1 & x_1u_1 & y_1z_1 & y_1u_1 & z_1u_1 \\ . & . & . & . & . & . & . & . & . & . \\ x_6^2 & y_6^2 & z_6^2 & u_6^2 & x_6y_6 & x_6z_6 & x_6u_6 & y_6z_6 & y_6u_6 & z_6u_6 \\ 2x & . & . & . & y & z & u & . & . & . \\ . & 2y & . & . & x & . & . & z & u & . \\ . & . & 2z & . & . & x & . & y & . & u \\ . & . & . & 2u & . & . & x & . & y & z \end{vmatrix}$$

$$= 2\{|\, x_1y_2z_3u \,|\,|\, x_2y_5z_6u \,|\,|\, x_1y_4z_5u \,|\,|\, x_3y_4z_6u \,| - |\, x_4y_5z_6u \,|\,|\, x_1y_3z_4u \,|\,|\, x_2y_3z_6u \,|\,|\, x_1y_2z_5u \,|\},$$

where x, y, z, u are the coordinates of the vertex of the cone, and x_1, y_1, z_1, u_1, ... , x_6, y_6, z_6, u_6 those of the six given points.

In the other paper in German he returns to the geometry of the plane, and with increased insight and skill now deduces from the single equation

$$\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & y_1z_1 & z_1x_1 & x_1y_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x_6^2 & y_6^2 & z_6^2 & y_6z_6 & z_6x_6 & x_6y_6 \end{vmatrix} = 0$$

all the related equations of his former memoir (*Hist.*, iii. pp. 191-2).

The paper in Magyar has practically been cut in two to furnish the material of the two papers in German; its style also is somewhat more diffuse.

SCOTT, R. F. (1881).

[On some forms of compound determinants. *Messenger of Math.*, xi. pp. 96–98.]

The main theorem here is that *the determinant whose* $(r, s)^{th}$ *element is the determinant got by bordering* $|a_{1n}|$ *by the* r^{th} *column of* $|b_{1n}|$ *and the* s^{th} *row of* $|c_{1n}|$ *is equal to*

$$(-1)^{\frac{n}{2}} \cdot |b_{1n}| \cdot |c_{1n}| \cdot |a_{1n}|^{n-1}.$$

It is another special case, similar to Muir's, of Sylvester's theorem of March 1851; but is not so viewed, and is proved quite independently. Sylvester's, as we know, may be said to concern the compound determinant whose elements are the $(n+1)$-line minors of $|a_{11}a_{22} \ldots a_{nn} \ldots a_{ww}|$ which contain $|a_{1n}|$ as a minor, the equivalent of the said compound being

$$|a_{11}a_{22} \ldots a_{nn}|^{w-n-1} \cdot |a_{11}a_{22} \ldots a_{nn} \ldots a_{ww}|.$$

What we have now come to is the case of this where $w = 2n$ and where $|a_{11}a_{22} \ldots a_{ww}|$ has nothing but zero elements in its last n-line minor, and consequently resolves into

$$(-1)^n \cdot |a_{1,n+1}a_{2,n+2} \ldots a_{n,2n}| \cdot |a_{n+1,1}a_{n+2,2} \ldots a_{2n,n}|.$$

The theorem is also extended by making each border consist of m lines instead of 1, the result then being

$$(-1)^{m \cdot n_m} \cdot \{|b_{1n}| \cdot |c_{1n}|\}^{(n-1)_{m-1}} \cdot |a_{1n}|^{(n-1)_m}.$$

HUNYADY, J. (1882).

[Néhány determinans-egyenletröl. *Értekezések a math. tud. Köréből*, ix. no. 10 ; 19 pp.]

[Ueber einige Determinantengleichungen. *Crelle's Journ.*, xciv. pp. 171–178.]

A return is here made to the consideration of various previously used relations between minors of a 3-by-6 array, the main object being still geometrical, and the difference in treatment being that now the number of the relations of each type is inquired into and all of them set down at full length.

KRETKOWSKI, W. (1882).

[Dowód pewnego twierdzenia tyczącego dwóch wyznaczników ogólnych. *Pamietnik Akad. . . . w Krakowie*, ix. pp. 45–47.]

What is given us here is another proof of Bazin's unextended theorem of 1851, our latest reference to which is in connection with Hunyady's third paper of 1880 (see above, p. 205).

KRONECKER, L. (1882).

(See under this heading in Chapter I.)

HUNYADY, J. (1882).

[Question 1416. *Nouv. Annales de Math.*, (3) i. p. 384.]

The proposition here set for proof concerns the determinant of the sum of a square matrix and its conjugate, and, in later phraseology, is to the effect that *a determinant and its adjugate bear to one another the same ratio as their duplicants* ; or, in symbols,

$$\text{dupl.}\,|\,A_{1n}\,| \;=\; |\,a_{1n}\,|^{n-2}\cdot\text{dupl.}\,|\,a_{1n}\,|.$$

No proof was elicited, and the question was forgotten.*

* The subject of the duplicant seems to have been neglected until taken up as new by Muir in 1914, when the theorem here stated was discovered and published along with other results in the *Messenger of Math.*, xliii. pp. 184–192. The priority of course rests with Hunyady.

SCOTT, R. F. (1882)

[On compound determinants. *Proceed. London Math. Soc.*, xiv. pp. 91–102.]

The first four paragraphs (pp. 91–95) concern Sylvester's theorems of March 1851, readily convincing proofs of them being given.* The next four paragraphs (pp. 95–98) make use of the theorems in the manner of Muir's paper of 1881 to obtain anew the results arrived at otherwise by the author the year before. The more general of these two results may be formulated thus : *The determinant,* F *say, whose* (r, s)th *element is the determinant got by bordering* $|\,a_{1n}\,|$ *by the* rth *set of* m *columns of* $|\,b_{1n}\,|$ *and by the* sth *set of* m *rows of* $|\,c_{1n}\,|$ *is equal to*

$$(-1)^{m\,.\,n_m}\,.\,|\,a_{1n}\,|^{(n-1)_m}\,.\,\{|\,b_{1n}\,|\,.\,|\,c_{1n}\,|\}^{(n-1)_{m-1}}.$$

He now notes a companion determinant, G say, whose $(r, s)^{th}$ element has for a border the columns of $|\,b_{1n}\,|$ and the rows of $|\,c_{1n}\,|$ which are not used in forming the $(r, s)^{th}$ element of F. This, he says, is equal to

$$(-1)^{(n-m)\,.\,n_m}\,.\,|\,a_{1n}\,|^{(n-1)_{m-1}}\,.\,\{|\,b_{1n}\,|\,.\,|\,c_{1n}\,|\}^{(n-1)_m}.$$

He might have added the suggestive deduction that

$$\mathrm{FG} \;=\; \{(-1)^n \mathrm{ABC}\}^{n_m}.$$

In the remaining four pages (pp. 99–102) fresh ground is broken, the subject being that which we have seen repeatedly arising in connection with such related pairs, namely, the ratio of a minor of F to the complementary of the corresponding minor in G. This ratio, when the minor of F is the first of the p-line minors, is found to be equal to

$$(-1)^{ny}\,.\,\mathrm{A}^x\,.\,(\mathrm{BC})^y \quad \text{where} \quad \begin{cases} x \;=\; p - (n-1)_{m-1} \\ y \;=\; p - (n-1)_m. \end{cases}$$

More than one proof is given.

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* This paper and Van Velzer's referred to on the opposite page helped to draw attention to only part of Sylvester's work of 1851 : they did nothing for the neglected theorem on sums of products of pairs of determinants (see above, p. 78).

BARBIER, E. (1883).

[Sur une formule de Lagrange déjà généralisée par Cauchy : nouvelle généralisation. *Comptes Rendus . . . Acad. des Sci.* (Paris), xcvi. pp. 1845–1849.]

[Généralisation du théorème de Jacobi sur les déterminants partiels du système adjoint. *Comptes Rendus . . . Acad. des Sci.* (Paris), xcvii. pp. 82–85.]

Neither of these communications is an advance, the first merely giving an example of a compound of $|\, a_1b_2c_3d_4 \,|$ and two compounds of $|\, a_1b_2c_3d_4e_5 \,|$, and the second giving examples equally simple of a minor of a compound.

VAN VELZER, C. A. (1883).

[Compound determinants. *American Journ. of Math.*, vi. pp. 164–172 : *Johns Hopkins Univ. Circ.*, i. (1880), p. 132.]

Although no really new theorems are reached here, the mode of investigation and the presentation have in them a welcome freshness. As in Sylvester's paper of 1879, more than ordinary use is made of geometrical diagrams in the exposition. Even the fresh terms introduced have a geometrical air about them ; for example, the law of extensible minors being unknown to the author, $|\, a_1b_2c_3d_4 \,|$, $|\, a_1b_2c_3d_4e_5 \,|$, $|\, a_1b_2c_3d_4e_5f_6 \,|$, . . . are not styled extensionals but 'gnomonics' of $|\, a_1b_2c_3 \,|$, because to form them a gnomon of elements is annexed to the array of $|\, a_1b_2c_3 \,|$ in each case : on the other hand, it is curious to note that $|\, a_1b_2c_3 \,|$ considered as a minor of its gnomonics is viewed as being formed from each 'by taking away an *aphaereton*' not a gnomon. The more important results arrived at are Muir's extension of Laplace's theorem (1879), Sylvester's theorem in compound determinants (1851), and an approximation to a 'principle of duality' depending on interchange of gnomonics and minors. This principle would have been more valuable had it been as readily applicable as the law of extensible minors. As it is, however, theorems obtained by the use of it are claimed to include some of Scott's theorems of the previous year

(see opposite page).

HAMBURGER, M. (1885).

[Anwendung einer gewissen Determinantenrelation auf die Integration partieller Differentialgleichungen. *Crelle's Journ.*, c. pp. 390–404.]

The first four pages of this paper are devoted to the consideration of a set of relations between the primary minors of an array of m rows and $m+h$ columns, say the array

$$\begin{array}{cccc} a_{11} & a_{12} & \ldots. & a_{1,\,m+h} \\ a_{21} & a_{22} & \ldots. & a_{2,\,m+h} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots. & a_{m,\,m+h}; \end{array}$$

and he opens with a theorem substantially identical with Bazin's unextended theorem of 1851, already repeatedly rediscovered. Hamburger's mode of proof is probably the simplest possible, consisting in merely multiplying column-wise any one of the said minors by the adjugate of the first of them, and noting that every product of two columns in such a multiplication is equal to one of the minors of the array; for example,

$$(a_{1r},\ a_{2r},\ \ldots,\ a_{mr} \mid A_{1k},\ A_{2k},\ \ldots,\ A_{mk})$$

is equal to the minor formable from the first of the minors by substituting for its k^{th} column the r^{th} column of the array. The connection of the theorem with Sylvester's of 1839 and 1851 is reached in the same way as by Rubini.

His next step takes him to d'Ovidio's result of 1877, namely, that the number of the relations in question that are independent is

$$(m+h)_m - mh - 1,$$

the reasoning being that $(m+h)_m$ is the total number of minors, and in expressing the whole of them by means of Bazin's theorem the number used is $mh+1$. In this connection it is recalled that the case where h is 2 had already been dealt with in 1875.

After the consideration of an unimportant converse theorem, the reader is reminded that throughout the discussion the first of the minors had been viewed as non-evanescent and is told what line should be taken in the opposite case.

MUIR, T. (1889).

[Note on Cayley's demonstration of Pascal's theorem. *Proceed. R. Soc. Edinburgh*, xvii. pp. 18–22.]

After the papers of Scholtz, Hunyady, etc., this has little or no interest save as showing how Cayley's early equation (1843, *Hist.*, ii. p. 13) could be simplified and made to take one of the later forms, namely, the form

$$123 \cdot 156 \cdot 425 \cdot 436 - 125 \cdot 136 \cdot 423 \cdot 456 = 0.$$

D'OVIDIO, E. (1890).

[Altra addizione alla nota sui determinanti di determinanti. *Atti . . . Accad. delle Sci.* (Torino), xxvi. pp. 131–133.]

This second supplement, called forth by the appearance of Barbier's two notes, passes quite fair criticism on them in showing their exact relation to theorems previously known. The author also takes occasion, with the same object in view, to refer shortly to Picquet's memoir, giving nothing more than reasonable prominence to his own work.

GRUSINTZEFF, A. P. (1891).

[Contribution to the theory of adjugate determinants (in Russian), *Communications Math. Soc. Kharkov*, (2) iii. pp. 94–102.]

The subject here is the iteration of the operation of forming the adjugate. In place of the author's main theorem, part of which seems inaccurate or misprinted, we prefer to give a more general theorem which ought to include it, namely, *If* A_0 *be any* n*-line determinant,* A_r *the adjugate of* A_{r-1}, *and* M_k *and* M'_{n-k} *be in their respective determinants complementary minors so far as position is concerned, the one being of the* k^{th} *order and the other of the* $(n-k)^{th}$, *then*

$$M'_{n-k} \text{ in } A_{2m} = A_0^{\frac{(n-1)^{2m}-1}{n}(n-k)} \cdot M'_{n-k} \text{ in } A_0,$$

$$M_k \text{ in } A_{2m+1} = A_0^{\left\{(n-1)^{2m+1}+1\right\}\frac{k}{n}-1} \cdot M'_{n-k} \text{ in } A_0.$$

By repeated use of the theorem regarding a minor of the adjugate, we have

$$\begin{aligned}
M_k \text{ in } A_1 &= A_0^{k-1} \cdot M'_{n-k} \text{ in } A_0 \\
M'_{n-k} \text{ in } A_2 &= A_1^{n-k-1} \cdot M_k \text{ in } A_1 \\
M_k \text{ in } A_3 &= A_2^{k-1} \cdot M'_{n-k} \text{ in } A_2 \\
M'_{n-k} \text{ in } A_4 &= A_3^{n-k-1} \cdot M_k \text{ in } A_3 \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

and therefore by multiplication and division

$$M'_{n-k} \text{ in } A_{2m} = (A_0 A_2 \ldots A_{2m-2})^{k-1} (A_1 A_3 \ldots A_{2m-1})^{n-k-1} \cdot M'_{n-k} \text{ in } A_0.$$

But since $A_r = A_0^{(n-1)^r}$, it follows that

$$\begin{aligned}
A_0 A_2 \ldots A_{2m-2} &= A_0^{1+(n-1)^2+\ldots+(n-1)^{2m-2}} \\
&= A_0^{\frac{(n-1)^{2m}-1}{(n-1^2)-1}} = A_0^{\frac{(n-1)^{2m}-1}{n(n-2)}},
\end{aligned}$$

and similarly

$$A_1 A_3 \ldots A_{2m-1} = A_0^{\frac{(n-1)^{2m}-1}{n(n-2)}(n-1)};$$

so that

$$\begin{aligned}
(A_0 A_2 \ldots A_{2m-2})^{k-1} (A_1 A_3 \ldots A_{2m-1})^{n-k-1} &= \left\{A_0^{\frac{(n-1)^{2m}-1}{n(n-2)}}\right\}^{k-1+(n-1)(n-k-1)} \\
&= \left\{A_0^{\frac{(n-1)^{2m}-1}{n(n-2)}}\right\}^{(n-2)(n-k)}
\end{aligned}$$

and consequently

$$M'_{n-k} \text{ in } A_{2m} = A_0^{\left\{(n-1)^{2m}-1\right\}\frac{n-k}{n}} \cdot M'_{n-k} \text{ in } A_0,$$

as desired. The second part of the theorem follows readily from the first; for, using again Jacobi's equality with which we started, we have

$$M_k \text{ in } A_{2m+1} = A_{2m}^{k-1} \cdot M'_{n-k} \text{ in } A_{2m},$$

which with two evident substitutions on the right takes the form given in the enunciation.

The interesting cases are where $k = 1$ and $k = n-1$. It is one of these that is considered in the paper under review.

We may add that an alternative to the above proof may be obtained by using the following lemma: *If* Δ_0 *stand for* $|\, a_{1n} \,|$ *and* Δ_r *be the adjugate of* Δ_{r-1}, *then the* $(r, s)^{th}$ *element of* Δ_{2m} *is*

$$a_{rs} \cdot \Delta_0^{\frac{(n-1)^{2m}-1}{n}}$$

and the (r, s)th *element of* Δ_{2m+1} *is*

$$A_{rs} \cdot \Delta_0^{\frac{(n-1)^{2m}-1}{n}(n-1)}.$$

For example, the 4th adjugate of $| a_1b_2c_3 |$ is

$$\left| \begin{matrix} a_1 | a_1b_2c_3 |^5 & a_2 | a_1b_2c_3 |^5 & a_3 | a_1b_2c_3 |^5 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{matrix} \right|,$$

and the 3rd adjugate of $| a_1b_2c_3d_4 |$ is

$$\left| \begin{matrix} A_1 | a_1b_2c_3d_4 |^6 & A_2 | a_1b_2c_3d_4 |^6 & A_3 | a_1b_2c_3d_4 |^6 & A_4 | a_1b_2c_3d_4 |^6 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{matrix} \right|.$$

RADOS, G. (1891).

[Az adjungált helyettesítések elméletéröl. *Math. és Term. tud. Értesitö*, x. pp. 34–42.]

[Zur Theorie der adjungierten Substitutionen. *Math.-u.-naturw. Berichte aus Ungarn*, x. pp. 98–107 : also in *Math. Annalen*, xlviii. pp. 417–424.]

As the 'substitution adjugate to a given substitution' is that whose determinant is a compound of the determinant of the latter, it is not unnatural to find in a paper with a title like this an interesting property of compound determinants. Put in different phraseology from the author's, the theorem is that *the latent roots of the* mth *compound of a determinant are the* m^{ary} *products of the latent roots of the latter.* An old closely-related theorem, so-called Francke's, also comes naturally in for attention, a proof of it being given.

LANDSBERG, G. (1892).

[Ueber relativ adjungirten Minoren. *Crelle's Journ.*, cix. pp. 225–230.]

In this paper, which in matter is comparable with Van Velzer's of 1883, there are no fresh results, the first of the two rediscovered theorems being Muir's extension of Laplace's expansion-theorem (1879), and the second being Sylvester's of 1851 on compound determinants. The idea of 'relatively adjugate minors' comes in merely as an aid to the enunciation of the former theorem ; it will

be understood when we say that $|a_1b_2c_3d_4e_5|$ and $|a_1b_2c_3f_6g_7h_8|$ are a pair of such minors, and that the justification given is that $|d_4e_5|$ and $|f_6g_7h_8|$, of which the previous pair are extensionals, are actual adjugates (*i.e.* complementaries) in $|d_4e_5f_6g_7h_8|$. It is strange, however, that the previous pair bear the designation along with the words 'in regard to $|a_1b_2c_3|$' of which neither is a minor.

NETTO, E. (1893).

[Zwei Determinantensätze. *Acta Math.*, xvii. pp. 199–204.]

The first theorem concerns a three-line compound determinant, the elements of which are primary minors of an $(n-2)$-by-n array. Elements of this kind being, as we have seen, sufficiently specified by their column-numbers, the theorem may conveniently be viewed as the extensional of

$$\begin{vmatrix} |1245| & |1246| & |1256| \\ |1345| & |1346| & |1356| \\ |2345| & |2346| & |2356| \end{vmatrix} = 0,$$

or

$$\| 12,45| \quad |13,46| \quad |23,56| \| = 0,$$

—a simple result of the type dealt with in Reiss' third section. Here it is deduced from consideration of a set of simultaneous equations, and also by the application of Laplace's expansion-theorem.

The second theorem is Sylvester's first of March 1851.

MUSSO, G. (1893, 1894).

[Sui determinanti reciproci. *Giornale di Mat.*, xxxi. pp. 201–209; xxxii. pp. 81–85.]

Here the second paper, which the writer uses to compare the rediscoveries of his first paper with D'Ovidio's work of 1876–7, is the more instructive.

VAHLEN, K. T. (1893).

(See under this heading in Chapter I.)

METZLER, W. H. (1893).

[Compound determinants. *American Journ. of Math.*, xvi. pp. 131–150.]

The main part (pp. 132–145) of this longish paper concerns the sums of certain sets of k-line minors of the m^{th} compound of a determinant; for example, coaxial minors and minors belonging to the same oblong array. In the opening cases the so-called Franke's theorem is used, followed by simple addition: in the others the law of extensionals could be employed with good effect.

Passing to the subject of latent roots, the author then (§ 3) affixes $-x$ to each diagonal element of $|a_{14}|$ and of its compounds, and expanding the resulting determinants in descending powers of x, is led to a rediscovery, namely, the probably general truth of Rados' theorem of 1891.

The remaining three pages concern axisymmetric and other special forms.

MÜLLER, E. (1894).

[Anwendung der Grassmann'schen Methoden auf . . . *Crelle's Journ.*, cxv. pp. 234–253.]

The only interest of this for us lies in the fact that geometrical results which we have seen Caspary, Pasch, and Hunyady obtain by means of determinant theorems are now arrived at in Grassmann's fashion, the 'outer product' being the moving agent.

FROBENIUS, G. (1894).

[Ueber das Trägheitsgesetz der quadratischen Formen. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), pp. 241–256, 407–431; or *Crelle's Journal*, cxiv. pp. 187–230.]

Sylvester's theorem of 1851, as being 'the basis of his whole investigation,' Frobenius again establishes (§ 1, pp. 242–245) as in his paper of 1878. He also points out, as Muir had done in 1881, its applicability to Kronecker's compound determinant of 1869, and thereby establishes the latter's theorem of that date regarding the evanescence of certain minors.

JAROCHENKO, S. P. (1894).

[Some theorems in determinants (in Russian language). *Mem. Univ. . . .* (Odessa), lxi. pp. 593–606.]

The main result is an extension of one of Netto's rediscoveries of 1893, but is also itself a rediscovery.

RADOS, G. (1896, 1897).

[Adjungierte quadratische Formen. *Math. u. naturw. Berichte aus Ungarn*, xiv. pp. 85–91, 116–127; but originally in Magyar, *Math. és Term. tud. Értesitö*, xiv. pp. 26–32. See also *Verhandl. d. internat. Math.-Kongresses* (Zürich), pp. 163–165.]

As a binary quadric 'adjugate to a given binary quadric' may be defined as that whose discriminant is a compound of the discriminant of the latter, there is the same room here for an incidental property of compound determinants as in the author's analogous paper of 1891. The nearest approach to such, however, is only when consideration is given to the special case where a number of the latent roots of the discriminant of the original quadric are zeros.

PASCAL, E. (1896).

[Sulle varie forme che possono darsi alle relazioni fra i determinanti di una matrice rettangolare. *Annali di Mat.*, (2) xxiv. pp. 241–253.]

The first of the four sections of this paper deals with the relations connecting the determinants of an oblong array, written, however, without knowledge of any previous work on the subject except Vahlen's paper of rediscoveries. It contains a fresh proof, comparable with Cayley's of 1843 (*Hist.*, i. pp. 10–11), of what is properly called the 'fundamental identity,' but not recognised as Sylvester's theorem of 1839 and 1851 *; for example, the vanishing of the expression

$$|a_1b_2c_3|\,|d_1e_2f_3| - |a_1b_2d_3|\,|c_1e_2f_3| + |a_1c_2d_3|\,|b_1e_2f_3| - |b_1c_2d_3|\,|a_1e_2f_3|$$

* The remark made above (p. 78, footnote †) regarding Scott's text-book applies also to Pascal's.

is based on the fact that it is an equivalent of

$$\begin{vmatrix} a_1 & a_2 & a_3 & |a_1e_2f_3| \\ b_1 & b_2 & b_3 & |b_1e_2f_3| \\ c_1 & c_2 & c_3 & |c_1e_2f_3| \\ d_1 & d_2 & d_3 & |d_1e_2f_3| \end{vmatrix}.$$

The second section concerns the vanishing 3-line compound determinant rediscovered by Netto, and gives a companion to it which is readily seen to be an extensional of a known result. They are here established by using thrice the above-named fundamental identity and then eliminating.

The subject of the third section is compound determinants whose elements are formed of columns taken from two given determinants. The inclusion of this is not out of keeping with the title of the paper, for the said elements may be equally justly viewed as minors of an n-by-$2n$ array. The previous writers referred to are Sylvester, d'Ovidio, and Picquet. First of all are given the two main theorems from Picquet regarding his pair of 'mixed compound.' Following on these are Bazin's theorem of 1851, though not recognised as such, and Sylvester's theorem of the same year. Then comes what is spoken of as a generalisation of the latter, but is perhaps more naturally viewable as the extensional of a case of Picquet's. It concerns a special minor of the 'mixed compound,' and is followed by a companion theorem the subject of which is the complementary minor in the twin compound. While the former is resolvable of course into powers of the two basic determinants, the latter has as an additional factor a power of a determinant formed of columns from both basic determinants.

In the fourth section Pascal resumes consideration of Laplace's expansion-theorem which he had dealt with in a previous paper of the same year. The two theorems brought forward in that paper are now generalised, expansions being given for

$$|a_1, a_2, \ldots, a_m| \cdot |a_1, a_2, \ldots, a_{m-\lambda}b_{m-\lambda+1}, \ldots, b_m|^{r-1}$$

and

$$|a_1, a_2, \ldots, a_m|^{r-1} \cdot |a_1, a_2, \ldots, a_{m-\lambda}b_{m-\lambda+1}, \ldots, b_m|,$$

where the a's and b's are the numbers of the columns in an m-by-$(m+\lambda)$ array, r is the number of factors in each term of

the expansions, and every determinant on both sides is of the m^{th} order and contains the columns $a_1, a_2, \ldots, a_{m-\lambda}$.*

ARANT, D. (1896).

[Question 1751. *Nouv. Annales de Math.*, (3) xvi. p. 583 ; (4) xvii. p. 246.]

The determinant here unsuccessfully drawn attention to is that whose elements are the determinants of three 2-by-3 arrays.

NANSON, E. J. (1897).

[On determinant notation. *Philos. Magazine*, xliv. pp. 396–400.]

It is here proposed to denote the oblong array

$$\begin{matrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{matrix} \quad \text{by} \quad (a_{pq}) \quad \begin{pmatrix} p = 1, 2, \ldots, m \\ q = 1, 2, \ldots, n \end{pmatrix},$$

and the proposal is supported by using the notation in stating Sylvester's two theorems of 1851 in regard to compound determinants, and Schweins' proposition regarding the equality of two aggregates of products of pairs of determinants. For example, the simpler of Sylvester's theorems is put in the form :

$$\textit{If} \qquad e_{rs} = \begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & d_{rs} \end{vmatrix} \quad (p, q = 1, 2, \ldots, m),$$

$$\textit{then} \quad |e_{rs}| = |a_{pq}|^{n-1} \cdot \begin{vmatrix} (a_{pq}) & (b_{ps}) \\ (c_{rq}) & (d_{rs}) \end{vmatrix} \quad (r, s = 1, 2, \ldots, n).$$

An interesting commentary on this and the other theorems is also given.

METZLER, W. H. (1897).

[Compound determinants. *American Journ. of Math.*, xx. pp. 253–272.]

The author resumes his study of 1893, relying now mainly on the law of complementaries and the law of extensionals. The first nine

* In connection with the third section see above, p. 205.

pages afford an interesting contrast with Van Velzer's paper of 1883, the theorems dealt with being the same. The next five pages concern minors of the m^{th} compound that are resolvable or are equal to zero, the former including some with *two* different factors, some with *three* different factors, and some with more than three like Muir's of 1879. The remaining pages (pp. 267–272) are occupied with illustrative examples taken from compounds of $|a_{15}|$. All these examples, however, with the exception of three, are merely extensionals of equalities pertaining to a determinant of lower degree than $|a_{15}|$. The three that primarily concern the 5^{th} order are :

$$\left\| \begin{vmatrix} 123 \\ 234 \end{vmatrix} \begin{vmatrix} 125 \\ 235 \end{vmatrix} \begin{vmatrix} 145 \\ 245 \end{vmatrix} \begin{vmatrix} 345 \\ 345 \end{vmatrix} \right\| = \begin{vmatrix} 1235 \\ 2345 \end{vmatrix} \begin{vmatrix} 1245 \\ 2345 \end{vmatrix} \begin{vmatrix} 1345 \\ 2345 \end{vmatrix},$$

$$\left\| \begin{vmatrix} 234 \\ 145 \end{vmatrix} \begin{vmatrix} 235 \\ 235 \end{vmatrix} \begin{vmatrix} 245 \\ 245 \end{vmatrix} \begin{vmatrix} 345 \\ 345 \end{vmatrix} \right\| = 0,$$

$$\left\| \begin{vmatrix} 145 \\ 135 \end{vmatrix} \begin{vmatrix} 234 \\ 145 \end{vmatrix} \begin{vmatrix} 235 \\ 235 \end{vmatrix} \begin{vmatrix} 245 \\ 245 \end{vmatrix} \begin{vmatrix} 345 \\ 345 \end{vmatrix} \right\| = 0,$$

and the second of these is pointed out to be one of the fifteen

$$\left\| \begin{array}{cccccc} \begin{vmatrix} 234 \\ 125 \end{vmatrix} & \begin{vmatrix} 234 \\ 135 \end{vmatrix} & \begin{vmatrix} 234 \\ 145 \end{vmatrix} & \begin{vmatrix} 234 \\ 235 \end{vmatrix} & \begin{vmatrix} 234 \\ 245 \end{vmatrix} & \begin{vmatrix} 234 \\ 345 \end{vmatrix} \\ \begin{vmatrix} 235 \\ 125 \end{vmatrix} & \begin{vmatrix} 235 \\ 135 \end{vmatrix} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & \begin{vmatrix} 235 \\ 345 \end{vmatrix} \\ \begin{vmatrix} 245 \\ 125 \end{vmatrix} & \begin{vmatrix} 245 \\ 135 \end{vmatrix} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & \begin{vmatrix} 245 \\ 345 \end{vmatrix} \\ \begin{vmatrix} 345 \\ 125 \end{vmatrix} & \begin{vmatrix} 345 \\ 135 \end{vmatrix} & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & \begin{vmatrix} 345 \\ 345 \end{vmatrix} \end{array} \right\| = 0.$$

It might have been added that the third hangs upon five of the same fifteen.

The notation used throughout the paper is not wanting in precision, but is distinctly cumbersome. Its basis is a symbol for the λ^{th} m-ary combination of the integers 1, 2, . . . , n, namely,

$$\binom{n \mid m}{\lambda},$$

for example,

$\binom{5 \mid 3}{7}$ stands for the combination 2, 3, 4.

As a comparatively simple instance of its use we may take from the first part of the paper the extension of Laplace's expansion-theorem, namely (p. 259),

$$\mathrm{A} \cdot \mathrm{A}_{\substack{(n\,|\,m)\\ \alpha\\ (n\,|\,m)\\ \gamma}} = \sum_{\beta \text{ or } \delta = 1}^{\beta \text{ or } \delta = \lambda} (-1)^{\nu} \mathrm{A}_{\substack{(n\,|\,m\,|\,l)\\ \alpha\ \ \beta\\ (n\,|\,m\,|\,l)\\ \gamma\ \ \delta}} \cdot \mathrm{A}_{\substack{(n\,|\,\overline{m}\,|\,l)\\ \alpha\ \ \beta\\ (n\,|\,\overline{m}\,|\,l)\\ \gamma\ \ \delta}},$$

the effective visualizing of which is likely to be found difficult, at least by the ordinary student.

IGEL, B. (1898).

[Beweis einiger Determinanten theoreme von Sylvester. *Monatshefte f. Math. u. Phys.*, ix. pp. 47–54.]

The theorems in question are Sylvester's of March 1851, already repeatedly proved, but here in their connection with the papers of Franke, Frobenius and Netto very clearly and profitably commented on. (See above, p. 76 and p. 205.)

STUDNIČKA, F. J. (1898).

[O novém druhu determinantů přidružených. *Věstnik České Akad.* . . . , vii. pp. 283–291.]

The 'new' determinants in question are the compounds. No fresh property is brought to light.

STUDNIČKA, F. J. (1898, 1899).

(See under this heading in Chap. I.)

WHITE, H. S. (1899).

[Two elementary geometrical applications of determinants. *Annals of Math.*, (2) i. pp. 103–107.]

The geometrical theorems are those of Pascal and Desargues which we have already seen more than once treated with the help of determinants. The author himself refers to Hunyady's second paper of 1879.

MUIR, T. (1899).

[On the eliminant of a set of general ternary quadrics. *Transac. R. Soc. Edinburgh*, xl. pp. 23–38.]

There enter here as auxiliaries two results worth noting, namely, the fact (p. 32) that the axisymmetric compound determinant

$$\begin{vmatrix} \cdot & |a_1b_2g_3| & |a_1b_2f_3| & |a_1b_2c_3| \\ |a_1b_2g_3| & |a_1f_2g_3| & |a_1b_2c_3| & |a_1f_2c_3| \\ |a_1b_2f_3| & |a_1b_2c_3| & |b_1f_2g_3| & |b_1c_2g_3| \\ |a_1b_2c_3| & |a_1f_2c_3| & |b_1c_2g_3| & |f_1c_2g_3| \end{vmatrix}$$

is the eliminant of

$$\left.\begin{aligned} a_1x^2 + b_1y^2 + c_1z^2 + f_1yz + g_1zx &= 0 \\ a_2x^2 + b_2y^2 + c_2z^2 + f_2yz + g_2zx &= 0 \\ a_3x^2 + b_3y^2 + c_3z^2 + f_3yz + g_3zx &= 0 \end{aligned}\right\};$$

and (2) the identity (pp. 35–36)

$$\begin{vmatrix} 147 & 248 & 349 \\ 157 & 258 & 359 \\ 167 & 268 & 369 \end{vmatrix} = 456 \begin{vmatrix} 137 & 238 \\ 179 & 289 \end{vmatrix} = 456 \begin{vmatrix} 218 & 319 \\ 287 & 397 \end{vmatrix} = \begin{vmatrix} 329 & 127 \\ 328 & 178 \end{vmatrix},$$

where the digits 1, 2, 3, . . . , 9 indicate the columns of any 3-by-3^2 array. With the latter may be compared identities given under Reiss and Scholtz (*Hist.*, iii. p. 186, p. 196).

CHAPTER VIII.

RECURRENTS, FROM 1880 TO 1898.

THE number of writings dealt with here is slightly in excess of the number for the preceding period.

GLASHAN, J. C. (1880).

[Change of the independent variable. *American Journ. of Math.*, iii. pp. 190–191.]

Here u and v are given functions of x, and it is required to find the successive differential coefficients of u with respect to v. Denoting

$$\frac{1}{r!}\frac{\partial^r u}{\partial x^r}, \quad \frac{1}{r!}\frac{\partial^r v}{\partial x^r} \quad \text{by} \quad u_r,\ v_r \text{ respectively,}$$

Glashan finds, for example,

$$\frac{1}{5!}\frac{\partial^5 u}{\partial v^5} = (-1)^5 \frac{\begin{vmatrix} u_1 & v_1 & . & . & . \\ u_2 & v_2 & v_1^2 & . & . \\ u_3 & v_3 & 2v_1v_2 & v_1^3 & . \\ u_4 & v_4 & 2v_1v_3+v_2^2 & 3v_1^2v_2 & v_1^4 \\ u_5 & v_5 & 2(v_1v_4+v_2v_3) & 3(v_1^2v_3+v_1v_2^2) & 4v_1^3v_2 \end{vmatrix}}{v_1 \cdot v_1^2 \cdot v_1^3 \cdot v_1^4 \cdot v_1^5}$$

and gives the general result.

McKENZIE, J. L. (1880): FARKAS, J. (1880).

[Question 6369. *Educ. Times*, xxxiii. p. 171: *Math. from Educ. Times*, xxxvii. p. 39.]

[Die Summe gleichartiger Potenzen von den Würzeln einer algebraischen Gleichung als Function der Coefficienten derselben Gleichung und umgekehrt. *Archiv d. Math. u. Phys.*, lxv. pp. 433–435.]

Nothing here is really new (see *Hist.*, iii. p. 209), save in the second paper the mode of obtaining the equation connecting the s's and the c's (*Hist.*, ii. p. 218).

CESÀRO, E. (1880).

[Question 604. *Nouv. Corresp. Math.*, vi. pp. 527–528.]

From its manner of statement Cesàro's theorem looks more important than it really is, becoming in fact self-evident when the functional symbols are got rid of, and the two complete columns of the recurrent are placed side by side. For example,

$$\begin{vmatrix} a_1 & a_2 & . & . & . & \dots \\ b_1 & b_2 & b_3 & . & . & \dots \\ c_1 & c_2 & c_3 & c_4 & . & \dots \\ d_1 & d_2 & d_3 & d_4 & d_5 & \dots \\ . & . & . & . & . & \dots \end{vmatrix} = a_2A_2 + a_1A_1,$$

$$= b_3B_3 + |\, a_1b_2 \,| \cdot \text{cof.},$$

$$= c_4C_4 + \begin{vmatrix} a_1 & a_2 & . \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \cdot \text{cof.},$$

$$= . \quad . \quad .$$

and, consequently, if the second term in any one of these expansions vanishes, the determinant equals a_2A_2 or b_3B_3 or c_4C_4 or ....

MOLLAME, V. (1881).

[Sulla somma della potenze simili di numeri qualunque in progressione aritmetica, ... *Atti dell' Accad. Gioenia* ..., (3) xv. pp. 261–272.]

Putting

$$S_k = a_1^k + a_2^k + \dots + a_s^k \quad \text{where} \quad a_r = a_{r-1} + d,$$

Mollame, evidently with an eye on Lucas' papers of 1876, 1877 (*Hist.*, iii. pp. 238–239), shows that

$$a_{s+1}^m - a_1{}^m = (m)_1 d\mathrm{S}_{m-1} + (m)_2 d^2\mathrm{S}_{m-2} + \ldots + (m)_m d^m\mathrm{S}_0$$
$$= (\mathrm{S}+d)_{\downarrow}^m - \mathrm{S}_m, \text{ say,}$$

and thence derives

$$m!\, d\mathrm{S}_{m-1} = (-1)^{m-1} \begin{vmatrix} (a_{s+1}-a_1)d^{m-1} & (1)_1 & \ldots & \cdot & \cdot \\ (a_{s+1}^2-a_1{}^2)d^{m-2} & (2)_2 & \ldots & \cdot & \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & & & & \\ (a_{s+1}^{m-1}-a_1{}^{m-1})d & (m-1)_{m-1} & \ldots & (m-1)_2 & (m-1)_1 \\ (a_{s+1}^m-a_1{}^m) & (m)_m & \ldots & (m)_3 & (m)_2 \end{vmatrix}.$$

His 'new expression' for a Bernoulli's number is essentially the same as one of Glaisher's (*Hist.*, iii. p. 235).

PINCHERLE, S. (1881, 1883).

[Sopra una formula di analisi. *Giornale di Mat.*, xix. pp. 385–386.]

[Una formola sui determinanti. *Rendic. . . . Accad. delle Sci. . . .* (Bologna), Anno 1882–3, pp. 105–106.]

The formula here twice published by Pincherle is a rediscovery, having been given in 1876 by Janni and by Glaisher (*Hist.*, iii. pp. 237, 460).

DAVID, J. M. (1882).

[Applications de la dérivation d'Arbogast à la solution de la partition des nombres et à d'autres problèmes. *Journ.* (*de Liouville*) *de Math.*, (3) viii. pp. 61–72.]

The operator D here used in developing

$$\phi(a+a_1x+a_2x^2+\ldots)$$

according to ascending powers of x, David defines by the recurrence-formula

$$\mathrm{D}^r\phi(a) = \phi^r(a)\cdot\frac{a_1{}^r}{r!} + \phi^{r-1}(a)\cdot\mathrm{D}\frac{a_1{}^{r-1}}{(r-1)!} + \phi^{r-2}(a)\cdot\mathrm{D}\frac{a_1{}^{r-2}}{(r-2)!} + \ldots.$$

In applying it the paragraphs specially interesting to us are §§ 4, 5, 6, where ϕ is the n^{th} power. The result obtained is

$$(a+a_1x+a_2x^2+\ldots)^n = a^n + x\mathrm{D}(a^n) + x^2\mathrm{D}^2(a^n) + \ldots,$$

$D^r(a^m)$ being expressed by means of a determinant : for example,

$$D^5\left(\frac{a^m}{m}\right) = \frac{a^{m-5}}{5!}\begin{vmatrix} a_1 & -a & . & . & . \\ 2a_2 & (m-1)a_1 & -2a & . & . \\ 3a_3 & (2m-1)a_2 & (m-2)a_1 & -3a & . \\ 4a_4 & (3m-1)a_3 & (2m-2)a_2 & (m-3)a_1 & -4a \\ 5a_5 & (4m-1)a_4 & (3m-2)a_3 & (2m-3)a_2 & (m-4)a_1 \end{vmatrix}.$$

Two special cases are considered, but the determinants occurring in connection therewith are rediscoveries.

CHERRIMAN, J. B. (1882).

[Note on application of a special determinant. *Proceed. and Transac. R. Soc. of Canada*, i. (3) pp. 13–14.]

From the determinants for s_r and $\aleph_r$ in terms of the coefficients of the equation·

$$x^n + a_1x^{n-1} + \ldots + a_n = 0$$

there is here obtained the relation

$$-s_r = a_1\aleph_{r-1} + 2a_2\aleph_{r-2} + \ldots + ra_r$$

almost formulated by Trudi in 1878.

SACHSE, A. (1882).

[Ueber die Darstellung der Bernoulli'schen und Euler'schen Zahlen durch Determinanten. *Archiv d. Math. u. Phys.*, lxviii. pp. 427–432.]

Sachse, adopting Lucas' procedure in connection with Bernoulli's numbers (*Hist.*, iii. pp. 238–239), obtains a similar expression for Euler's numbers, namely,

$$E_n = \frac{1}{(2n)!}\begin{vmatrix} \frac{1}{2!} & 1 & . & . & \ldots \\ \frac{1}{4!} & \frac{1}{2!} & 1 & . & \ldots \\ \frac{1}{6!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \ldots \\ . & . & . & . & . \end{vmatrix}_n$$

He also expresses any one of either set of numbers in terms of preceding numbers of the same set.

AMANZIO, D. (1883).

[Di alcune trasformazioni del simbolo d'operazione

$$V\frac{\partial}{\partial x}\cdot U\frac{\partial}{\partial x}\cdots Z\frac{\partial}{\partial x}\cdot Y\frac{\partial}{\partial x}\cdot X\frac{\partial}{\partial x},$$

e proprietà di alcuni determinanti che derivano da queste trasformazioni. *Giornale di Mat.*, xxi. pp. 110–144.]

There are four determinants here evaluated with much labour, but the fourth

$$\begin{vmatrix}
\frac{1}{k(k+1)} & m & . & \cdots & . \\
\frac{1}{(k+1)(k+2)} & \frac{n-m}{1\cdot 2} & m-1 & \cdots & . \\
\frac{1}{(k+2)(k+3)} & \frac{n-m}{2\cdot 3} & \frac{n-m+1}{1\cdot 2} & \cdots & . \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{(k+m-1)(k+m)} & \frac{n-m}{(m-1)m} & \frac{n-m+1}{(m-2)(m-1)} & \cdots & 1 \\
\frac{1}{(k+m)(k+m+1)} & \frac{n-m}{m(m+1)} & \frac{n-m+1}{(m-1)m} & \cdots & \frac{n-1}{1\cdot 2}
\end{vmatrix}$$

includes the three others, the third being the case where $k = 1$, the second the case where $m = n-k-1$, and the first the case where both these specializations are made. Save in the simplest case,—where the value is $1/n$,—the evaluation is only effected by retaining in use a difference-operator, the reason doubtless being that it is only in the said case that the constitution of the first column is in keeping with the others, as tested by the law of progression in the diagonals. The result reached is given in the form

$$\frac{m!}{n!\,k}\,\delta^{(k)}\left[\frac{\Delta^{(n-k-m-1)}x^n - \delta^{(n-k-m-1)}(x+1)^n}{x-1}\right],$$

where $\Delta^{(k)}$ denotes the operation of taking the k^{th} difference with respect to x, $\delta^{(k)}$ the same operation followed by putting x equal to 0.

BRUNO, F. FAÀ DI (1883): HAGEN, J. (1883): MACMAHON, P. A. (1883).

[Sur le développement des fonctions rationnelles. *American Journ. of Math.*, v. pp. 238–240.]

[On division of series. *American Journ. of Math.*, v. pp. 236–237.]

[Note on the development of an algebraic fraction. *American Journ. of Math.*, vi. pp. 287–288.]

The double result meant to be given by Bruno is that *the coefficient of* x^p *in the expansion of*

$$(a_0+a_1x+a_2x^2+\ldots+a_nx^n)^{-1}$$

$$= \frac{1}{p!\,a_0}\begin{vmatrix} s_{-1} & -1 & . & . & \ldots. \\ s_{-2} & s_{-1} & -2 & . & \ldots. \\ s_{-3} & s_{-2} & s_{-1} & -3 & \ldots. \\ . & . & . & . & . \end{vmatrix}_p = (-1)^p \frac{1}{a_0^{p+1}} \begin{vmatrix} a_1 & a_0 & . & . & \ldots. \\ a_2 & a_1 & a_0 & . & \ldots. \\ a_3 & a_2 & a_1 & a_0 & \ldots. \\ . & . & . & . & . \end{vmatrix}_p.$$

Neither right-hand equivalent is really new: both are reached with much trouble: and both are incorrectly stated. The desideratum being the coefficient of x^p in

$$\frac{1}{a_0}\left(1+\frac{a_1}{a_0}x+\ldots+\frac{a_n}{a_0}x^n\right)^{-1}$$

is $\frac{1}{a_0}\cdot p^{\text{th}}$ aleph function of the reciprocals of the roots of the equation

$$a_nx^n+\ldots+a_1x+a_0=0,$$

and therefore is $\frac{1}{a_0}\cdot p^{\text{th}}$ aleph function of the roots of the equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0.$$

The two forms of expression, the one in terms of the s's and the other in terms of the a's, are thus known from Brioschi's paper of 1857 (*Hist.*, ii. pp. 216–217). The second, however, is really the simpler, and goes back to Faure (1855).

Hagen's result is still more manifestly Faure's.

In MacMahon's note the order of thought is different, the development being the second of the above with the coefficients in non-determinant as well as determinant form. A note appended to it by Franklin leads back to the old way of obtaining the latter form by equatement of coefficients.

LODGE, A. (1883).

[Note on the coefficients in a transformed equation. *Messenger of Math.*, xii. pp. 178–179.]

By putting $x = y+h$ and $a_0h+a_1 = 0$, the equation

$$a_0{}^n + na_1x^{n-1} + (n)_2a_2x^{n-2} + \ldots + a_n = 0,$$

or

$$(a_0,\ a_1,\ a_2,\ \ldots,\ a_n\}(x+1)^n = 0,$$

is transformed into one having 0 for the coefficient of the second term, the result in fact being

$$a_0y^n + n\mathrm{U}_1y^{n-1} + (n)_2\mathrm{U}_2y^{n-2} + \ldots + \mathrm{U}_n = 0$$

where

$$\mathrm{U}_r = (a_0,\ a_1,\ a_2,\ \ldots,\ a_n\}(h+1)^r,$$

$$= \left(a_0,\ a_1,\ a_2,\ \ldots,\ a_n\right\}\left(-\frac{a_1}{a_0}+1\right)^r.$$

Lodge considers it worth while to express the latter form of U_r as a determinant, performing the substitution $h = -a_1/a_0$ by solving for U_r the set of equations

$$\left.\begin{array}{rl} a_0h^r + ra_1h^{r-1} + (r)_2a_2h^{r-2} + \ldots + ra_{r-1}h + a_r &= \mathrm{U}_r \\ a_0h^r + a_1h^{r-1} &= 0 \\ a_0h^{r-1} + a_1h^{r-2} &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_0h + a_1 &= 0 \end{array}\right\}$$

with the result

$$\mathrm{U}_r = \frac{1}{(-a_0)^{r-1}}\begin{vmatrix} (r-1)a_1 & (r)_2a_2 & (r)_3a_3 & \ldots & ra_{r-1} & a_r \\ a_0 & a_1 & \cdot & \ldots & \cdot & \cdot \\ \cdot & a_0 & a_1 & \ldots & \cdot & \cdot \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & & & \\ \cdot & \cdot & \cdot & \ldots & a_0 & a_1 \end{vmatrix}.$$

SYLVESTER, J. J.: FRANKLIN, F. (1883).

[On Crocchi's theorem. *Johns Hopkins Univ. Circ.*, ii. pp. 2, 24.]

The theorem so called is one of those of 1879 (*Hist.*, iii. pp. 246–7); but determinants are not used.

MARCHAND, . (1883): STUDNIČKA, F. J. (1884).

[Note sur la manière d'écrire $(x+h)^n$ sous forme de determinant. *Journ. de Math. Spéc.*, (2) vii. pp. 248–250.]

[Úloha 22. *Časopis pro pěstování math. a fys.*, xiii. pp. 149, 279–281.]

The kind of determinant used in the first paper can, through multiplication by h, be made a persymmetric recurrent, the identity given being in effect

$$\begin{vmatrix} h & -1 & . & . & . & \dots \\ hx & h & -1 & . & . & \dots \\ hx^2 & hx & h & -1 & . & \dots \\ hx^3 & hx^2 & hx & h & -1 & \dots \\ . & . & . & . & . & \dots \end{vmatrix}_{m+1} = h(x+h)^m.$$

Studničkа's result is Pincherle's rediscovery of 1881. (See above, p. 226.)

JEŽEK, O. (1884).

[Ueber das formale Bildungsgesetz der Coefficienten des Quotienten zweier Potenzreihen. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), pp. 127–144.]

A useful account (pp. 135–139) of the subject, but with insufficient reference to previous workers, for example, Faure (1855), Brioschi (1858), Glaisher (1879). It is preceded (pp. 127–134) by a reinvestigation of the formulae for the evaluation of the four well-known recurrents that connect the simpler symmetric functions; and is followed (pp. 140–144) by three tables giving a limited number of the results of such evaluations.

GLAISHER, J. W. L. (1884).

[Application of Möbius's theorem on the reversion of certain series. *Philos. Magazine*, (5) xviii. pp. 518–540.]

Eight pages of this (pp. 531–539) are devoted to determinant expressions for the sum of the r^{th} powers of the divisors of an integer, the sum of the r^{th} powers of the uneven divisors, and so forth. These expressions are even less attractive and less promising than those connected with partitions which the same author gave with a caution in 1879.

CROCCHI, L. (1885).

[Question 52. *Giornale di Mat.*, xxiii. p. 33.]

The interesting result here given is that

$$\begin{vmatrix} \sigma_1 & -1 & . & . & \dots \\ \sigma_2 & \sigma_1 & -1 & . & \dots \\ \sigma_3 & \sigma_2 & \sigma_1 & -1 & \dots \\ . & . & . & . & . \end{vmatrix}_r = n^{n+r} - n_1(n-1)^{n+r} + n_2(n-2)^{n+r} - \dots,$$

where $\sigma_p = 1^p + 2^p + 3^p + \dots + n^p$. No proof appears to have been elicited.

It may be well to draw attention to the fact that the determinant is essentially positive.

STUDNIČKA, F. J. (1886).

[O novém neodvislém vyjádření čísel Bernujských a o vlastnostech přislušného determinantu. *Časopis pro pěstování math. a fys.* xv. pp. 97–102.]

[O nejjednodušším odvození součinitelů řady, převratnou hodnotu polynomu stupně n-tého ojedné proměnné představující. *Časopis pro pěstování math. a fys.*, xv. pp. 170–178.]

The result in the first paper is Hammond's of 1875. The second paper obtains in the original way the simpler of Bruno's rediscovered results of 1883 and gives easy illustrations. (*Hist.*, iii. pp. 232–233; iv. p. 229.)

DICKSTEIN, S. (1886).

[O niektórych własnościach funkcyj *alef*. *Pamietnik Akad. . . . w Krakowie*, xii. pp. 35–40.]

[O twierdzeniu Crocchiego. *Pamietnik Akad. . . . w Krakowie*, xii. pp. 41–44.]

[Dowód dwóch wzorów Wronskiego. *Pamietnik Akad. . . . w Krakowie*, xii. pp. 87–92.]

The second and third of these notes, like the first, concern the aleph functions. Nothing fresh regarding the determinant forms of the functions is brought forward.

GORDON, A. (1887).

[Question 9016. *Educ. Times*, xl. p. 135; or *Math. from Educ. Times*, lix. pp. 78–79.]

An old result (see above, p. 225).

MACKENZIE,* J. L. (1887).

[A theorem in algebra. *Proceed. Edinburgh Math. Soc.*, v. pp. 59–61.]

In illustrating the solution of a problem in the construction of equations, the simple example is given that, if the $s_1, s_2, \ldots, s_n$ of an equation of the n^{th} degree be known, the equation is obtained by eliminating the a's from

$$\left.\begin{array}{ll} x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n & = 0 \\ s_1 + a_1 & = 0 \\ s_2 + a_1 s_1 + 2a_2 & = 0 \\ \ldots\ldots\ldots\ldots\ldots & \end{array}\right\}.$$

In other instances where the conditions are more complex but where the resulting determinants are still recurrents evaluation is carried out.

LAISANT, C. A. (1889).

[Sur un déterminant remarquable. *Bull. Soc. Math. de France*, xvii. pp. 104–107.]

The determinant is Spottiswoode's of 1853, representing a binary quantic (*Hist.*, ii. p. 211). Laisant makes an addition regarding the differential-coefficients of the quantic.

DUPUIS, N. F. (1889).

[Expression of the general Bernoullian number as a combinational determinant. *Proceed. and Tr. R. Soc. Canada*, vii. (3) pp. 20–21.]

The expression is essentially the same as one of Nägelsbach's of 1874 (*Hist.*, iii. p. 232), the full equivalence of the two resting on the curious fact that the multiplication of

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*Same writer as McKenzie in *Educ. Times.*

$$\begin{vmatrix} 1 & (3)_2 & . & . & \dots \\ 3 & (5)_2 & (5)_4 & . & \dots \\ 5 & (7)_2 & (7)_4 & (7)_6 & \dots \\ 7 & (9)_2 & (9)_4 & (9)_6 & \dots \\ . & . & . & . & . \end{vmatrix}_n$$

by 2^{2n-2} is effected by changing the first column into

$$1, 2, 3, 4, \dots .$$

GUBLER, E. (1890).

[Ueber eine Determinante, welche bei der Berechnung symmetrischer Funktionen vorkommt. *Vierteljahrschrift d. naturf. Ges.* (Zürich), xxxv. pp. 79–82.]

The result here given is

$$\begin{vmatrix} (m+2)_1 & 1 & . & . & \dots & . \\ (m+4)_2 & (m+4)_1 & 1 & . & \dots & . \\ (m+6)_3 & (m+6)_2 & (m+6)_1 & 1 & \dots & . \\ . & . & . & . & . & . \end{vmatrix}_n = \frac{(m+2n)(m+n-1)!}{m!\,n!}$$

the evaluation being reached by repeatedly lowering the grade of the determinant and removing a factor.

DUPORCQ, E. (1890): TEIXEIRA, J. P. (1893).

[Sur la somme des puissances semblables des n premiers nombres. *Nouv. Annales de Math.*, (3) ix. pp. 594–595; *Periodico di Mat.*, vi. pp. 97–98.]

[Sur les nombres Bernoulliens. *Jorn. de Sci. Math.* . . . (Lisboa) (2) iii. pp. 73–75.]

The result in the first paper here is

$$1^n + 2^n + 3^n + \dots + (x-1)^n = (-1)^n \begin{vmatrix} x & 1 & . & . & . & \dots \\ x^2 & 1 & 2 & . & . & \dots \\ x^3 & 1 & (3)_1 & (3)_2 & . & \dots \\ x^4 & 1 & (4)_1 & (4)_2 & (4)_3 & \dots \\ . & . & . & . & . & . \end{vmatrix}_{n+1} \div (n+1)!$$

and obvious use is made of it for the same purpose as that of the second paper. In neither case is there any fresh outcome.

SCHENDEL, L. (1891).

[Mathematische Miscellen. *Zeitschrift f. Math. u. Phys.*, xxxvi. pp. 302–308.]

What is here given is a not very helpful way of stating Faure's result of 1855 regarding the partial quotient of

$$c_0x^m + c_1x^{m-1} + \dots . \quad \text{by} \quad a_0x^n + a_1x^{n-1} + \dots .$$

and a still less helpful way of expressing the remainder.

GAMBIOLI, D. (1891).

[Sopra alcune relazioni fra le funzioni simmetriche e sopra particolari loro caratteri invariantivi. *Giornale di Mat.*, xxix. pp. 41–60.]

[Sopra lo sviluppo secondo le potenze crescenti di x della frazione $1/(a_0+a_1x+\dots+a_nx^n)$ ove è $a_0 = 1$. *Giornale di Mat.*, xxx. pp. 35–39.]

The first paper deals at great length with the relations between the three sets of functions, the c's and the s's and the alephs; but so far at least as the six determinants are concerned gives nothing new. The latter remark applies also to the other paper. (See *Hist.*, iii. pp. 246–247.)

The second part of the subject is continued in the same journal, vol. xxx., pp. 192–206.

MARTONE, M. (1891).

[Le funzioni Alef di Hoëne Wronski. 20 pp. Catanzaro.]

This is an attempt to construct a theory of a subsection of algebra with a view to making better known the merits of Wronski. In regard to the related determinants nothing new is brought forward, but the pamphlet has the merit of being a clearly-written compendium.

ESCHERICH, G. v. (1892).

[Bestimmung einer Determinante. *Monatshefte f. Math. u. Phys.*, iii. pp. 19–20.]

The determinant referred to differs from Spottiswoode's special

recurrent of 1853 in having the x's all different and the y's all different: for example,

$$\begin{vmatrix} a & -y_1 & . & . \\ b & x_1 & -y_2 & . \\ c & . & x_2 & -y_3 \\ d & . & . & x_3 \end{vmatrix} = \begin{matrix} ax_1x_2x_3 \\ +by_1x_2x_3 \\ +cy_1y_2x_3 \\ +dy_1y_2y_3. \end{matrix}$$

SEGAR, H. W. (1892).

[On the multinomial determinant. *Messenger of Math.*, (2) xxi. pp. 177–188.]

The fundamental theorem here is that if

$$(a_0 + a_1x + a_2x^2 + \ldots)^n = (A_0 + A_1x + A_2x^2 + \ldots)^m$$

then

$$A_r = \frac{a_0^{n/m-r}}{r!\,m^r} \begin{vmatrix} na_1 & -ma_0 & . & . & \ldots \\ 2na_2 & (n-m)a_1 & -2ma_0 & . & \ldots \\ 3na_3 & (2n-m)a_2 & (n-2m)a_1 & -3ma_0 & \ldots \\ 4na_4 & (3n-m)a_3 & (2n-2m)a_2 & (n-3m)a_1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_r$$

We must note, however, that it is essentially included in David's of 1882, as may be seen on extracting the m^{th} root of both sides of the given equality and distributing the divisor m^r over the rows, one m to each row. On the other hand, it is much more readily seen to include David's, which is got from it by merely putting m equal to 1.

In illustrations and deductions the author is very fertile. From the equality

$$\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right)^{-n} = \left(1 + \frac{E_1}{2!}x + \frac{E_2}{4!}x^2 + \ldots\right)^n$$

he derives

$$(-)^r \begin{vmatrix} \frac{n}{2!} & 1 & . & \ldots \\ \frac{2n}{4!} & \frac{n+1}{2!} & 2 & \ldots \\ \frac{3n}{6!} & \frac{2n+1}{4!} & \frac{n+2}{2!} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}_r = \begin{vmatrix} \frac{n}{2!}E_1 & -1 & . & \ldots \\ \frac{2n}{4!}E_2 & \frac{n-1}{2!}E_1 & -2 & \ldots \\ \frac{3n}{6!}E_3 & \frac{2n-1}{4!}E_2 & \frac{n-2}{2!}E_1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

with, of course, a companion derived from the equality

$$\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots.\right)^{-n} = \left(1-\frac{1}{2}x+\frac{B_1}{2!}x^2-\frac{B_2}{4!}x^4+\ldots\right)^n.$$

By means of similar determinants he gives expressions for

$$\mathrm{L}_{x=0}\frac{d^{2r}}{dx^{2r}}\left(\frac{\sin x}{x}\right)^n, \ldots.,$$

for the first $n+1$ terms of

$$\exp(a_0+a_1x+a_2x^2+\ldots), \quad \ldots.,$$

and for the zonal and tesseral harmonics. His second section he bases on a similar expression for

$$\frac{d^r}{dx^r}\{f(x)\}^n,$$

and in his third section he deals in like manner with

$$\frac{d^r}{dx^r}\{\mathrm{E}(x)\mathrm{X}^n(x)\}.$$

REICH, K. (1892).

[Zur Theorie der quadratischen Reste. *Archiv d. Math. u. Phys.*, (2) xi. pp. 176–192.]

Incidentally here there are evaluated several recurrents, the main two of which are specially interesting, namely,

$$\begin{vmatrix} 1 & -2 & . & . & . & \ldots. \\ (3)_1 & 1 & -4 & . & . & \ldots. \\ (5)_2 & (3)_1 & 1 & -6 & . & \ldots. \\ (7)_3 & (5)_2 & (3)_1 & 1 & -8 & \ldots. \\ . & . & . & . & . & . \end{vmatrix}_n$$

and the one differing from this merely in having no negative elements. Calling the former R_n and the latter S_n, the author shows that

$$(2n+1)\mathrm{R}_n = (4n-3)(4n-1)\mathrm{R}_{n-1},$$

whence $$\mathrm{R}_n = (2n+3)(2n+5)\ldots.(4n-1):$$

and similarly that

$$(2n-1)\mathrm{S}_n = -(4n-5)(4n-3)\mathrm{S}_{n-1},$$

whence $$\mathrm{S}_n = (-1)^{n-1}(2n+1)(2n+3)\ldots.(4n-3).$$

SEGAR, H. W. (1892).

(See under this heading in Chap. XIII.)

SAALSCHÜTZ, L. (1892, 1893).

[Kettenbrüche, Bernoulli'sche Zahlen und Determinanten. *Bericht . . . d. phys.-ökon Ges.* (Königsberg), 1892, pp. 44–47.]

[Vorlesungen über die Bernoulli'schen Zahlen, ... viii.+208 pp. Berlin.]

The first of these devotes a page or so to 'the formation of determinants which can be evaluated with the help of Bernoulli's numbers.' This, however, is really not anything different from what we have seen others call 'expressing Bernoulli's numbers by means of a determinant.' As the recurrent which is reached involves a variable, a variety of such expressions is available.

The Lectures include these results, and, of course, much else unconnected with determinants.

CALDARERA, G. (1893).

[Sviluppo di un determinante particolare ad n variabili. *Atti dell' Accad. Gioenia . . .* (Catania), (4) vii., no. 8, 15 pp.]

The determinant here is the persymmetric recurrent of Brioschi (1858) and Trudi (1862). By the method employed the evaluation of the six-line instance occupies two pages. (*Hist.*, iii. pp. 208–209 and p. 214.)

HAUSSNER, R. (1894).

[Independente Darstellung der Bernoulli'schen und Euler'schen Zahlen durch Determinanten. *Zeitschrift f. Math. u. Phys.*, xxxix. pp. 183–188.]

This may be viewed as a determinantal appendix to the author's lengthy paper * of the previous year on the general theory of Bernoulli's numbers. The main results, however, which he gives date back at least to 1875 (Hammond).

* *Nachrichten . . . Ges. d. Wiss.* (Göttingen), 1893, pp. 777–809.

VERNIER, P. (1895): JAMET, V. (1898).

[Question 448. *L'Intermédiaire des Math.*, ii. p. 15; iii. pp. 162–163.]

[Sur la division des polynômes entiers. *Annales . . . Fac. des Sci.* (Marseilles), viii. pp. 151–161.]

In neither of these is there anything of fresh interest. The result in the first is a determinant expression for $(h+x)^n$ and included in Pincherle's rediscovery of 1881 (see above, p. 226). The subject of the second is that dealt with by Faure (1885) and his successors, to some of whom only a few pages back considerable space has been granted. (See above, p. 229.)

STUDNIČKA, F. J. (1897).

(See under this heading in Chap. VI.)

MANGEOT, S. (1897).

[Sur un mode de développement en série des fonctions algébriques explicites. *Annales de l'Ec. Norm.*, (3) xiv. pp. 247–250.]

The form of the function here in effect developed is

$$\{f_1(x)\}^{m_1}\{f_2(x)\}^{m_2}\ldots\{f_r(x)\}^{m_r},$$

and as examples there are given

$$(a_0+a_1x+a_2x^2+\ldots)^{-1} = \frac{1}{a_0}-\frac{G_1}{{a_0}^2}x+\frac{G_2}{2!\,{a_0}^3}x^2-\frac{G_3}{3!\,{a_0}^4}x^3+\ldots,$$

where

$$G_n \equiv \begin{vmatrix} 2a_1 & a_0 & . & \ldots & . \\ 4a_2 & 3a_1 & 2a_0 & \ldots & . \\ 6a_3 & 5a_2 & 4a_1 & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (2n-2)a_{n-1} & (2n-3)a_{n-2} & (2n-4)a_{n-3} & \ldots & (n-1)a_0 \\ na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \ldots & a_1 \end{vmatrix};$$

and

$$(a_0+a_1x+a_2x^2+\ldots)^{\frac{1}{2}} = \sqrt{a_0}+\frac{H_1}{1!\,2^1a^{1-\frac{1}{2}}}x-\frac{H_2}{2!\,2^2a^{2-\frac{1}{2}}}x^2+\frac{H_3}{3!\,2^3a^{3-\frac{1}{2}}}x^3-\ldots,$$

where

$$H_n \equiv \begin{vmatrix} 1a_1 & 2a_0 & . & \ldots. & . \\ 2a_2 & 3a_1 & 4a_0 & \ldots. & . \\ 3a_3 & 4a_2 & 5a_1 & \ldots. & . \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (n-1)a_{n-1} & na_{n-2} & (n+1)a_{n-3} & \ldots. & (2n-2)a_0 \\ na_n & (n-1)a_{n-1} & (n-2)a_{n-2} & \ldots. & 1a_1 \end{vmatrix}.$$

It must be noted that G_n differs markedly in form from that obtained by Faure (1855), and that there thus arises a very curious equality.

BOURLET, C. (1897).

[Sur un déterminant remarquable. *Nouv. Annales de Math.*, (3) xvi. pp. 369–373.]

To include the result

$$\begin{vmatrix} n!\,a_0 & -n & . & . & \ldots. & . & . \\ (n-1)!\,a_1 & x & -(n-1) & . & \ldots. & . & . \\ (n-2)!\,a_2 & . & x & -(n-2) & \ldots. & . & . \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 2!\,a_{n-2} & . & . & . & \ldots. & -2 & . \\ a_{n-1} & . & . & . & \ldots. & -x & -1 \\ a_n & . & . & . & \ldots. & . & x \end{vmatrix} = n!(a_0x^n+a_1x^{n-1}+\ldots+a_n),$$

and Laisant's (so-called) of 1889, a generalisation is here made, which for the 4th order is

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ -\beta_1 & \beta_1x-\gamma_1 & \gamma_1x-\delta_1 & \delta_1x \\ -\beta_2 & \beta_2x-\gamma_2 & \gamma_2x-\delta_2 & \delta_2x \\ -\beta_3 & \beta_3x-\gamma_3 & \gamma_3x-\delta_3 & \delta_3x \end{vmatrix} = |\beta_1\gamma_2\delta_3|.(a_0x^3+a_1x^2+a_2x+a_3).$$

CAZZANIGA, T. (1898).

[Sul calculo di qualche determinante numerico. *Giornale di Mat.*, xxxvi. pp. 362–367.]

Note is here made that Mangeot's developments being sometimes obtainable otherwise, the values of his determinants in such cases may be known: for example, when $a_r=1/r!$ his G_n, H_n are 1, $(-1)^{n+1}$ respectively.

LAISANT, C. A. (1898).

[Application géométrique d'une proposition d'algèbre. *Assoc. franç. pour l'avancem. des sci.* ii. pp. 73–76.]

The proposition referred to concerns the recurrent

$$\begin{vmatrix} f^{(n-1)} & f^{(n)} & . & . & \dots \\ 2f^{(n-2)} & f^{(n-1)} & f^{(n)} & . & \dots \\ 3f^{(n-3)} & f^{(n-2)} & f^{(n-1)} & f^{(n)} & \dots \\ . & . & . & . & . \end{vmatrix}_n$$

which is independent of x when $f(x)$ is of the same degree as the determinant.

CHAPTER IX.

WRONSKIANS, FROM 1880 TO 1899.

CONSIDERABLY increased interest under this heading has to be reported, the number of writings being threefold the number for the period of the preceding volume.

CASORATI, F. (1880).

[Il calcolo delle differenze finite interpretato ed accresciuto di nuovi teoremi. . . . *Annali di Mat.*, (2) x. pp. 10–45, 154–157.]

The second chapter (pp. 19–22) of this interesting memoir is occupied with the subject of linearly-related functions of a complex variable, the theorem arrived at being that *the necessary and sufficient condition for* $y_1, y_2, \ldots, y_n$ *being connected by a linear homogeneous relation with 'monotropic' coefficients* is

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ Ey_1 & Ey_2 & \cdots & Ey_n \\ \cdots & \cdots & \cdots & \cdots \\ E^{n-1}y_1 & E^{n-1}y_2 & \cdots & E^{n-1}y_n \end{vmatrix} = 0.$$

As the operator E is equivalent to $1+\Delta$, we may put the condition also in the form

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \Delta y_1 & \Delta y_2 & \cdots & \Delta y_n \\ \cdots & \cdots & \cdots & \cdots \\ \Delta^{n-1}y_1 & \Delta^{n-1}y_2 & \cdots & \Delta^{n-1}y_n \end{vmatrix} = 0,$$

which perhaps makes more readily evident the nature of the generalization effected. In regard to the proof we need only note that the

less easy part of it,—the deduction of the relation from the vanishing of the determinant,—is accomplished by establishing first the case where n is 2, and then using Hermite's condensation-theorem of 1849 to make any one case dependent on the case before it.

BURNSIDE, W. S. AND PANTON, A. W. (1881).

[The Theory of Equations, . . . xvi.+387 pp. Dublin.]

The easily generalizable result on p. 259 may be worth noting namely,

$$\mathrm{W}(a_1x^3+3a_2x^2+3a_3x+a_4,\ b_1x^3+\ldots,\ c_1x^3+\ldots) = -18\begin{vmatrix} 1 & -x & x^2 & -x^3 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

MALET, J. C. (1882).

[Question 7008. *Educ. Times*, xxxv. p. 105: *Math. from Educ. Times*, xxxvii. pp. 119–120.]

The result established here is that if

$$c_1y_1 + c_2y_2, \quad c_3y_3 + c_4y_4$$

be the solutions of

$$\frac{\partial^2 y}{\partial x^2} + \mathrm{P}y = 0, \quad \frac{d^2y}{\partial x^2} + \mathrm{Q}y = 0$$

respectively, then

$$\mathrm{W}(y_1, y_2, y_3, y_4) = (\mathrm{P} - \mathrm{Q})^2 \times \text{constant}.$$

MANSION, P. (1884).

[Rapport sur la "Démonstration élémentaire de la loi suprème de Wronski" par C. Lagrange. *Bulletin . . . Acad. roy. de Belgique*, (3) viii. pp. 165–172.]

The report regarding the demonstration is itself interesting, and especially when read along with Cayley's proof of 1872 (*Hist.*, iii. pp. 249–250); it is, however, a theorem contained in an appended 'Note' that really calls for record. In effect this is that *if in the triangular array*

$$\begin{array}{ccccccc} x & y & z & u & v & \cdots \\ & y_1 & z_1 & u_1 & v_1 & \cdots \\ & & z_2 & u_2 & v_2 & \cdots \\ & & & u_3 & v_3 & \cdots \\ & & & & v_4 & \cdots \\ & & & & & \cdots \end{array}$$

the elements of the first row be mutually independent functions of t, *and any other element be got by dividing the derivate of the element above it in the immediately preceding row by the derivate of the first element of that row, then each derived element is expressible as the quotient of two wronskians of the derivates of elements of the first row.* By merely performing the requisite operations it is readily found that

$$z_2,\ u_2,\ v_2 = \frac{W(x', z')}{W(x', y')},\ \frac{W(x', u')}{W(x', y')},\ \frac{W(x', v')}{W(x', y')},$$

and supposing that in the same way

$$u_3,\ v_3 = \frac{W(x', y', u')}{W(x', y', z')},\ \frac{W(x', y', v')}{W(x', y', z')},$$

Mansion shows, as follows, that

$$v_4 = \frac{W(x', y', z', v')}{W(x', y', z', u')}.$$

For the wronskians that express u_3, v_3 he uses temporarily U, V, R and thus has

$$(u_3)' = \frac{RU' - UR'}{R^2},\quad (v_3)' = \frac{RV' - VR'}{R^2},$$

and consequently by definition

$$v_4 = \frac{RV' - VR'}{RU' - UR'}.$$

But

$$RV' - VR' = \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix} \cdot \begin{vmatrix} x' & y' & v' \\ x'' & y'' & v'' \\ x^{iv} & y^{iv} & v^{iv} \end{vmatrix} - \begin{vmatrix} x' & y' & v' \\ x'' & y'' & v'' \\ x''' & y''' & v''' \end{vmatrix} \cdot \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}$$

$$= \begin{vmatrix} x' & y' & z' & v' \\ x'' & y'' & z'' & v'' \\ x''' & y''' & z''' & v''' \\ x^{iv} & y^{iv} & z^{iv} & v^{iv} \end{vmatrix} \cdot \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}$$

$$= W(x', y', z', v') \cdot W(x', y'):$$

and similarly

$$RU' - UR' = W(x', y', z', u') \cdot W(x', y').$$

By division there is thus obtained

$$v_4 = W(x', y', z', v')/W(x', y', z', u')$$

as desired.

STARKOFF, A. (1884).

[Two formulae in the theory of determinants (in Russian). *Annals Univ. Kasan.*, iii. pp. 175–179.]

The first is the old theorem included in Schweins' of 1825 (*Hist.*, i. p. 484) regarding the differentiation of a wronskian; and the second which concerns the differentiation of

$$W(y_1, \ldots, y_{m-1}, y_{m+1})/W(y_1, \ldots, y_{m-1}, y_m),$$

is also already known, being included in a theorem of Frobenius of 1873 (*Hist.*, iii. p. 252).

DICKSTEIN, S. (1888, 1897).

[Wlasności i niektore zastosowania wrońskianów. *Prace math.-fiz.*, i. pp. 5–25.]

[Propriedades y algunas relaciones de las wronskianas. *Archivo de Mat.*, ii. pp. 102–108, 128–132, 141–145.]

What we have here is just such a methodical exposition of the main known properties as would suitably fill the place of a chapter of a largish text-book of determinants. The second paper is a translation of the first.

PEANO, G. (1889).

[Sur le déterminant Wronskien. *Mathesis*, ix. pp. 75–76, 110–112.]

The subject here is the inaccuracy of the usual form of the theorem regarding the vanishing of a Wronskian,—a theorem first referred to by us under Puiseux (1851). One of the examples given as sufficient disproof is

$$W(x^2, x \bmod x).$$

Peano would accept as a substitute the proposition : *If*

$$W(y_1, y_2, \ldots, y_n)$$

vanishes identically, so also does the determinant of the array got by giving any n *values to the independent variable in* $y_1, y_2, \ldots, y_n$. He is apparently unaware of Christoffel's enunciation of 1857 (*Hist.*, ii. pp. 225–228).

KONIGSBERGER, L. (1889).

[Ueber eine Determinantenbeziehung in der Theorie der Differentialgleichungen. *Crelle's Journ.*, cv. pp. 170–179.]

The subject here is the successive differentiation of a Wronskian, $W(y_1, y_2, \ldots, y_{n+1})$. As the y's are the same throughout, it suffices to specify only the differentiations, W being thus denoted by

$$|\, 0,\ 1,\ \ldots,\ n{-}1,\ n\,|.$$

We then readily obtain

$$\frac{dW}{dx} = |\, 0,\ 1,\ \ldots,\ n-1,\ n+1\,|,$$

$$\frac{d^2W}{dx^2} = |\, 0,\ 1,\ \ldots,\ n-2,\ n,\ n+1\,| + |\, 0,\ 1,\ \ldots,\ n-1,\ n+2\,|,$$

$$\frac{d^3W}{dx^3} = |\, 0,\ 1,\ \ldots,\ n-3,\ n-1,\ n,\ n+1\,| + 2\,|\, 0,\ 1,\ \ldots,\ n-2, n,\ n+2\,| + |\, 0,\ 1,\ \ldots,\ n-1,\ n+3\,|,$$

and foresee with the author what the succeeding differentiations will bring, although the statement of the general result may be difficult to put concisely.

DEMOULIN, A. (1889, 1897) : NEUBERG, J. (1894).

[Remarque sur une propriété fondamentale des Wronskiens. *Mathesis*, ix. p. 136.]

[Démonstration de la propriété fondamentale des Wronskiens. *Mathesis*, (2) vii. pp. 62–63.]

[Sur les Wronskiens. *Mathesis*, (2) iv. p. 165.]

These concern an already known property dealt with by Christoffel and Frobenius.

SEGAR, H. W. (1891).

[A theorem in determinants. *Messenger of Math.*, (2) xx. pp. 141–142.]

The theorem is that *if* f(t) *be a rational integral algebraical expression of the* n^th *degree, then*

$$\begin{vmatrix} \mathrm{D}\frac{1}{\mathrm{f(t)}} & \mathrm{D}\frac{\mathrm{t}}{\mathrm{f(t)}} & \mathrm{D}\frac{\mathrm{t}^2}{\mathrm{f(t)}} & \cdots \\ \mathrm{D}^2\frac{1}{\mathrm{f(t)}} & \mathrm{D}^2\frac{\mathrm{t}}{\mathrm{f(t)}} & \mathrm{D}^2\frac{\mathrm{t}^2}{\mathrm{f(t)}} & \cdots \\ \mathrm{D}^3\frac{1}{\mathrm{f(t)}} & \mathrm{D}^3\frac{\mathrm{t}}{\mathrm{f(t)}} & \mathrm{D}^3\frac{\mathrm{t}^2}{\mathrm{f(t)}} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{\mathrm{n}} = (-1)^{\mathrm{n}} \frac{\mathrm{n!(n-1)!(n-2)!} \cdots}{\{\mathrm{f(t)}\}^{\mathrm{n+1}}}.$$

It is reached by investigating the case of Borchardt's theorem of 1855 (*Hist.*, ii. pp. 173–175) where $t_1 = t_2 = \ldots = t_n = t$. We have inserted the necessary sign-factor.

LAURENT, H. (1888): MARTONE, M. (1891): ECHOLS, W. H. (1893).

[Série de Wronski. *Traité d'Analyse*, iii. pp. 352–356.]

[I determinanti Wronskiani e la legge suprema. 41 pp. Catanzaro.]

[Wronski's expansion. *Bull. New York Math. Soc.*, ii. pp. 178–184.]

The third writer throws some light on the constitution of the coefficients of the expansion and of the determinants involved in the same: but the paper has considerable additional interest in being a short, sane and fair account of the whole of Wronski's mathematical work.

Martone's pamphlet I have not seen.

TORELLI, G. (1893).

[Sui determinanti di funzioni. *Rendic. del Circolo Mat.*, vii. pp. 75–84.]

Keeping in view his generalization regarding Jacobians, Torelli takes into consideration the corresponding Wronskian

$$\mathrm{W}(u_1v_1,\ u_2v_2,\ \ldots,\ u_nv_n),$$

but only for the case where the u's are all powers of a common function, and even then only deals with the question of the divisibility of the Wronskian by a power of the said common function.

LECORNU, L. (1894).

[Sur les déterminants wronskiens. *Assoc. Franç.* . . . (Caen), xxiii. pp. 282–285.]

The author here replies to a criticism made by Peano (*Intermédiaire des Math.*, i. pp. 12–13, 157–158) on similar lines to those above indicated (pp. 245–246), the object being not to refute utterly but to minimize.

GRÉVY, A. (1894.)

[Étude sur les équations fonctionnelles. *Annales sci. de l'Éc. Norm. Sup.*, (3) xi. pp. 249–323.]

As the solutions of functional equations have in form an analogy with the solutions of linear differential equations, and the analogy continues to persist when the *relations* between the solutions come to be studied, one is prepared for the appearance also of an analogue to the Wronskian. On this account a reference to Chap. VI. (pp. 301–317) of Grévy's extensive memoir is considered desirable.

BORTOLOTTI, E. (1896).

[Sui determinanti di funzioni nel calcolo alle differenze finite. *Rendic.* . . . *Accad. dei Lincei*, (5) v. (1) pp. 254–261.]

Being aware of the study of the properties of $\mathrm{W}(y_1, \ldots, y_n)$ the author here thinks to take a fresh step by investigating the properties of

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ \Delta y_1 & \Delta y_2 & \cdots & \Delta y_n \\ \cdots & \cdots & \cdots & \cdots \\ \Delta^{n-1} y_1 & \Delta^{n-1} y_2 & \cdots & \Delta^{n-1} y_n \end{vmatrix}, \quad \text{or} \quad \mathrm{V}(y_1, y_2, \ldots, y_n) \text{ say,}$$

not knowing, apparently, of Casorati's work of 1880, or that even Wronski began with finite differences and was led to differentials later. His aim clearly is to discover theorems regarding V corresponding to those regarding W : indeed Frobenius' paper of 1873 on Wronskians is kept steadily before him as a source of suggestion and as a model. He thus begins with

$$\mathrm{V}(yy_1, yy_2, \ldots, yy_n) = y \cdot \mathrm{E}y \cdot \mathrm{E}^2 y \ldots \mathrm{E}^{n-1} y \cdot \mathrm{V}(y_1, y_2, \ldots, y_n)$$

where E is the operation $1 + \Delta$; passes on to

$$\mathrm{V}(u_1, u_2, \ldots, u_\mu, v_1, v_2, \ldots, u_\nu)$$

$$= \frac{\mathrm{V}\{\mathrm{V}(u_1, \ldots, u_\mu, v_1),\ \mathrm{V}(u_1, \ldots, u_\mu, v_2), \ldots,\ \mathrm{V}(u_1, \ldots, u_\mu, v_\nu)\}}{\mathrm{EV}(u_1, \ldots, u_\mu) \cdot \mathrm{E}^2\mathrm{V}(u_1, \ldots, u_\mu) \ldots \mathrm{E}^{\nu-1}\mathrm{V}(u_1, \ldots, u_\mu)} :$$

and then deals with the set of functions defined by

$$z_k \equiv (-1)^{n+k} \cdot \mathrm{V}(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)/\mathrm{V}(y_1, \ldots, y_n),$$

justifying the application of the name *adjunct* to the set, and establishing results which differ from their analogues merely in having V for W.

PINCHERLE, S. (1897).

[Sulla generalizzazione della proprietà del determinante Wronskiano. *Atti . . . Accad. dei Lincei . . . Rendic.* (5) vi. (1) pp. 301–307.]

From a study of the proofs of the various known condition-theorems regarding linearly-related functions the author here notes that in each case the operator prominently involved (d/dx, Δ, S) besides being distributive admits of a common type of multiplication-theorem. He thereupon takes advantage of the opening thus suggested towards generalization, and succeeds in showing that a more general operator than d/dx may be substituted in the original theorem regarding the Wronskian, and likewise a more general operator than S in Grévy's theorem. The combined result put in one enunciation is to the effect that if any operator O which admits of the said form of multiplication-theorem take the place of d/dx in the Wronskian, the coefficients in the linear relation have to be altered to 'coefficients periodic in O.'

PEANO, G. (1897): VIVANTI, G. (1898).

[Sul determinante Wronskiano. *Atti . . . Accad. dei Lincei . . . Rendic.*, (5) vi. (1) pp. 413–415; vii. (1) pp. 194–197.]

After again restating his objection of 1889, Peano substitutes the theorem that *if for all the values of the independent variable belonging to a certain interval the Wronskian vanishes, and no value within the said interval causes the cofactors of all the elements of the last row of*

the Wronskian to vanish, then there exists between the functions a homogeneous linear relation with coefficients that are constant and not all zero. The proof given is unnecessarily far-fetched and is not strictly determinantal.

The object of Vivanti's paper is to supply a proof not open to this objection. There is also given in it a variant form of the theorem.

BORTOLOTTI, E. (1898).

[Sulla generalizzazione della proprietà del determinante Wronskiano. *Atti . . . Accad. dei, Lincei . . . Rendic.*, (5) vii. (1) pp. 45–50.]

It is here sought to improve on Pincherle's result of the preceding year, and with the help of several preparatory lemmas proof is given that *if* $\phi_1, \phi_2, \ldots, \phi_n$ *be analytic functions having a common domain of convergence, and* O *be a functional operation that is distributive and uniquely determinate, the necessary and sufficient condition for the functions being connected by a linear homogeneous relation with coefficients constant with respect to* O *is*

$$|\phi_1 \cdot \mathrm{O}\phi_2 \cdot \mathrm{O}^2\phi_3 \ldots . \mathrm{O}^{n-1}\phi_n| \equiv 0.$$

STUDNIČKA, F. J. (1899).

[Ueber eine neue Art von Derivations-Determinanten. *Monatshefte f. Math. u. Phys.*, x. pp. 338–342.]

This is again a rediscovery, the determinant referred to being Sylvester's persymmetric Wronskian of 1862. Several special functions of x are taken in succession as the basic function—the $(1, 1)^{\text{th}}$ element—and evaluation effected. Of these the interesting one is x^{-m}, in connection with which the author is led to the result

$$\mathrm{P}(1,\ a^{1/d},\ a^{2/d},\ \ldots,\ a^{2n-2/d}) = d^{(n)_2} \prod_{k=1}^{n-1} k!\, a^{k/d},$$

where there is put

$$a^{k/d} \text{ for } a(a+d)(a+2d)\ldots\{a+(k-1)d\}.$$

The paper seems to be a second edition of the third part of a paper published in Czech in 1898 (*Věstník České Akad. . . .* vii. pp. 477–493).

CHAPTER X.

JACOBIANS, FROM 1880 TO 1899.

This is one of the very few special forms which in the period under review received less attention than in the preceding period, the number of papers being about a third fewer.

BRUNO, F. FAÀ DI (1880).

[Notes on modern algebra. 3. Sur une propriété du Jacobien. *American Journ. of Math.*, iii. pp. 154–163.]

[Trois notes sur la théorie des formes : 2. Sur le Jacobien des formes binaires. *Math. Annalen*, xviii. pp. (280–288) 283–286.]

The third note is occupied with a proof of a theorem of Clebsch's, namely, that *the square of the Jacobian of two binary quantics is a homogeneous quadratic function of the quantics.* In the main the proof depends on Euler's theorems regarding the differentiation of homogeneous functions. The given quantics, u and v, being of the m^{th} and n^{th} degrees respectively, we have

$$mn\mathrm{J} = mn\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} xu_{xx}+yu_{xy}+u_x & xu_{xy}+yu_{yy}+u_y \\ xv_{xx}+yv_{xy}+v_x & xv_{xy}+yv_{yy}+v_y \end{vmatrix}$$

$$= x^2\begin{vmatrix} u_{xx} & u_{xy} \\ v_{xx} & v_{xy} \end{vmatrix} + xy\begin{vmatrix} u_{xx} & u_{yy} \\ v_{xx} & v_{yy} \end{vmatrix} + y^2\begin{vmatrix} u_{xy} & u_{yy} \\ v_{xy} & v_{yy} \end{vmatrix}$$

$$+ x\begin{vmatrix} u_{xx} & u_y \\ v_{xx} & v_y \end{vmatrix} + x\begin{vmatrix} u_x & u_{xy} \\ v_x & v_{xy} \end{vmatrix} + y\begin{vmatrix} u_{xy} & u_y \\ v_{xy} & v_y \end{vmatrix} + y\begin{vmatrix} u_x & u_{yy} \\ v_x & v_{yy} \end{vmatrix}$$

$$+ \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} + yx\begin{vmatrix} u_{xy} & u_{xy} \\ v_{xy} & v_{xy} \end{vmatrix}$$

$$= \begin{vmatrix} y^2 & -xy & x^2 \\ u_{xx} & u_{xy} & u_{yy} \\ v_{xx} & v_{xy} & v_{yy} \end{vmatrix} + (m+n-2)\,\mathrm{J} + \mathrm{J},$$

so that the three-line determinant here is equal to $(m-1)(n-1)\mathrm{J}$. It only then remains to multiply this determinant by

$$\begin{vmatrix} x^2 & 2xy & y^2 \\ u_{yy} & -2u_{xy} & u_{xx} \\ v_{yy} & -2v_{xy} & v_{xx} \end{vmatrix}$$

when we obtain for $-2(m-1)^2(n-1)^2\mathrm{J}^2$ an axisymmetric determinant having for its first row

$$0 \quad m(m-1)u \quad n(n-1)v$$

and therefore being of the form desired.

GILBERT, PH. (1880).

[Sur une propriété de la fonction de Poisson, ... *Comptes Rendus . . . Acad. des Sci.* (Paris), xci. pp. 541–544.]

All that need be said in regard to this is that for the statement of the property in question the use of secondary minors of a Jacobian is essential.

PAIGE, C. LE (1881).

[Sur le déterminant fonctionnel d'un nombre quelconque de formes binaires. *Comptes Rendus . . . Acad. des Sci.* (Paris), xcii. pp. 688–690: *Bull. . . . Acad. . . . de Belgique*, (3) i. pp. 490–499, 461–462.]

Le Paige's subject is the same as Bruno's, and his method is quite similar, but in his hands Clebsch's theorem is widely generalized.* Further, there is the interesting point that when the number of quantics is odd the determinant obtained for J^2 is zero-axial skew, and thus can have its root extracted. His two theorems are in effect:

1. *The square of the Jacobian of* 2k *binary quantics each of degree not less than* 2k *is a homogeneous quadratic function of the quantics, the coefficients of the function being sums of products of related covariants.*

* Readers specially interested in the algebra of quantics will find extensions by d'Ovidio, Torelli, and Gerbaldi in

Atti . . . Accad. . . . Torino, xiv. (1879) pp. 963–972;
Rendic . . . Accad. . . . Napoli, xxv. (1886) pp. 125–134;
Giornale di Mat., xxvii. (1889) pp. 33–39.

Connected also with the algebra of quantics is a paper by Giordano in

Giornale di Mat., xxxv. (1897) pp. 349–353.

2. *The Jacobian of* 2k+1 *binary quantics each of degree not less than* 2k+1 *is a homogeneous linear function of the quantics, the coefficients of the function being sums of products of related covariants.*

By way of proof the second theorem is verified for the case where the quantics are

$$u \equiv (a_0, a_1, \ldots \S x, y)^6,$$
$$v \equiv (b_0, b_1, \ldots \S x, y)^5,$$
$$w \equiv (c_0, c_1, \ldots \S x, y)^4.$$

In regard to this, however, it has to be pointed out that the serious work of squaring the multiple

$$\begin{vmatrix} y^3 & (a_0, \ldots \S x, y)^3 & (b_0, \ldots \S x, y)^2 & c_0x + c_1y \\ -y^2x & (a_1, \ldots \S x, y)^3 & (b_1, \ldots \S x, y)^2 & c_1x + c_2y \\ yx^2 & (a_2, \ldots \S x, y)^3 & (b_2, \ldots \S x, y)^2 & c_2x + c_3y \\ x^3 & (a_3, \ldots \S x, y)^3 & (b_3, \ldots \S x, y)^2 & c_3x + c_4y \end{vmatrix}$$

of the Jacobian is not at all necessary, as the operation

$$x^3\,\mathrm{col}_1 + 3x^2y\,\mathrm{col}_2 + 3xy^2\,\mathrm{col}_3 + \mathrm{col}_4$$

performed on this determinant changes the first column into

$$0, \quad u, \quad v, \quad w.$$

PASCH, M. (1881).

[Notiz über ternäre Formen mit verschwindender Functionaldeterminante. *Math. Annalen*, xviii. pp. 93–94.]

The result here is that if the quantics have no common factor and be of the n^{th} degree, there exists between them an irreducible equation of the ν^{th} degree where ν is a factor of n.

MUIR, T. (1882).

[Question 6961. *Educ. Times*, xxxv. pp. 25, 104 : *Math. from Educ. Times*, xxxvii. pp. 58–59.]

In accordance with the fact that the Jacobian of the functions

$$\begin{aligned} V_1 &\equiv \quad -ay - bz - cu - dv, \\ V_2 &\equiv ax \quad\quad - ez - fu - gv, \\ V_3 &\equiv bx + ey \quad\quad - hu - kv, \\ V_4 &\equiv cx + fy + hz \quad\quad - lv, \\ V_5 &\equiv dx + gy + kz + lu \end{aligned}$$

vanishes, it is shown that a relation exists between the functions, this being

$$\begin{vmatrix} a & b & c & d & V_1 \\ & e & f & g & V_2 \\ & & h & k & V_3 \\ & & & l & V_4 \\ & & & & V_5 \end{vmatrix} = 0.$$

JACOBI, C. G. J. (1884).

[Vorlesungen über Dynamik. Zweite, revidirte Ausgabe. viii+300 pp. (*Gesammelte Werke*, Supplementband.)]

The thirteenth lecture (pp. 100–106) concerns those properties of the Jacobian which are familiar in the special case of the ordinary differential-coefficient.

It must be noted that the first edition appeared in 1866, and that the lectures were originally delivered in 1842–3.

KRAUS, L. (1884).

[Ueber Functionaldeterminanten. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), xc. pp. 813–826.]

The object here is to establish the following theorem and its converse: *If the Jacobian of two binary quantics be a non-zero constant, then the independent variables are rational integral functions of the quantics.*

PEANO, G. (1884).

[Calcolo Differenziale, e principii di calcolo integrale: (per Angelo Gennocchi), pubblicato con aggiunte dal Dr. Giuseppe Peano. xxxii+338 pp. Torino.]

In giving Bertrand's definition of a Jacobian (*Hist.*, ii. pp. 237–238) we carefully left the burden of legitimizing it on the author. Here, on p. xxvii. of his *Annotazioni*, Peano pointedly draws attention to its inaccuracy. Taking the simplest case, namely, where the quotient to be considered is

$$\begin{vmatrix} f_1(x_1+h_1, x_2+h_2) - f_1(x_1, x_2) & f_2(x_1+h_1, x_2+h_2) - f_2(x_1, x_2) \\ f_1(x_1+k_1, x_2+k_2) - f_1(x_1, x_2) & f_2(x_1+k_1, x_2+k_2) - f_2(x_1, x_2) \end{vmatrix} \div \begin{vmatrix} h_1 & h_2 \\ k_1 & k_2 \end{vmatrix}$$

he points out that, if h_2 were taken equal to h_1 and k_2 equal to k_1, the denominator would be permanently 0, whereas the numerator would not in general be so, and that therefore the quotient would not then tend to any limit at all. He adds, however, that his objection would not hold if the f's were connected by a linear relation with constant coefficients, or were quotients of linear functions of the x's and had a common denominator.

MANSION, P. (1884).

[Discours sur les travaux mathématiques de M. Eugène-Charles Catalan : III. Déterminants et intégrales multiples. *Mém. . . . Soc. roy. des Sci.* (Liège), (2) xii. No. 1, 38 pp.]

A statement of Catalan's contributions to the subject, including a pronouncement (pp. 10–12) on a question of priority as between Catalan and Jacobi (*Hist.*, i. pp. 354–358.)

STIELTJES, T. J. (1885).

[Sur une généralisation de la série de Lagrange. *Annales de l'Éc. norm. sup.*, (3) ii. pp. 93–98.]

In the present connection this is only interesting because the left-hand member of Lagrange's equality in the form

$$f(\mathrm{X})\frac{d\mathrm{X}}{dx} = \sum_0^\infty \frac{a^m}{m!}\frac{d^m}{dx^m}[f(x)\phi^m(x)],$$

where $\mathrm{X} = x + a\phi(x)$, is extended to

$$f(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \ldots)\frac{\partial(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \ldots)}{\partial(x, y, z, \ldots)}$$

where $\mathrm{X} = x + a\phi(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \ldots)$, $\mathrm{Y} = y + b\psi(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \ldots)$, $\mathrm{Z} = z + c\chi(\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \ldots)$, etc.

PEANO, G. (1889).

[Su d'una proposizione riferentesi ai determinanti jacobiani. *Giornale di Mat.*, xxvii. pp. 226–228.]

Peano resumes here his criticism of Bertrand's definition; and, besides supporting his former contention, offers three substitutes, the least geometrical being in effect : *If* u_1, u_2, u_3 *be, each of them, functions of* x_1, x_2, x_3, *and the determinant of the increments of the* x*'s be*

$$\begin{vmatrix} h_1 t & h_2 t & h_3 t \\ h_1' t' & h_2' t' & h_3' t' \\ h_1'' t'' & h_2'' t'' & h_3'' t'' \end{vmatrix}$$

where the h*'s are fixed quantities, and* $| h_1 \; h_2' \; h_3'' |$ *is not zero, then the limit of the ratio of the dependent to the independent determinant of increments as* t, t′, t″ *tend to zero is the Jacobian*

$$\partial(u_1, u_2, u_3)/\partial(x_1, x_2, x_3).$$

TEIXEIRA, F. G. (1890).

[Extension d'un théorème de Jacobi. *Monatshefte f. Math. u. Phys.*, i. pp. 481–484.]

The extension simply but carefully dealt with is included in Trzaska's of 1871 (*Hist.*, iii. p. 267).

PRIME, F. (1892).

[Sur un théorème de Jacobi. *Mathesis*, (2) ii. p. 227.]

This in only a variant of § 18 of the *De determinantibus functionalibus* (*Hist.*, i. pp. 391–2).

TORELLI, G. (1893).

[Sui determinanti di funzioni. *Rendic. del Circolo Mat.*, vii. pp. 75–84.]

Torelli carries on Casorati's work of 1874. Of his six theorems, all attended by corollaries, the first may be viewed as fundamental, the others being either direct deductions from it or analogues suggested by it. It is that

$$\mathrm{J}(u_1 v_1, u_2 v_2, \ldots, u_n v_n)$$

$$= \begin{vmatrix} u_1 & . & \ldots & . & -v_1 & . & \ldots & . \\ . & u_2 & \ldots & . & . & -v_2 & \ldots & . \\ . & . & \ldots & . & . & . & \ldots & . \\ . & . & \ldots & u_n & . & . & \ldots & -v_n \\ \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_1} & \ldots & \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial v_1}{\partial x_1} & \dfrac{\partial v_2}{\partial x_1} & \ldots & \dfrac{\partial v_n}{\partial x_1} \\ . & . & \ldots & . & . & . & \ldots & . \\ \dfrac{\partial u_1}{\partial x_n} & \dfrac{\partial u_2}{\partial x_n} & \ldots & \dfrac{\partial u_n}{\partial x_n} & \dfrac{\partial v_1}{\partial x_n} & \dfrac{\partial v_2}{\partial x_n} & \ldots & \dfrac{\partial v_n}{\partial x_n} \end{vmatrix}$$

where the determinant array is best looked at *in quarters*, two of which are the arrays of

$$\mathrm{J}(u_1,\ u_2,\ \ldots,\ u_n),\quad \mathrm{J}(v_1,\ v_2,\ \ldots,\ v_n).$$

When all the u's are powers of a common function w, the order falls to the $(n+1)^{\text{th}}$, and a factor is separable ; also, further specialization leads to a case where the order falls to the n^{th}, the determinant factor being indeed the Jacobian of the v's.

The first analogue concerns the pre-jacobian

$$\mathrm{K}(u_0v_0,\ u_1v_1,\ \ldots,\ u_nv_n),$$

where the number of the u's and of the v's is now $n+1$, but the independent variables are the same as before. Similar specializations are made.

Like Casorati, Torelli also goes on to the Wronskian.

We note for ourselves here a curious-looking equality, which for the sixth degree is

$$\begin{vmatrix} \lambda & . & . & -l & . & . \\ . & \mu & . & . & -\mu & . \\ . & . & \nu & . & . & -\nu \\ a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & \beta_1 & \beta_2 & \beta_3 \\ c_1 & c_2 & c_3 & \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} = \begin{Vmatrix} l & . & . & \lambda & . & . \\ . & m & . & . & \mu & . \\ . & . & n & . & . & \nu \end{Vmatrix} \cdot \begin{Vmatrix} a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 & \beta_1 & \beta_2 & \beta_3 \\ c_1 & c_2 & c_3 & \gamma_1 & \gamma_2 & \gamma_3 \end{Vmatrix},$$

both members being equal to

$$\begin{vmatrix} la_1 + \lambda a_1 & ma_2 + \mu a_2 & na_3 + \nu a_3 \\ lb_1 + \lambda\beta_1 & mb_2 + \mu\beta_2 & nb_3 + \nu\beta_3 \\ lc_1 + \lambda\gamma_1 & mc_2 + \mu\gamma_2 & nc_3 + \nu\gamma_3 \end{vmatrix}.$$

JACOBI, C. G. J. (1896).

[Ueber die Functionaldeterminanten. Herausg. von P. Stäckel. 72 pp. Leipzig.]

At the close of this translation of the *De determinantibus functionalibus* the editor appends 13 pages of useful notes. The volume belongs to the same series as that noted on pages 63 and 189 above.

AUTONNE, L. (1897).

[Sur un certain jacobien. *Nouv. Annales de Math.*, (3) xvi. pp. 376–379.]

The n functions here are

$$(a_{10}+a_{11}x_1+\ldots+a_{1n}x_n)/(a_{00}+a_{01}x_1+\ldots+a_{0n}x_n),$$
$$(a_{20}+a_{21}x_1+\ldots+a_{2n}x_n)/(a_{00}+a_{01}x_1+\ldots+a_{0n}x_n),$$
$$(a_{30}+a_{31}x_1+\ldots+a_{3n}x_n)/(a_{00}+a_{01}x_1+\ldots+a_{0n}x_n),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

and their Jacobian is found to be

$$|\,a_{00}a_{11}a_{22}\ldots a_{nn}\,|/(a_{00}+a_{01}x_1+\ldots+a_{0n}x_n)^{n+1}.$$

In the proof Chio's condensation-theorem of 1853 (*Hist.*, ii. p. 80) might have been used with effect.

BERRY, A. (1898): CRAWFORD, L. (1899).

[On the evaluation of a certain determinant which occurs in the mathematical theory of statistics and in that of elliptic geometry of any number of dimensions. *Proceed. Cambridge Philos. Soc.*, x. pp. 2–10.]

[On the evaluation of a certain determinant. *Proceed. Edinburgh Math. Soc.*, xviii. pp. 25–27.]

The determinant in question is the third in order of generation from a unit-axial axisymmetric determinant $|\,r_{1n}\,|$, the first being the adjugate $|\,\mathrm{R}_{1n}\,|$, the second $|\,\mathrm{R}'_{1n}\,|$ where $\mathrm{R}'_{ij} \equiv \mathrm{R}_{ij}/\sqrt{\mathrm{R}_{ii}\mathrm{R}_{jj}}$, and the third the Jacobian

$$\frac{\partial(\mathrm{R}'_{12},\ \mathrm{R}'_{13},\ \ldots,\ \mathrm{R}'_{n-1,\,n})}{\partial(r_{12},\ r_{13},\ \ldots,\ r_{n-1,\,n})},$$

the order of which is $\frac{1}{2}n(n-1)$. The peculiarity of Crawford's treatment of the case where n is 3 lies in taking for the non-diagonal elements $\cos a$, $\cos b$, $\cos c$, where a, b, c are the sides of a spherical triangle. Berry seeks the analogue of this in n-dimensional space, and attains a solution, the value found for the Jacobian being

$$(-1)^{\frac{1}{2}n(n-1)}\cdot\left\{|\,r_{1n}\,|^{n-2}/\mathrm{R}_{11}\mathrm{R}_{22}\ldots\mathrm{R}_{nn}\right\}^{\frac{1}{2}(n+1)}.$$

The main point of the procedure consists in showing that it is equal to

$$\frac{\partial(\mathrm{R}'_{12},\ \mathrm{R}'_{13},\ \ldots,\ \mathrm{R}'_{n-2,\,n-1})}{\partial(r_{12},\ r_{13},\ \ldots,\ r_{n-2,\,n-1})}\ \div\ \frac{\partial(r_{1n},\ r_{2n},\ \ldots,\ r_{n-1,\,n})}{\partial(\mathrm{R}_{1n},\ \mathrm{R}'_{2n},\ \ldots,\ \mathrm{R}'_{n-1,\,n})}.$$

CHAPTER XI.

SKEW DETERMINANTS AND PFAFFIANS, FROM 1880 TO 1899.

THE subject of this chapter, quite unlike that of the preceding, shows a great advance in popularity, the number of writings for the twenty-year period having quite doubled. It is also worthy of note that the number of languages employed in them has more than doubled, there being now Italian, Russian, Polish, Czech in addition to English, German, French.

ZAJACZKOWSKI, W. (1880).

[O pewnéj wlasnosci pfafiana. *Rozpravy . . . Akad. . . . Krakowie*, viii. pp. 67–74.]

[A theorem relating to Pfaffians. *Messenger of Math.*, x. pp. 36–37.]

This is the identity given and verified by Tanner in support of his asserted theorem of 1878, and is the exact analogue of the determinant identity which in 1873–74 Muir called * 'extension of Laplace's expansion-theorem'—that is to say, it expresses *the product of a Pfaffian and one of its own minors as a sum of products of pairs of minors.* An example in the old notation is

$$\begin{aligned}[1, 2, 3, \ldots, 8]\,[5, 6, 7, 8] \;=\;& [3, 4, 5, 6, 7, 8]\,[1, 2, 5, 6, 7, 8] \\ &- [2, 4, 5, 6, 7, 8]\,[1, 3, 5, 6, 7, 8] \\ &+ [2, 3, 5, 6, 7, 8]\,[1, 4, 5, 6, 7, 8],\end{aligned}$$

and another in later notation is

$$\begin{vmatrix} a & b & c & d & e \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & o \end{vmatrix} \cdot o = \begin{vmatrix} j & k & l \\ & m & n \\ & & o \end{vmatrix} \cdot \begin{vmatrix} a & d & e \\ & h & i \\ & & o \end{vmatrix} - \begin{vmatrix} g & h & i \\ & m & n \\ & & o \end{vmatrix} \cdot \begin{vmatrix} b & d & e \\ & k & l \\ & & o \end{vmatrix} + \begin{vmatrix} f & g & i \\ & k & l \\ & & n \end{vmatrix} \cdot \begin{vmatrix} c & d & e \\ & m & n \\ & & o \end{vmatrix}.$$

* *Proceed. R. Soc. Edinburgh*, viii. p. 23 ; *Transac.*, xxix. pp. 47–51.

In Zajaczkowski's proof the Pfaffian is viewed as the square root of a zero-axial skew determinant, Cayley having suggested that such a proof would be possible though difficult.

GUNTHER, S. (1880): LENGAUER, . (1880).

[Ueber einen Satz von den symmetralen Determinanten. *Blätter f. d. bairischen Gymn.-und-Realschulwesen*, xvi. pp. 310–314.]

[Zum Beweise des Cayley'schen Satzes von den symmetralen Determinanten. *Blätter . . .*, xvi. pp. 423–424.]

Both papers concern Cayley's familiar theorem. By the first writer a six-line example is taken and, without altering the diagonal, is transformed so as to have zeros in the $(2, 4)^{th}$, $(3, 4)^{th}$, $(4, 5)^{th}$, $(4, 6)^{th}$, $(3, 5)^{th}$, $(5, 6)^{th}$ places and the places conjugate to these, the resulting determinant being thus expressible as the product of a pair of 3-line determinants which are found to differ at most in sign only. The second writer improves on the work of the first, but still further improvement is readily seen to be possible. And such is worth making, because of the fresh and instructive character of the procedure.

MUIR, T. (1881).

[On Professor Cayley's theorem regarding a bordered skew symmetric determinant. *Quart. Journ. of Math.*, xviii. pp. 46–49.]

Here the subject is for the first time treated independently of Jacobi's 'umbral' notation, the variables, in whatever way they may be symbolized, being put in full evidence. It is also incidentally stated that a Pfaffian as thus written can be defined exactly after the manner in which a determinant is ordinarily defined, and that there thence can be developed a complete theory of the subject in perfect analogy with the theory of determinants.

Start is made with the manifest identity

$$\begin{vmatrix} a+\alpha & b+\beta & c+\gamma & d+\delta & e+\epsilon \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & o \end{vmatrix} = \begin{vmatrix} a & b & c & d & e \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & o \end{vmatrix} + \begin{vmatrix} \alpha & \beta & \gamma & \delta & \epsilon \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & o \end{vmatrix},$$

or say

$$\mathrm{F} = f + \phi;$$

and both sides being squared, the determinant equivalents are substituted for F^2, f^2, ϕ^2. Then, the substitute for F^2, having a row and a column containing binomial elements, is partitioned into four determinants, when it is found that one is that substituted for f^2, one that substituted for ϕ^2, and that the other two are equal. There is thus at once obtained

$$\begin{vmatrix} . & a & b & c & d & e \\ -a & . & f & g & h & i \\ -\beta & -f & . & j & k & l \\ -\gamma & -g & -j & . & m & n \\ -\delta & -h & -k & -m & . & o \\ -\epsilon & -i & -l & -n & o & . \end{vmatrix} = \begin{vmatrix} a & b & c & d & e \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & o \end{vmatrix} \cdot \begin{vmatrix} a & \beta & \gamma & \delta & \epsilon \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & o \end{vmatrix}.$$

The bordering of the determinant with a row and column having *zero* for their common element indicates no want of generality, as the cofactor of the said common element is itself zero. Further, it is pointed out that on putting $a, \beta, \gamma, \delta, \epsilon = 0, 0, 0, 0, 1$, we obtain

$$\begin{vmatrix} e & a & b & c & d \\ i & . & f & g & h \\ l & -f & . & j & k \\ n & -g & -j & . & m \\ o & -h & -k & -m & . \end{vmatrix} = \begin{vmatrix} a & b & c & d & e \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & o \end{vmatrix} \cdot \begin{vmatrix} f & g & h \\ & j & k \\ & & m \end{vmatrix},$$

which is the so-called second case. In it the one Pfaffian is a primary minor of the other, being, indeed, the Pfaffian of which the given four-line determinant is the square; and, if the first column of the determinant be removed to the last place, the arrays of the two Pfaffians are seen to be separated by the line of zeros.

SYLVESTER, J. J. (1881-1889).

[Questions 6795, 6826. *Educ. Times*, xxxiv. pp. 217, 245; *Math. from Educ. Times*, xxxvi. pp. 97–98, 117–118.]

Proof is here given that the series

$$1 + 1 \cdot \mathrm{C}_{n,1} + 1 \cdot 5 \mathrm{C}_{n,2} + 1 \cdot 5 \cdot 9 \mathrm{C}_{n,3} + \ldots .$$

used in Sylvester's previous paper on the subject is, when summed, divisible by 2^n, the quotient being the determinant

$$\begin{vmatrix} 1 & 1 & . & . & .... \\ 1 & 3 & 2 & . & .... \\ . & 1 & 5 & 3 & .... \\ . & . & 1 & 7 & .... \\ ..... \end{vmatrix} \quad \text{or} \quad v_n \text{ say.}$$

Also that the greatest common factor of v_n and v_{n+1} is 2^k, where k is the greatest integer in $\frac{1}{4}(n+2)$.

MUIR, T. (1881).

[On new and recently discovered properties of certain symmetric determinants. *Quart. Journ. of Math.*, xviii. pp. 166–177.]

The product of two n-line Pfaffians is here (§ 12) expressed as a skew symmetric Pfaffian of the $2n^{\text{th}}$ order ; for example,

$$\begin{vmatrix} a_2 & a_3 & a_4 \\ & \beta_3 & \beta_4 \\ & & \gamma_4 \end{vmatrix} \cdot \begin{vmatrix} x_2 & x_3 & x_4 \\ & y_3 & y_4 \\ & & z_4 \end{vmatrix}$$

$$= \frac{1}{2^4} \begin{vmatrix} a_2+x_2 & a_3+x_3 & a_4+x_4 & a_4-x_4 & a_3-x_3 & a_2-x_2 & 0 \\ & \beta_3+y_3 & \beta_4+y_4 & \beta_4-y_4 & \beta_3-y_3 & 0 & x_2-a_2 \\ & & \gamma_4+z_4 & \gamma_4-z_4 & 0 & y_3-\beta_3 & x_3-a_3 \\ & & & 0 & z_4-\gamma_4 & y_4-\beta_4 & x_4-a_4 \\ & & & & -z_4-\gamma_4 & -y_4-\beta_4 & -x_4-a_4 \\ & & & & & -y_3-\beta_3 & -x_3-a_3 \\ & & & & & & -x_2-a_2 \end{vmatrix}$$

and, conversely,

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & 0 \\ & b_3 & b_4 & b_5 & b_6 & 0 & -a_7 \\ & & c_4 & c_5 & 0 & -b_6 & -a_6 \\ & & & 0 & -c_5 & -b_5 & -a_5 \\ & & & & -c_4 & -b_4 & -a_4 \\ & & & & & -b_3 & -a_3 \\ & & & & & & -a_2 \end{vmatrix} = \begin{vmatrix} a_2+a_7 & a_3+a_6 & a_4+a_5 \\ & b_3+b_6 & b_4+b_5 \\ & & c_4+c_5 \end{vmatrix}$$

$$\times \begin{vmatrix} a_2-a_7 & a_3-a_6 & a_4-a_5 \\ & b_3-b_6 & b_4-b_5 \\ & & c_4-c_5 \end{vmatrix}.$$

From the latter two deductions are made, namely, *A skew symmetric Pfaffian of odd order vanishes*, and, *A skew symmetric Pfaffian of even order is equal to the difference of two squares.**

MUIR, T. (1881).

[On skew determinants. *Philos. Magazine*, (5) xii. pp. 391–394.]

The main result here is *the representation of any determinant as a Pfaffian*; for example,

$$|a_1b_2c_3| = \begin{vmatrix} \frac{1}{2}(a_2-b_1) & \frac{1}{2}(a_3-c_1) & \frac{1}{2}(a_3+c_1) & \frac{1}{2}(a_2+b_1) & a_1 \\ & \frac{1}{2}(b_3-c_2) & \frac{1}{2}(b_3+c_2) & b_2 & \frac{1}{2}(a_2+b_1) \\ & & & c_3 & \frac{1}{2}(b_3+c_2) & \frac{1}{2}(a_3+c_1) \\ & & & & \frac{1}{2}(b_3-c_2) & \frac{1}{2}(a_3-c_1) \\ & & & & & \frac{1}{2}(a_2-b_1) \end{vmatrix}.$$

It is obtained from a wider result by the same author giving an expression for the product of any two determinants of the n^{th} order as a determinant of the $2n^{\text{th}}$ order † (p. 12 above). From Brioschi's result and the present it is deduced that *an axisymmetric Pfaffian of the* $2n^{th}$ *order is expressible as a Pfaffian of the* n^{th} *order*: and there is the further deduction that *the product of two axisymmetric Pfaffians is expressible as an axisymmetric Pfaffian.*

Centre-skew determinants are then taken up. First it is noted as self-evident that a centre-skew determinant of the $2n^{\text{th}}$ order is simply a centro-symmetric determinant with the sign-factor $(-1)^n$ attached: and it is then proved that *when of the* $(2m+1)^{th}$ *order*

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* 'Order' as here used implies that $2n$-line Pfaffians are of the n^{th} order: for later usage see below, p. 265.

† The converse proposition is this time taken for granted; it is given merely as an exercise in the author's text-book of the same date (p. 206, ex. 17). We may add the example:

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & b_4 & a_4 \\ & & & b_3 & a_3 \\ & & & & a_2 \end{vmatrix} = \begin{vmatrix} a_6 & a_5+a_2 & a_4+a_3 \\ a_5-a_2 & b_5 & b_4+b_3 \\ a_4-a_3 & b_4-b_3 & c_4 \end{vmatrix},$$

where, be it observed, the matrix of the determinant is the sum of an axisymmetric and a zero-axial skew matrix.

the determinant is expressible as the product of the central element and two m*-line determinants*, for example,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 \\ d_1 & d_2 & d_3 & \omega & -d_3 & -d_2 & -d_1 \\ -c_7 & -c_6 & -c_5 & -c_4 & -c_3 & -c_2 & -c_1 \\ -b_7 & -b_6 & -b_5 & -b_4 & -b_3 & -b_2 & -b_1 \\ -a_7 & -a_6 & -a_5 & -a_4 & -a_3 & -a_2 & -a_1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1-a_7 & a_2-a_6 & a_3-a_5 \\ b_1-b_7 & b_2-b_6 & b_3-b_5 \\ c_1-c_7 & c_2-c_6 & c_3-c_5 \end{vmatrix} \cdot \omega \cdot (-1)^3 \begin{vmatrix} a_1+a_7 & a_2+a_6 & a_3+a_5 \\ b_1+b_7 & b_2+b_6 & b_3+b_5 \\ c_1+c_7 & c_2+c_6 & c_3+c_5 \end{vmatrix}.$$

From this it follows that *An odd-ordered determinant which is skew with respect to a zero-centre vanishes*, and that *a centre-skew determinant of odd order is not altered by making all the elements of the middle row and middle column zero with the exception of the centre element.*

MUIR, T. (1881).

[A Treatise on the Theory of Determinants, with graduated sets of exercises, ... vii+240 pp. London.]

The fourteen-page chapter here devoted to skew determinants is considerably different from the corresponding chapters of previous text-books, the reason mainly being that Pfaffians are viewed more in the light of independent functions having properties similar to those of determinants, and therefore being best treated by the use of similar notation and nomenclature. Thus the Pfaffian

$$af - be + cd,$$

which is equal to

$$\begin{vmatrix} . & a & b & c \\ -a & . & d & e \\ -b & -d & . & f \\ -c & -e & -f & . \end{vmatrix}^{\frac{1}{2}},$$ is denoted by $$\begin{vmatrix} a & b & c \\ & d & e \\ & & f \end{vmatrix};$$

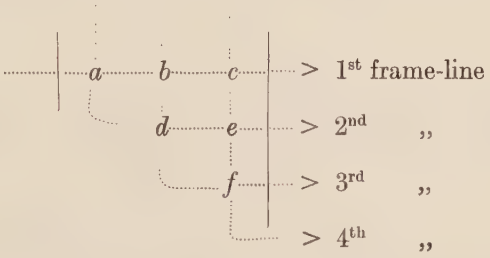
the Pfaffian $a_2c_4 - a_3b_4 + a_4b_3$ in the same way or more shortly by

$$^1|\, a_2b_3c_4 \,|\,;$$

and the Pfaffian $a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}$ in either of these ways, or still more shortly by

$$\| a_{14} |$$

—all in as close analogy as possible with the notation of determinants. Rows and columns, however, are not spoken of but *frame-lines*; for example, the frame-lines of the first of the above-written Pfaffians are



the Pfaffian of the *second degree* being thus a *four-line* Pfaffian. The place of any element is indicated by the numbers of the frame-lines passing through it; for example, d in the above occupies the $(2, 3)^{\text{th}}$ place. The terms *minor*, *adjugate*, *compound*, and others are taken over, and analogous notation adopted in connection with them. The *complementary minor* of an element is obtained by deleting the frame-lines to which the element belongs; for example, the complementary minors of e and g in

$$\begin{vmatrix} a & b & c & d & e \\ & f & g & h & i \\ & & j & k & l \\ & & & m & n \\ & & & & o \end{vmatrix}$$

are $\begin{vmatrix} f & g & h \\ & j & k \\ & & m \end{vmatrix}$ and $\begin{vmatrix} b & d & e \\ & k & l \\ & & o \end{vmatrix}$

respectively. Such complementaries are denoted by the corresponding capital letters, and the adjugate of $\| a_{12n} |$ by

$$\| A_{12n} |.$$

The recurrent law of formation thus comes to be written (ff being the suggested analogue for D or Δ)

$$ff(a_{1,2n}) = a_{12}ff_{(1,2} - a_{13}ff_{(1,3} + \ldots,$$

or
$$||\, a_{1,2n} \,|| = a_{12}\mathrm{A}_{12} + a_{13}\mathrm{A}_{13} + \ldots,$$

or
$$ff(a_{1,2n}) = a_{12}\frac{\partial ff}{\partial a_{12}} + a_{13}\frac{\partial ff}{\partial a_{13}} + \ldots;$$

the fundamental relation between determinants and Pfaffians appears as

$$|\, a_{1,2n} \,|_{\substack{a_{rs}=-a_{sr} \\ a_{rr}=0}} = ||\, a_{1,2n} \,||^2;$$

and Cayley's theorem regarding a primary minor of a zero-axial skew determinant takes the form:

If $\mathrm{D} = |\, a_{1,2n} \,|_{\substack{a_{rs}=-a_{sr} \\ a_{rr}=0}}$, then $\mathrm{D}^{(h}_{(k} = ff \cdot ff_{(h,k} \quad (h<k)$;

but if $\mathrm{D} = |\, a_{1,2n+1} \,|_{\substack{a_{rs}=-a_{sr} \\ a_{rr}=0}}$, then $\mathrm{D}^{(h}_{(k} = ff_{(h} \cdot ff_{(k}$.

Finally, the analogy with determinants is again insisted on, instances being given where the likeness is complete, and instances where more or less divergence in the two enunciations occurs. Among the latter, for example, appears the proposition that *if two adjacent frame-lines of a Pfaffian be transposed and their common element be changed in sign, the new Pfaffian differs only in sign from the original.*

At the end of the chapter there is a somewhat lengthy collection of exercises involving results like the following:

(5) $$\begin{vmatrix} a & b & c \\ & d & e \\ & & f \end{vmatrix} \cdot (\mu_1+\mu_2+\mu_3+\mu_4)$$

$$= \begin{vmatrix} a\mu_2 & b\mu_3 & c\mu_4 \\ & d & e \\ & & f \end{vmatrix} + \begin{vmatrix} a\mu_1 & b & c \\ & d\mu_3 & e\mu_4 \\ & & f \end{vmatrix} + \begin{vmatrix} a & b\mu_1 & c \\ & d\mu_2 & e \\ & & f\mu_4 \end{vmatrix} + \begin{vmatrix} a & b & c\mu_1 \\ & d & e\mu_2 \\ & & f\mu_3 \end{vmatrix}$$

(6) The axial term (the term on the perpendicular bisecting the hypotenuse) of a Pfaffian is always positive.

(10) The last frame-line of a Pfaffian may be passed over the others to occupy the first place without altering the value of the Pfaffian.

(12) The cofactor of a_{rs} in $ff(a_{1,2n})$ is $(-1)^{r+s-1} ff_{(r,s)}$.

(20)
$$\begin{vmatrix} x & a & b & c \\ -a & -n_1n_2x & n_1c & n_2b \\ -b & -n_1c & n_1n_3x & n_3a \\ -c & -n_2b & -n_3a & -n_2n_3x \end{vmatrix} = (n_3a^2 - n_2b^2 + n_1c^2 - n_1n_2n_3x^2)^2.$$

(21) If D_{pq} denote the determinant got from

$$\begin{vmatrix} . & a & b & c \\ -a & . & d & e \\ -b & -d & . & f \\ -c & -e & -f & . \end{vmatrix}$$

by multiplying the p^{th} row and the q^{th} column by $\mu_1, \mu_2, \mu_3, \mu_4$, then

$$D_{12} + D_{13} + D_{14}$$
$$= \begin{vmatrix} a\mu_2 & b\mu_3 & c\mu_4 \\ & d & e \\ & & f \end{vmatrix} \cdot \left\{ \begin{vmatrix} a\mu_1 & b & c \\ & d\mu_3 & e\mu_4 \\ & & f \end{vmatrix} + \begin{vmatrix} a & b\mu_1 & c \\ & d\mu_2 & e \\ & & f\mu_4 \end{vmatrix} + \begin{vmatrix} a & b & c\mu_1 \\ & d & e\mu_2 \\ & & f\mu_3 \end{vmatrix} \right\}.$$

MUIR, T. (1882).

[On a determinant formed by bordering the product of two determinants. *Messenger of Math.*, xi. pp. 161–165.]

As deductions from a general theorem we here learn that *if an odd-ordered zero-axial skew determinant be bordered symmetrically with* 0, x_1, x_2, x_3, ... , *and its square be bordered in exactly the same way, the latter resulting determinant contains the former as a factor*; for example, when the given determinant is

$$\begin{vmatrix} . & a & b & c & d \\ -a & . & e & f & g \\ -b & -e & . & h & k \\ -c & -f & -h & . & l \\ -d & -g & -k & -l & . \end{vmatrix}$$

the cofactor is

$$-\begin{vmatrix} e & f & g \\ & h & k \\ & & l \end{vmatrix}^2 - \begin{vmatrix} b & c & d \\ & h & k \\ & & l \end{vmatrix}^2 - \begin{vmatrix} a & c & d \\ & f & g \\ & & l \end{vmatrix}^2 - \begin{vmatrix} a & b & d \\ & e & g \\ & & k \end{vmatrix}^2 - \begin{vmatrix} a & b & c \\ & e & f \\ & & h \end{vmatrix}^2.$$

Also, *if an even-ordered zero-axial skew determinant be bordered symmetrically with* 0, x_1, x_2, x_3, . . . *the resulting determinant contains the original skew determinant as a factor* ; for example, when the given determinant is

$$\begin{vmatrix} . & a & b & c \\ -a & . & d & e \\ -b & -d & . & f \\ -c & -e & -f & . \end{vmatrix}$$

the cofactor is

$$-\begin{vmatrix} x_2 & x_3 & x_4 \\ & d & e \\ & & f \end{vmatrix}^2 - \begin{vmatrix} x_1 & x_3 & x_4 \\ & b & c \\ & & f \end{vmatrix}^2 - \begin{vmatrix} x_1 & x_2 & x_4 \\ & a & c \\ & & e \end{vmatrix}^2 - \begin{vmatrix} x_1 & x_2 & x_3 \\ & a & b \\ & & d \end{vmatrix}^2 .$$

The Pfaffians occurring in the two cofactors are conveniently viewed as minors got by deleting a frame-line of a quasi-Pfaffian.

MUIR, T. (1882).

[Note on the condensation of skew determinants which are partially zero-axial. *Proceed. London Math. Soc.*, xiii. pp. 161–164.]

The writer here starts with the skew determinant

$$\begin{vmatrix} x_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ -a_2 & x_2 & b_3 & b_4 & b_5 & b_6 \\ -a_3 & -b_3 & x_3 & c_4 & c_5 & c_6 \\ -a_4 & -b_4 & -c_4 & x_4 & d_5 & d_6 \\ -a_5 & -b_5 & -c_5 & -d_5 & x_5 & e_6 \\ -a_6 & -b_6 & -c_6 & -d_6 & -e_5 & . \end{vmatrix}, \quad \text{or } \mathrm{D}_1 \text{ say,}$$

having *one* zero in the diagonal ; changes it into

$$\frac{1}{a_6^4} \begin{vmatrix} x_1 & a_2a_6 & a_3a_6 & a_4a_6 & a_5a_6 & a_6 \\ -a_2 & x_2a_6 & b_3a_6 & b_4a_6 & b_5a_6 & b_6 \\ -a_3 & -b_3a_6 & x_3a_6 & c_4a_6 & c_5a_6 & c_6 \\ -a_4 & -b_4a_6 & -c_4a_6 & x_4a_6 & d_5a_6 & d_6 \\ -a_5 & -b_5a_6 & -c_5a_6 & -d_5a_6 & x_5a_6 & e_6 \\ -a_6 & -b_6a_6 & -c_6a_6 & -d_6a_6 & -e_6a_6 & . \end{vmatrix}$$

and multiplies it by x_1 in the form

$$\begin{vmatrix} 1 & . & . & . & . & . \\ . & . & . & . & . & x_1 \\ -b_6 & 1 & . & . & . & -a_2 \\ -c_6 & . & 1 & . & . & -a_3 \\ -d_6 & . & . & 1 & . & -a_4 \\ -e_6 & . & . & . & 1 & -a_5 \end{vmatrix},$$

obtaining the equality

$$D_1 = \frac{1}{x_1 a_6^4} \begin{vmatrix} x_1 & a_6x_1 & -b_6x_1 & -c_6x_1 & -d_6x_1 & -e_6x_1 \\ -a_2 & b_6x_1 & a_6x_2 & |a_2b_3c_6| & |a_2b_4d_6| & |a_2b_5e_6| \\ -a_3 & c_6x_1 & -|a_2b_3c_6| & a_6x_3 & |a_3c_4d_6| & |a_3c_5e_6| \\ -a_4 & d_6x_1 & -|a_2b_4d_6| & -|a_3c_4d_6| & a_6x_4 & |a_4d_5e_6| \\ -a_5 & e_6x_1 & -|a_2b_5e_6| & -|a_3c_5e_6| & -|a_4d_5e_6| & a_6x_5 \\ -a_6 & . & . & . & . & . \end{vmatrix},$$

and thence the desired result

$$x_1a_6^3D_1 = \begin{vmatrix} a_6x_1 & b_6x_1 & c_6x_1 & d_6x_1 & e_6x_1 \\ -b_6x_1 & a_6x_2 & |a_2b_3c_6| & |a_2b_4d_6| & |a_2b_5e_6| \\ -c_6x_1 & -|a_2b_3c_6| & a_6x_3 & |a_3c_4d_6| & |a_3c_5e_6| \\ -d_6x_1 & -|a_2b_4d_6| & -|a_3c_4d_6| & a_6x_4 & |a_4d_5e_6| \\ -e_6x_1 & -|a_2b_5e_6| & -|a_3c_5e_6| & -|a_4d_5e_6| & a_6x_5 \end{vmatrix}, \qquad (I)$$

where the new five-line determinant is still skew.

Next he takes a determinant, D_2, with *two* zeros in the diagonal, namely, the determinant got from D_1 on making $x_5 = 0$; and using (I) obtains at once for $x_1a_6^3D_2$ a five-line skew determinant with a zero in the last place. This is then treated like D_1, with the result

$$\frac{1}{a_6x_1\,(e_6x_1)^2} \begin{vmatrix} a_6e_6x_1^2 & a_6x_1|a_2b_5e_6| & a_6x_1|a_3b_5e_6| & a_6x_1|a_4b_5e_6| \\ -a_6x_1|a_2b_5e_6| & a_6e_6x_1x_2 & P & Q \\ -a_6x_1|a_3b_5e_6| & -P & a_6e_6x_1x_3 & R \\ -a_6x_1|a_4b_5e_6| & -Q & -R & a_6e_6x_1x_4 \end{vmatrix},$$

where P, Q, R are put temporarily for the compound Pfaffians

$$\begin{vmatrix} b_6x_1 & cx_1 & e_6x_1 \\ & |a_2b_3e_6| & |a_2b_5e_6| \\ & & |a_3c_5e_6| \end{vmatrix}, \quad \begin{vmatrix} b_6x_1 & d_6x_1 & e_6x_1 \\ & |a_2b_4d_6| & |a_2b_5e_6| \\ & & |a_4d_5e_6| \end{vmatrix}, \quad \begin{vmatrix} c_6x_1 & d_6x_1 & e_6x_1 \\ & |a_3c_4d_6| & |a_3c_5e_6| \\ & & |a_4d_5e_6| \end{vmatrix},$$

whose values are

$$x_1a_6|\,b_3c_5e_6\,|,\quad x_1a_6|\,b_4d_5e_6\,|,\quad x_1a_6|\,c_4d_5e_6\,|.$$

Substitution and simplification give finally

$$e_6{}^2\mathrm{D}_2 = \begin{vmatrix} e_6x_1 & |\,a_2b_5e_6\,| & |\,a_3c_5e_6\,| & |\,a_4d_5e_6\,| \\ -|\,a_2b_5e_6\,| & e_6x_2 & |\,b_3c_5e_6\,| & |\,b_4d_5e_6\,| \\ -|\,a_3c_5e_6\,| & -|\,b_3c_5e_6\,| & e_6x_3 & |\,c_4d_5e_6\,| \\ -|\,a_4d_5e_6\,| & -|\,b_4d_5e_6\,| & -|\,c_4d_5e_6\,| & e_6x_4 \end{vmatrix} \qquad \text{(II)}$$

Similar procedure leads to

$$x_1|\,a_4d_5e_6|\,\mathrm{D}_3 = \begin{vmatrix} x_1|\,a_4d_5e_6\,| & x_1|\,b_4d_5e_6\,| & x_1|\,c_4d_5e_6\,| \\ -x_1|\,b_4d_5e_6\,| & x_2|\,a_4d_5e_6\,| & |\,a_2b_3c_4d_5e_6\,| \\ -x_1|\,c_4d_5e_6\,| & -|\,a_2b_3c_4d_5e_6\,| & x_3|\,a_4d_5e_6\,| \end{vmatrix} \qquad \text{(III)}$$

and

$$\mathrm{D}_4 = \begin{vmatrix} x_1|\,c_4d_5e_6\,| & |\,a_2b_3c_4d_5e_6\,| \\ -|\,a_2b_3c_4d_5e_6\,| & x_2|\,c_4d_5e_6\,| \end{vmatrix}, \qquad \text{(IV)}$$

D_5 and D_6 needing only to be mentioned.

Attention is drawn to an interesting bye-product got from (II), by putting $x_1 = x_2 = x_3 = x_4 = 0$ and extracting the square root, namely,

$$e_6\,|\,a_2\ a_3\ a_4\ a_5\ a_6\,| = \begin{vmatrix} |\,a_2b_5e_6\,| & |\,a_3c_5e_6\,| & |\,a_4d_5e_6\,| \\ & |\,b_3c_5e_6\,| & |\,b_4d_5e_6\,| \\ & & |\,c_4d_5e_6\,| \end{vmatrix}$$

$$\begin{matrix} b_3 & b_4 & b_5 & b_6 \\ & c_4 & c_5 & c_6 \\ & & d_5 & d_6 \\ & & & e_6 \end{matrix}$$

$$= \begin{vmatrix} \mathrm{C}_4 & -\mathrm{B}_4 & \mathrm{A}_4 \\ & \mathrm{B}_3 & -\mathrm{A}_3 \\ & & \mathrm{A}_2 \end{vmatrix},$$

where the compound Pfaffian on the right is a minor of the adjugate of $|\,a_2b_3c_4d_5e_6\,|$, and the equality corresponds to Jacobi's theorem regarding such a minor in determinants.*

* For years after this paper appeared its contents were viewed as entirely new: it was only in 1900 when the third part of Cayley's paper of 1854 came to be interpreted by Muir that the former's identity was seen to be not essentially different from the first case above (*Hist.*, ii. pp. 270–272).

MUIR. T. (1885).

[Question 8059. *Educ. Times*, xxxviii. p. 36 ; lvii. p. 527 : *Math. from Educ. Times*, (2) vii. p. 97.]

The identity established here is

$$\begin{vmatrix} a_1+a_2 & a_1+a_3 & a_1+a_4 & \dots & a_1+a_{2n} \\ & a_2+a_3 & a_2+a_4 & \dots & a_2+a_{2n} \\ & & a_3+a_4 & \dots & a_3+a_{2n} \\ & & \cdot & \cdot & \cdot \\ & & & & a_{2n-1}+a_{2n} \end{vmatrix} = 2^{n-1}(a_1a_3 \dots a_{2n-1}+a_2a_4 \dots a_{2n}).$$

From consideration of the construction of the frame-lines Nanson infers that the result must be of a certain form, and then from consideration of the related zero-axial determinant he evaluates the unknowns of the form.

By splitting the binomials of the first frame-line the Pfaffian may be partitioned into two, one of which has a_1 for a factor ; and taking the other, we may in like manner express it as the sum of two, one of which has a_2 for a factor and the other vanishes.

TORELLI, G. (1886).

[Quistione 64. *Giornale di Mat.*, xxiv. p. 377.]

The identity here given is essentially that noted in Scott's text-book (p. 117), and, with the Pfaffian notation, in Muir's (pp. 206, 240). It is noteworthy as involving a fresh expression for the difference-product of an even number of quantities. It is

$$\begin{vmatrix} (a_1-a_2)^{2n-1} & (a_1-a_3)^{2n-1} & \dots & (a_1-a_{2n})^{2n-1} \\ & (a_2-a_3)^{2n-1} & \dots & (a_2-a_{2n})^{2n-1} \\ & & \cdot & \cdot \\ & & & (a_{2n-1}-a_{2n})^{2n-1} \end{vmatrix}$$

$$= (-1)^{\frac{1}{2}n(n+1)} \cdot (2n-1)_1(2n-1)_2 \dots (2n-1)_{n-1} \cdot |\, a_1^0 a_2^1 \dots a_{2n}^{2n-1} |.$$

In the absence of proof, we may point out that it can be derived from Zehfuss' theorem of 1859 (*Hist.*, ii. p. 190) by taking therein the same even number of variables for each set and the sum of every two corresponding variables equal to 0. It may be even more appropriate, however, to view it as a case of Brioschi's of 1855 (*Hist.*, ii. pp. 276–277).

MERTENS, F. (1887).

[Ueber windschiefe Determinanten. *Sitzungsb. . . . Akad.* (Wien), xcvi. II A, pp. 1245–1255.]

Still another demonstrator of Cayley's familiar theorem, the aim being like Veltmann's of 1871, namely, to derive it by direct consideration of the ordinary final development of the determinant (*Hist.*, iii. p. 273).

CATALAN, E. C. (1888).

[Calculs de déterminants. *Mém. . . . Soc. R. des Sci.* (Liège), (2) xv. pp. 176–178 ; or *Mélanges Math.*, iii. pp. 176–178.]

It is not observed here that the determinant

$$\begin{vmatrix} -h & f' & . & g \\ f & -h' & g & . \\ . & g' & h & f \\ g' & . & f' & h' \end{vmatrix}$$

can, by mere translation of rows and multiplication by $(-1)^2$, be written as zero-axial skew, and therefore is equal to

$$(gg'+hh'-ff')^2.$$

FOURET, G. (1889).

[Sur deux déterminants numériques. *Nouv. Annales de Math.*, (3) vii. pp. 82–85.]

The determinants here are

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ -1 & . & 1 & \dots & 1 \\ -1 & -1 & . & \dots & 1 \\ . & . & . & . & . \\ -1 & -1 & -1 & \dots & . \end{vmatrix}_n \qquad \begin{vmatrix} . & 1 & 1 & \dots & 1 \\ -1 & . & 1 & \dots & 1 \\ -1 & -1 & . & \dots & 1 \\ . & . & . & . & . \\ -1 & -1 & -1 & \dots & . \end{vmatrix}_n.$$

Calling them A_n and B_n respectively, we already know that when n is odd B_n is 0, and we can easily show that when n is even B_n is 1. From this it follows that A_n, being equal to B_n+B_{n-1}, is 1 whatever n may be.

DERUYTS, F. (1889).

[Sur une propriété des déterminants symétriques gauches. *Mém. . . . Soc. R. des Sci.* (Liège), (2) xvii. No. 2, 6 pp.]

The property in question is the third of those given by Frobenius in 1876 (*Hist.*, iii. pp. 275–277) as corollaries to a theorem regarding the vanishing of equigrade minors in a general determinant. An essential part of any proof is of course the fact that in such determinants, quite apart from other considerations, all the coaxial minors of odd order vanish.

BALL, R. (1890).

[Note on a determinant in the theory of screws. *Proceed. R. Irish Acad.*, (3) i. pp. 375–378.]

The results as stated are partly geometrical in form, namely, if (r, s) be the angle between the r^{th} and s_i^{th} screws of reference, then

$$\begin{vmatrix} 1 & \cos(12) & \cos(13) & \dots & \cos(16) \\ \cos(21) & 1 & \cos(23) & \dots & \cos(26) \\ \cos(31) & \cos(32) & 1 & \dots & \cos(36) \\ \dots & \dots & \dots & \dots & \dots \\ \cos(61) & \cos(62) & \cos(63) & \dots & 1 \end{vmatrix}$$

and its coaxial minors of the 4^{th} and 5^{th} orders all vanish.

TORELLI, G. (1891).

[Appunti sulla teoria delle forme binarie. *Annali del R. Ist. Tecn. e Naut.* (Napoli). Anno 1891, 10 pp.]

What is effected in the second of Torelli's two notes is a fresh generalization of the old identity

$$|a_1b_2|\,|c_1d_2| - |a_1c_2|\,|b_1d_2| + |a_1d_2|\,|b_1c_2| = 0$$

by viewing the left-hand member of this as a Pfaffian, namely,

$$\begin{vmatrix} |a_1b_2| & |a_1c_2| & |a_1d_2| \\ & |b_1c_2| & |b_1d_2| \\ & & |c_1d_2| \end{vmatrix}.$$

The next case concerns the elements of a 2-by-6 array, and to obtain it he multiplies together the two derived arrays

$$\begin{Vmatrix} a_1^3 & a_1^2a_2 & a_1a_2^2 & a_2^3 \\ b_1^3 & b_1^2b_2 & b_1b_2^2 & b_2^3 \\ \cdot & \cdot & \cdot & \cdot \\ f_1^3 & f_1^2f_2 & f_1f_2^2 & f_2^3 \end{Vmatrix} \quad \begin{Vmatrix} a_2^3 & -3a_2^2a_1 & 3a_2a_1^2 & -a_1^3 \\ b_2^3 & -3b_2^2b_1 & 3b_2b_1^2 & -b_1^3 \\ \cdot & \cdot & \cdot & \cdot \\ f_2^3 & -3f_2^2f_1 & 3f_2f_1^2 & -f_1^3 \end{Vmatrix},$$

the result being

$$0 = \text{a 6-line zero-axial skew determinant,}$$

whence, on extracting the square root, there follows

$$\begin{vmatrix} |a_1b_2|^3 & |a_1c_2|^3 & \dots & |a_1f_2|^3 \\ & |b_1c_2|^3 & \dots & |b_1f_2|^3 \\ & & \cdot \quad \cdot \quad \cdot & \\ & & & |c_1f_2|^3 \end{vmatrix} = 0.$$

When the elements concerned form a 2-by-$2r$ array, the common exponent appearing in the elements of the Pfaffian is $2r-3$.

METZLER, W. H. (1893).

[Compound determinants. *American Journ. of Math.*, xvi. pp. 131–150.]

The theorem here established (p. 148) is that *All the minors of order* n−m *in an* n*-line zero-axial skew determinant will vanish if for this and each of the higher orders the sum of the coaxial minors vanishes.* As an immediate consequence it is deduced that *every zero-axial skew determinant "has a nullity equal to its vacuity."* (See below, p. 286.)

HILL, M. J. M. (1895).

[A property of skew determinants. *Proceed. London Math. Soc.*, xxvi. pp. 341–345.]

The property in question is that given by Spottiswoode in 1853, proposed for proof by Cremona in 1864, and satisfactorily established by Torelli in 1865 (*Hist.*, ii. pp. 289-291, 314–315). Hill's proof is of the nature of a verification, and is more direct than Torelli's, though not so concise,

RUSSIAN, C. (1896-7).

[Teorya przekształcenia Pfaffa. *Prace mat.-fiz.*, viii. pp. 61–98 ; ix. pp. 61–102.]

The preliminary pages devoted to determinants (pp. 69–76) do not contain fresh matter relating to Pfaffians. The theorems there collected for convenience are merely known results afterwards made use of in dealing with the actual subject of the paper,—the Pfaffian transformation.

MUIR, T. (1896).

[The expression of any bordered skew determinant as a sum of products of Pfaffians. *Proceed. R. Soc. Edinburgh*, xxi. pp. 243–359.]

The main portion (§1–§10) of this lengthy paper is occupied with a thorough investigation of a theorem exemplified by Cayley in May 1854 and November 1857, such being needed because the two statements of one of the examples do not agree, and because on neither occasion is anything in the way of proof advanced. Further, for forty years the theorem had been neglected, and when the *Collected Mathematical Papers* came to be published, the example in question appeared in a form different from both of the originals.*

A quite general theorem is first established, which in the case of the 5th order is

$$|a_1b_2c_3d_4e_5| = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & . & b_3 & b_4 & b_5 \\ c_1 & c_2 & . & c_4 & c_5 \\ d_1 & d_2 & d_3 & . & d_5 \\ e_1 & e_2 & e_3 & e_4 & . \end{vmatrix} + \sum(b_2 . \begin{vmatrix} a_1 & a_3 & a_4 & a_5 \\ c_1 & . & c_4 & c_5 \\ d_1 & d_2 & . & d_5 \\ e_1 & e_2 & e_4 & . \end{vmatrix})$$

$$+ \sum(b_2c_3 . \begin{vmatrix} a_1 & a_4 & a_5 \\ d_1 & . & d_5 \\ e_1 & e_4 & . \end{vmatrix}) + \sum(b_2c_3d_4 . \begin{vmatrix} a_1 & a_5 \\ e_1 & . \end{vmatrix}) + b_2c_3d_4e_5a_1,$$

* Cayley's two papers in question will be found reported on in our second volume (pp. 267–272, 278–279).

and this, when applied to the bordered skew determinant

$$\begin{vmatrix} m_1 & h_1 & h_2 & h_3 & h_4 \\ -k_1 & m_2 & a_1 & a_2 & a_3 \\ -k_2 & -a_1 & m_3 & \beta_1 & \beta_2 \\ -k_3 & -a_2 & -\beta_1 & m_4 & \gamma_1 \\ -k_4 & -a_3 & -\beta_2 & -\gamma_1 & m_5 \end{vmatrix}, \quad \text{or } D_5 \text{ say,}$$

and certain substitutions made, gives the first of Cayley's identities in its correct form, namely,

$$\begin{vmatrix} & a_1 & a_2 & a_3 \\ & & \beta_1 & \beta_2 \\ & & & \gamma_1 \end{vmatrix} \cdot \begin{vmatrix} m_1 & h_1 & h_2 & h_3 & h_4 \\ & k_1 & k_2 & k_3 & k_4 \\ & & a_1 & a_2 & a_3 \\ & & & \beta_1 & \beta_2 \\ & & & & \gamma_1 \end{vmatrix} + \sum \left(m_2 \cdot \begin{vmatrix} h_2 & h_3 & h_4 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{vmatrix} \cdot \begin{vmatrix} k_2 & k_3 & k_4 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{vmatrix} \right)$$

$$+ \sum (m_2 m_3 \cdot \gamma_1 \begin{vmatrix} m_1 & h_3 & h_4 \\ & k_3 & k_4 \\ & & \gamma_1 \end{vmatrix}) + \sum (m_2 m_3 m_4 \cdot h_4 k_4) + m_1 m_2 m_3 m_4 m_5.$$

On putting in this

$$h_4 = a_3 = \beta_2 = \gamma_1 = 0, \qquad m_5 = 1,$$

there is obtained the corresponding identity for the case of the 4th order, namely,

$$\begin{vmatrix} m_1 & h_1 & h_2 & h_3 \\ -k_1 & m_2 & a_1 & a_2 \\ -k_2 & -a_1 & m_3 & \beta_1 \\ -k_3 & -a_2 & -\beta_1 & m_4 \end{vmatrix} = \begin{vmatrix} h_1 & h_2 & h_3 \\ & a_1 & a_2 \\ & & \beta_1 \end{vmatrix} \cdot \begin{vmatrix} k_1 & k_2 & k_3 \\ & a_1 & a_2 \\ & & \beta_1 \end{vmatrix}$$

$$+ m_2\beta_1 \begin{vmatrix} m_1 & h_2 & h_3 \\ & k_2 & k_3 \\ & & \beta_1 \end{vmatrix} + m_3 a_2 \begin{vmatrix} m_1 & h_1 & h_3 \\ & k_1 & k_3 \\ & & a_2 \end{vmatrix}$$

$$+ m_4 a_1 \begin{vmatrix} m_1 & h_1 & h_2 \\ & k_1 & k_2 \\ & & a_1 \end{vmatrix} + m_2 m_3 h_3 k_3$$

$$+ m_2 m_4 h_2 k_2 + m_3 m_4 h_1 k_1 + m_1 m_2 m_3 m_4.$$

In order to make clear the law of development, all the axial elements except the first are made 1,—a specialization shown to be more

apparent than real—the order of the terms in the developments is reversed, and another example is added to aid comparison, namely,

$$\begin{vmatrix} m & h_1 & h_2 & h_3 & h_4 & h_5 \\ -k_1 & 1 & a_1 & a_2 & a_3 & a_4 \\ -k_2 & -a_1 & 1 & \beta_1 & \beta_2 & \beta_3 \\ -k_3 & -a_2 & -\beta_1 & 1 & \gamma_1 & \gamma_2 \\ -k_4 & -a_3 & -\beta_2 & -\gamma_1 & 1 & \delta_1 \\ -k_5 & -a_4 & -\beta_3 & -\gamma_2 & -\delta_1 & 1 \end{vmatrix} = m + (h_1k_1 + h_2k_2 + \dots)$$

$$+ \sum\left(a_1 \cdot \begin{vmatrix} m & h_1 & h_2 \\ & k_1 & k_2 \\ & & a_1 \end{vmatrix}\right) \quad (10 \text{ terms})$$

$$+ \sum\left(\begin{vmatrix} h_1 & h_2 & h_3 \\ a_1 & a_2 & \\ \beta_1 & & \end{vmatrix} \cdot \begin{vmatrix} k_1 & k_2 & k_3 \\ a_1 & a_2 & \\ \beta_1 & & \end{vmatrix}\right) \quad (10 \text{ terms})$$

$$+ \sum\left(\begin{vmatrix} a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \\ \gamma_1 & & \end{vmatrix} \cdot \begin{vmatrix} m & h_1 & h_2 & h_3 & h_4 \\ & k_1 & k_2 & k_3 & k_4 \\ & & a_1 & a_2 & a_3 \\ & & & \beta_1 & \beta_2 \\ & & & & \gamma_1 \end{vmatrix}\right) \quad (5 \text{ terms})$$

$$+ \begin{vmatrix} h_1 & h_2 & h_3 & h_4 & h_5 \\ a_1 & a_2 & a_3 & a_4 & \\ \beta_1 & \beta_2 & \beta_3 & & \\ \gamma_1 & \gamma_2 & & & \\ \delta_1 & & & & \end{vmatrix} \cdot \begin{vmatrix} k_1 & k_2 & k_3 & k_4 & k_5 \\ a_1 & a_2 & a_3 & a_4 & \\ \beta_1 & \beta_2 & \beta_3 & & \\ \gamma_1 & \gamma_2 & & & \\ \delta_1 & & & & \end{vmatrix}.$$

Here it is important to observe (1) that the 1st and 3rd groups of Pfaffian products resemble each other, as do also the 2nd and 4th; (2) that in the 1st and 3rd groups of Pfaffian products the one Pfaffian is a minor of the other,—is, in fact, the cofactor of m in that other; (3) that in the 1st and 3rd groups of Pfaffian products the first line of the Pfaffian of higher order is got by taking m and an even number of h's; (4) that in the 2nd and 4th groups of Pfaffian products the Pfaffians are of like order, the second Pfaffian differing from the first merely in having k's for h's; (5) that in the 2nd and 4th groups of Pfaffian products the first line of the first Pfaffian is got by taking an odd number of h's.

A paragraph or two is then devoted to the question of the number of distinct terms in such determinants, the more interesting of the two formulae obtained being

$$(\mathrm{BS})_n = \mathrm{S}_{n-1} + (n-1)\mathrm{S}_{n-2} + (n-1)(n-2)\mathrm{S}_{n-3} + \dots + (n-1)(n-2)\dots 3\cdot 2\mathrm{S}_1 + (n-1)!$$

where S_n and $(BS)_n$ stand for the number of distinct terms in an n-line skew and bordered skew determinant respectively. The values of $(BS)_n$ for $n = 3, 4, 5, 6, 7, \ldots$ are given as

$$6, \quad 22, \quad 101, \quad 546, \quad 3502, \quad \ldots \quad .$$

Incidentally, too, a fresh formula is found for S_n, namely,

$$1 + C_{n,2} + C_{n,4} \cdot \frac{3(1+3)}{2} + C_{n,6} \cdot \frac{3\cdot5(1+3\cdot5)}{2} + C_{n,8} \cdot \frac{3\cdot5\cdot7(1+3\cdot5\cdot7)}{2} + \ldots .$$

The rest of the paper (§§ 16–19, pp. 356–359) is occupied with a new theorem, giving an expansion of the sum of m times a unit-axial skew determinant D_n and the bilinear quadric B_n of which D_n is the discriminant. The first three cases are

$$m \begin{vmatrix} 1 & a_1 \\ -a_1 & 1 \end{vmatrix} + \begin{array}{|cc|c} h_1 & h_2 & \\ \hline 1 & a_1 & k_1 \\ -a_1 & 1 & k_2 \end{array} = m + (h_1k_1 + h_2k_2) + a_1 \begin{vmatrix} m & h_1 & h_2 \\ & k_1 & k_2 \\ & & a_1 \end{vmatrix},$$

$$m \begin{vmatrix} 1 & a_1 & a_2 \\ -a_1 & 1 & \beta_1 \\ -a_2 & -\beta_1 & 1 \end{vmatrix} + \begin{array}{|ccc|c} h_1 & h_2 & h_3 & \\ \hline 1 & a_1 & a_2 & k_1 \\ -a_1 & 1 & \beta_1 & k_2 \\ -a_2 & -\beta_1 & 1 & k_3 \end{array} = m + (h_1k_1 + h_2k_2 + h_3k_3) + \sum a_1 \begin{vmatrix} m & h_1 & h_2 \\ & k_1 & k_2 \\ & & a_1 \end{vmatrix},$$

$$mD_4 + B_4 = m + \Sigma h_1k_1 + \sum (a_1 \begin{vmatrix} m & h_1 & h_2 \\ & k_1 & k_2 \\ & & a_1 \end{vmatrix}) + m \begin{vmatrix} a_1 & a_2 & a_3 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{vmatrix}^2;$$

and it is shown how with the help of the last of these the next case can be established, namely,

$$mD_5 + B_5 = m + \Sigma h_1k_1 + \sum (a_1 \begin{vmatrix} m & h_1 & h_2 \\ & k_1 & k_2 \\ & & a_1 \end{vmatrix}) + \sum (m \begin{vmatrix} a_1 & a_2 & a_3 \\ & \beta_1 & \beta_2 \\ & & \gamma_1 \end{vmatrix}^2).$$

FONTENÉ, G. (1896): MICHEL, CH. (1896): STUYVAERT, M. (1897).

[Expression de la quantité $p(u_1+u_2+\ldots+u_n)$ au moyen d'un Pfaffien. *Annales de l'éc. norm. sup.*, (3) xiii. pp. 469–487.]

[Le déterminant symétrique gauche d'ordre pair. *Journ. de Math. Spéc.*, xx. pp. 127–129.]

[Question 1101. *Mathesis*, (2) vii. pp. 31, 51; (3) x. pp. 246–252.]

The first two papers here involve nothing fresh in reference to our subject. In the third there is incidentally given the evaluation of the $2n$-line Pfaffian

$$\begin{vmatrix} x & -1 & . & . & . & \ldots \\ & x & -1 & . & . & \ldots \\ & & x & -1 & . & \ldots \\ & & & x & -1 & \ldots \\ & & & & x & \ldots \end{vmatrix},$$

which is found to equal

$$x^n - C_{n-1,1}x^{n-2} + C_{n-2,2}x^{n-4} - \ldots.$$

None of the writers interested in the question, however, seems to have observed the curious fact that the said Pfaffian is identical with the n-line continuant

$$\begin{vmatrix} x & 1 & . & \ldots \\ 1 & x & 1 & \ldots \\ . & 1 & x & \ldots \\ . & . & . & . \end{vmatrix}.$$

SADUN, ELCIA (1896).

[Sulla trasformazione d'un prodotto di due somme di n quadrati in una somma di n quadrati. *Periodico di Mat.*, xiv. pp. 125–139.]

This is of general interest in connection with a question raised in Roberts' paper of 1879 (*Hist.*, iii. pp. 280–281); but there is also a section of it (§ 2, pp. 128–131) which is directly relevant to our subject in that it points out an oversight of Brioschi's when using his theorem of 1855 (*Hist.*, ii. pp. 276–277) to deal with the case where n is 8. An attempt, too, is made to show how the oversight

may be rectified; but, unfortunately, the determinant proposed, namely,

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ -b & a & -d & c & -f & e & -h & g \\ -c & d & a & -b & g & -h & -e & f \\ -d & -c & b & a & -h & -g & f & e \\ -e & f & -g & h & a & -b & c & -d \\ -f & -e & h & g & b & a & -d & -c \\ -g & h & e & -f & -c & d & a & -b \\ -h & -g & -f & -e & d & c & b & a \end{vmatrix},$$

is itself open to the same objection as Brioschi's, in that it does not satisfy the required conditions. Further, if a multiplier different from Brioschi's had to be used, it would have been simpler to take the proposed determinant itself unaltered. When this is done, the said determinant is seen to be a skew orthogonant equal to $(\Sigma a^2)^4$.

CAZZANIGA, T. (1897).

[Sopra i determinanti gobbi. *Rendic. R. Ist. Lombardo* (Milano), (2) xxx. pp. 1303–1308.]

Notwithstanding the title, this paper only concerns one theorem, namely, Cayley's regarding a bordered zero-axial skew determinant. It involves no new result, but the exposition given is very clear and full. The form of the deduction from it regarding primary minors of a zero-axial skew determinant is slightly better than Scheibner's, and agrees exactly with that on p. 200 of Muir's text-book of 1882, namely, *If* $|a_{in}|$ *be zero-axial skew, and* M_{pq} *be the minor got by deleting the* p^{th} *row and* q^{th} *column, then*

$$M_{pq} = (1, 2, \ldots, p-1, p+1, \ldots, n) \cdot (1, 2, \ldots, q-1, q+1, \ldots, n),$$

when n *is odd,*

and

$$M_{pq} = (1, 2, \ldots, n) \cdot (1, 2, \ldots, p-1, p+1, \ldots, q-1, q+1, \ldots, n),$$

when n *is even and* $p > q$.

When n is even and $p < q$, we can make use of the fact that

$$M_{pq} = -M_{qp}.$$

We may note that if the non-umbral notation be employed, the Pfaffians in the said deduction are in the first of the two cases got from the quasi-Pfaffian of the determinant by deleting the p^{th} and q^{th} frame-lines separately, and in the second case the one is the Pfaffian of the determinant and the other is the minor got from it by deleting the p^{th} and q^{th} frame-lines separately. Thus when the given determinants are

$$\begin{vmatrix} . & a & b & c & d \\ -a & . & e & f & g \\ -b & -e & . & h & i \\ -c & -f & -h & . & j \\ -d & -g & -i & -j & . \end{vmatrix}, \qquad \begin{vmatrix} . & a_2 & a_3 & a_4 & a_5 & a_6 \\ -a_2 & . & b_3 & b_4 & b_5 & b_6 \\ -a_3 & -b_3 & . & c_4 & c_5 & c_6 \\ -a_4 & -b_4 & -c_4 & . & d_5 & d_6 \\ -a_5 & -b_5 & -c_5 & -d_5 & . & e_6 \\ -a_6 & -b_6 & -c_6 & -d_6 & -e_6 & . \end{vmatrix}$$

and $p, q = 2, 3$, we have

$$\begin{vmatrix} . & a & c & d \\ -b & -e & h & i \\ -c & -f & . & j \\ -d & -g & -j & . \end{vmatrix} = \begin{vmatrix} b & c & d \\ & h & i \\ & & j \end{vmatrix} \cdot \begin{vmatrix} a & c & d \\ & f & g \\ & & j \end{vmatrix},$$

and

$$\begin{vmatrix} . & a_2 & a_4 & a_5 & a_6 \\ -a_3 & -b_3 & c_4 & c_5 & c_6 \\ -a_4 & -b_4 & . & d_5 & d_6 \\ -a_5 & -b_5 & -d_5 & . & e_6 \\ -a_6 & -b_6 & -d_6 & -e_6 & . \end{vmatrix} = - \begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ & b_3 & b_4 & b_5 & b_6 \\ & & c_4 & c_5 & c_6 \\ & & & d_5 & d_6 \\ & & & & e_6 \end{vmatrix} \cdot \begin{vmatrix} a_4 & a_5 & a_6 \\ & d_5 & d_6 \\ & & e_6 \end{vmatrix}.$$

VIVANTI, G. (1897).

[Sulle trasformazioni infinitesime che lasciano invariata un' eguazione Pfaffiana. *Rendic. Circolo Mat.* (Palermo), xii. pp. 1–20.]

An interesting section (§ 2, pp. 3–9) is here devoted to the properties of zero-axial skew determinants. None of the twelve properties are really new, and those that take the form of relations between Pfaffians lose in width and value by being viewed as explicitly connected with determinants. Frobenius' theorem (*Hist.*, iii. p. 276) that *the vanishing of a zero-axial skew determinant of even*

order entails the vanishing of all its primary minors is simply and freshly proved for the non-coaxial minors by noting that the square root of the vanishing determinant is a factor of each one of them.

BAKER, H. F. (1897).

[Note on a property of Pfaffians. *Proceed. London Math. Soc.*, xxix. pp. 141–142 ; or *Amer. Journ. of Math.*, xx. pp. 360–362.]

Of the two theorems here formulated, the first is a rediscovery, being included in Lloyd-Tanner's theorem of 1878. It may be viewed as a simple extensional of the identity

$$[1234\ldots] = [12]\cdot[34\ldots] - [13]\cdot[24\ldots] + [14]\cdot[23\ldots] - \ldots,$$

for example,

$$[xy][xy1234] = [xy12][xy34] - [xy13][xy24] + [xy14][xy23].$$

It degenerates into the second when all the elements of one of the frame-lines of $[xy12\ldots2n]$ vanish except one : thus, putting

$$x4,\ y4,\ 14,\ 24,\ 34 = 0,\ -1,\ 0,\ 0,\ 0$$

in the foregoing, we obtain

$$[xy][x123] = [xy12][x3] - [xy13][x2] + [x1][xy23].$$

(See Muir's text-book, p. 205, ex. 2.)

STUDNIČKA, F. J. (1898).

[Příspěvek ke nauce o determinantech symmetrálných neboli protiměrných. *Věstnik České Akad.* . . . , vii. pp. 235–239.]

The proof here given of Cayley's theorem regarding an even-ordered zero-axial skew determinant is essentially gradational. For example, the theorem being known to hold for the 4th order, the determinant of the 6th order is seen to be

$$= \{[1,1][6,6] - [1,6][6,1]\} \div \text{central 4-line minor}$$

$$= \{0\cdot0 + [1,6]^2\} \div \text{a square,}$$

where $[r,s]$ stands for the cofactor of the element in the $(r,s)^{\text{th}}_{\iota}$ place.

The division indicated here is performed, and an expression got for the Pfaffian in terms of 3-line and other minors of the determinant.

YOUNG, W. H. (1898).

[On the null-spaces of a one-system. *Proceed. London Math. Soc.*, xxx. pp. 33–53.]

The portion of this (§ 2, p. 34) devoted to notation gives an account of Pfaffians under the designation 'square brackets.' Thus, the theorem regarding a primary minor of the adjugate appears in words thus : "*The square brackets of dimensions* m—1 *in the* b*'s are proportional to the square brackets of dimensions* 1 *in the* a*'s*," where the *b*'s stand for the cofactors of the *a*'s in the original 'square bracket.' The same appears in symbols, thus :

$$[34 \ldots 2m]_b = \{[12 \ldots 2m]_a\}^{m-2} \cdot a_{12}.$$

SCARPIS, U. (1899).

[Una proprietà dei determinanti dedotta dal concetto di sostituzione. *Giornale di Mat.*, xxxvii. pp. 73–79.]

The theorem is Cayley's regarding a zero-axial skew determinant of even order, but the proof does not commend itself.

RUSSIAN, C. K. (1899).

[Some theorems on symbolic determinants. (In Russian.) *Mem. . . . Univ. . . .* (Odessa), lxxvii. pp. 323–348.]

The symbolic determinants in question are those introduced by Tanner in 1878 (*Hist.*, iii. pp. 278–280) and used by him two years later (*Proceed. London Math. Soc.*, xi. pp. 131–139) in formulating a generalization of Pfaff's problem. The application of the determinants for this purpose, and especially for the formulation of conditions, is what the author is concerned with here and in several other papers. We may also note that part of such a paper in the next volume is occupied (pp. 417–429) with a useful collection of determinant theorems used by him in the course of the investigation.

CHAPTER XII.

ORTHOGONANTS AND LATENT ROOTS, FROM 1880 TO 1899.

THE two determinantal matters that arise out of the study of linear transformation (*Hist.*, iii. p. 284) are here left unseparated as in the preceding volumes, this course being on the whole the more convenient for all concerned. The number of papers is considerably in excess of a half more than it was in the preceding period.

LAURENT, H. (1880).

[Sur la reduction des polynomes du second degré homogènes à des sommes de carrés. *Nouv. Annales de Math.*, (2) xix. pp. 12–27.]

A portion of this (pp. 16–22) is occupied with Lagrange's determinantal equation. The treatment, however, is not an improvement on Gravelaar's of 1875.

HAZZIDAKIS, J. N. (1880).

(See under this heading in Chap. IV.)

SCOTT, R. F. (1880).

[A Treatise on the Theory of Determinants, ... xi+251 pp. Cambridge.]

To Siacci's fourth theorem of 1872 Scott prefixes (p. 233) a result which perhaps is best presented thus : *If the Cayleyan orthogonant obtained from* $|\,a_{1n}\,|$ *have* 1 *subtracted from each of its diagonal elements, the resulting determinant is equal to*

$$\frac{2^n\,|\,a_{1n}\,|_0}{|\,a_{1n}\,|},$$

where $|\,a_{1n}\,|_0$ *stands for what* $|\,a_{1n}\,|$ *becomes when each of its diagonal elements is changed to* 0. And the mode of proof which we should prefer would be to remove 2^n from the determinant in question and multiply the cofactor columnwise by $|\,a_{1n}\,|$. The fact that $|\,a_{1n}\,|_0$ vanishes when n is odd establishes part of Brioschi's theorem of 1854, namely, that every Cayleyan orthogonant of odd order has 1 for a latent root (*Hist.*, ii. pp. 317–318).

WALECKI, . (1882).

[Équation en s de degré m, et décomposition d'une forme quadratique en carrés. *Nouv. Annales de Math.*, (3) i. pp. 401–409, 556–560.]

An elementary proof given here is worth recording, although no part of it is really new. With a little further simplification it is in effect as follows. If a be a root which occurs more than once in Lagrange's determinantal equation $\Delta(x) = 0$, then $\Delta'(a)$ must vanish. Now, by the differentiation-theorem of determinants, $\Delta'(x)$ is readily found to be the negative sum of the coaxial primary minors of $\Delta(x)$, and therefore this sum must vanish when x in it is put equal to a. But then another theorem—regarding a two-line coaxial minor of the adjugate—shows that on account of the vanishing of $\Delta(a)$ all these minors must have the same sign when x equals a. The conclusion thus is that they all vanish individually, as was to be proved.

SYLVESTER, J. J. (1884).

[Lectures on the principles of universal algebra. *American Journ. of Math.*, vi. pp. 270–286; or *Coll. Math. Papers*, iv. pp. 208–224.]

The subject of these lectures, so far as they are here printed, is *Cayleyan matrices*,—a subject which, as we know, dates back to 1854 (*Hist.*, ii. pp. 85–87), and to which a special memoir was devoted in 1857.* Sylvester appears to have taken up the study of such matrices about 1881; and in the years 1883, 1884 published

* Cayley, A. A memoir on the theory of matrices. *Philos. Transac. R. Soc. London*, cxlviii. pp. 17–37; or *Coll. Math. Papers*, ii. pp. 475–496. The result of a first attempt to give a List of Writings on the Theory of Matrices from 1857 to 1893 is published in the *American Journ. of Math.*, xx. pp. 225–228.

numerous notes on the subject in the *Comptes Rendus* of the French Academy. So far as determinants proper are concerned, there is little or nothing to be noted. Of the class of theorems that may not unnaturally be attached to either theory, there is one important instance (p. 278), which we may state as follows : *If the roots of the equation*

$$\begin{vmatrix} a_{11}-x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-x & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn}-x \end{vmatrix} = 0.$$

be $\rho_1, \rho_2, \dots, \rho_n$, *and the Lagrangian determinantal function on the left be denoted by* $L(x)$, *then*

$$L(\rho_1) \cdot L(\rho_2) \cdot \dots \cdot L(\rho_n) = \begin{vmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{vmatrix},$$

the multiplication of the L's *being row-by-column.* In this connection we may note as a matter of terminology that if one of the latent roots of $|\, a_{1n} \,|$ be zero, so that $|\, a_{1n} \,| = 0$, Sylvester speaks of the 'content' (*i.e.* determinant) of the matrix being vacuous ; and, if k of the roots be zeros, the matrix is said to have vacuity k, k being thus a sort of index of so-called vacuity. What it amounts to, of course, is the vanishing of the last k coefficients in the expansion of the determinantal function according to descending powers of x.

STIELTJES, T. J. (1884).

[Un théorème d'algèbre. *Acta Math.*, vi. pp. 319–320.]

In a letter to Hermite, Stieltjes notes the fact, which he had verified, that *if* $|\, a_1 b_2 c_3 \,|$ *and* $|\, \alpha_1 \beta_2 \gamma_3 \,|$ *be positive unit orthogonants, and the determinant*

$$\begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 \\ b_1+\beta_1 & b_2+\beta_2 & b_3+\beta_3 \\ c_1+\gamma_1 & c_2+\gamma_2 & c_3+\gamma_3 \end{vmatrix} = 0,$$

then must every primary minor of the latter vanish also. He adds that he was inclined to believe that the corresponding statement in the case of the fourth order was likewise true.

ROUTH, E. J. (1884).

[On Lagrange's determinant. *Dynamics of a System of Rigid Bodies.* Part II. pp. 36–41.]

Routh considers the determinantal equation in the form

$$\begin{vmatrix} a_{11}x+b_{11} & a_{12}x+b_{12} & \dots & a_{1n}x+b_{1n} \\ a_{12}x+b_{12} & a_{22}x+b_{22} & \dots & a_{2n}x+b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{1n}x+b_{1n} & a_{2n}x+b_{2n} & \dots & a_{nn}x+b_{nn} \end{vmatrix} = 0,$$

and examines its roots in the so-called Salmon manner. Denoting the determinant by Δ_n, and the minor of it got by deleting the first r rows and first r columns by Δ_{n-r}, he proves that *the roots of each member of the series* Δ_n, Δ_{n-1}, Δ_{n-2}, ... *are all real*, and that *the roots of each member separate* (*or lie between*) *the roots of the member next before it in the series.* From this his next proposition readily follows, namely, that *if* Δ_n *has* m *of its roots equal, every primary minor of it has* m-1 *roots equal to each of these, every secondary minor* m-2 *roots equal to the same, and so on.* He then borders Δ_n axisymmetrically with the elements $0, c_1, c_2, \dots, c_n$, and proves that *the roots of the new determinant are also all real, and that they separate those of the original.* Naturally, also, he continues the process of bordering, as we have already seen it done in the simpler case where the x's of the determinant are confined to the diagonal.

Another possible mode of discussing the subject is indicated in the handling of two examples at the close (§ 71).

MUIR, T. (1884).

[Note on the determinantal equation connected with the investigation of the small oscillations of a system about a position of equilibrium. *Messenger of Math.*, xiv. pp. 141–143.]

This deals with the reality of the roots of the equation

$$\begin{vmatrix} ax+\alpha & bx+\beta & cx+\gamma \\ bx+\beta & ex+\epsilon & fx+\phi \\ cx+\gamma & fx+\phi & kx+\kappa \end{vmatrix} = 0$$

as predicated by Sylvester's theorem of 1853 (*Hist.*, ii. p. 314), the aim now being to show that the said theorem is a direct consequence

of the original theorem whose foundations were laid by Lagrange. This is effected by taking the more general determinant

$$\begin{vmatrix} ax+\alpha & bx+\beta & cx+\gamma \\ dx+\delta & ex+\epsilon & fx+\phi \\ gx+\eta & hx+\theta & kx+\kappa \end{vmatrix}$$

and axisymmetrically transforming it so as to have no x's outside the diagonal, the result being

$$\left| \begin{array}{lll} ax+\alpha & \begin{vmatrix} a & b \\ \alpha & \beta \end{vmatrix} & \begin{vmatrix} a & b & c \\ d & e & f \\ \alpha & \beta & \gamma \end{vmatrix} \\ \begin{vmatrix} a & \alpha \\ d & \delta \end{vmatrix} & a\begin{vmatrix} a & b \\ d & e \end{vmatrix} x - \begin{vmatrix} . & a & b \\ a & \alpha & \beta \\ d & \delta & \epsilon \end{vmatrix} & \begin{vmatrix} . & a & b & c \\ . & d & e & f \\ a & \alpha & \beta & \gamma \\ d & \delta & \epsilon & \phi \end{vmatrix} \\ \begin{vmatrix} a & b & a \\ d & e & \delta \\ g & h & \eta \end{vmatrix} & \begin{vmatrix} . & . & a & b \\ a & b & \alpha & \beta \\ d & e & \delta & \epsilon \\ g & h & \eta & \theta \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} x + \mathrm{Z} \end{array} \right| \div \mathrm{P},$$

where P and Z stand for

$$a^2 \begin{vmatrix} a & b \\ d & e \end{vmatrix}^2 \quad \text{and} \quad \begin{vmatrix} . & . & a & b & c \\ . & . & d & e & f \\ a & b & \alpha & \beta & \gamma \\ d & e & \delta & \epsilon & \phi \\ g & h & \eta & \theta & \kappa \end{vmatrix}$$

respectively. On putting in this

$$d,\ g,\ h = b,\ c,\ f \quad \text{and} \quad \delta,\ \eta,\ \theta = \beta,\ \gamma,\ \phi,$$

it is seen that the determinant of the equation with which we started has an axisymmetric equivalent of the form desired, save that the coefficients of x are in general not 1,—a point of difference, however, which is easily cleared away by dividing each element of the row and of the column in which one of the said coefficients stands by the square root of that coefficient.

BUCHHEIM, A. (1884).

(See under this heading in Chap. IV.)

HOPPE, R. (1885).

[Neue Relationen innerhalb eines Orthogonalcoefficientensystems. *Archiv d. Math. u. Phys.*, (2) ii. pp. 413–416.]

If $|\ \alpha_1\beta_2\gamma_3\ |$ be a positive unit orthogonant, the relations referred to are typified by the following :

$$\begin{aligned}
(\beta_2+\gamma_3)^2 + (\gamma_2-\beta_3)^2 &= (1+\alpha_1)^2,\\
(\alpha_2-\beta_1)(\gamma_1-\alpha_3) &= (\beta_3+\gamma_2)(1+\alpha_1+\beta_2+\gamma_3),\\
(\alpha_2-\beta_1)^2 + (\gamma_1-\alpha_3)^2 &= 2(1-\alpha_1)(1+\alpha_1+\beta_2+\gamma_3),\\
(\alpha_2-\beta_1)^2 - (\gamma_1-\alpha_3)^2 &= 2(\gamma_3-\beta_2)(1+\alpha_1+\beta_2+\gamma_3),\\
(\alpha_2-\beta_1+\gamma_1-\alpha_3)^2 &= 2(1-\alpha_1+\beta_3+\gamma_2)(1+\alpha_1+\beta_2+\gamma_3),\\
(\alpha_2-\beta_1)(\alpha_2-\beta_1+\gamma_1-\alpha_3) &= (1-\alpha_1+\beta_2+\gamma_3+\beta_3+\gamma_2)(1+\alpha_1+\beta_2+\gamma_3).
\end{aligned}$$

All are readily verifiable, and especially so if taken in the order here followed.

LIPSCHITZ, R. (1886).

[Untersuchungen ueber die Summen von Quadraten. 148 pp. Bonn.]

At the outset of this, consideration is given to those orthogonants that may be obtained from a given positive unit orthogonant by changing the signs of an even number of columns. The resulting theorem, which is attributed to Hurwitz,* may be formulated thus : *If the orthogonant* $|\ a_{1n}\ | = 1$, *and every possible determinant be formed from* $|\ a_{1n}\ |$ *by changing an even number of its columns, then among these* 2^{n-1} *determinants there is at least one such that the addition of positive unity to every diagonal element of it produces a determinant that does not vanish,* or, in other words, *one that does not have* -1 *for a latent root.* The proof amounts to showing that if $+1$ be added to each diagonal element of the series of determinants, the sum of the resulting series is 2^n.

Hurwitz' theorem, we may note, concerns a different sum, and his mode of proof would be inapplicable here.

* *Crelle's Journ.*, xciv. (1882) p. 7.

LORIA, G. (1886).

[Su una proprietà del determinante di una sostituzione ortogonale. *Jorn. de Sci. math. e astron.*, vii. pp. 129–132.]

What is proposed to be effected here is the extension of Siacci's fourth theorem of 1872 to orthogonants of *odd* order, a single proof being considered sufficient for both orders. The basis of the proof is a pair of separate multiplications of one of the primary minors in question, M_1 say, by the determinant, D say, of which it is a minor, it being overlooked that for odd orders D vanishes. Addition is then found to give, properly enough, $2M_1D = -D^2$. No distinction is made between positive and negative orthogonants.

NETTO, E. (1886).

[Ueber orthogonale Substitutionen. *Acta Math.*, ix. pp. 295–300.]

This is an interesting examination of Stieltjes' suggested theorem of 1884.

It opens with an account of the construction of Cayley's orthogonant, and deduces therefrom a corollary which we may formulate for ourselves thus : *If* 1 *be added to each of the diagonal elements of Cayley's orthogonant as formed from an* n*-line skew determinant* B *with diagonal elements all equal to* ω, *the determinant so obtained is equal to*

$$2^n\omega^n/B,$$

and vanishes when, and only when, ω *vanishes : further, when* ω *vanishes, all the minors down to and including those of order* 2 *vanish, and in the case when* n *is even all the individual elements as well.* The first portion is a companion to Scott's corollary of 1880 ; the second follows readily from the expansion of B in descending powers of ω, there being either before or after the removal of the first power of ω from numerator and denominator a term in B independent of ω.

Netto then takes the two orthogonal substitutions

$$x_k = b_{k1}y_1 + b_{k2}y_2 + \ldots + b_{kn}y_n,$$

$$\xi_k = \beta_{k1}y + \beta_{k2}y_2 + \ldots + \beta_{kn}y_n,$$

and deriving from the second,

$$y_k = \beta_{1k}\xi_1 + \beta_{2k}\xi_2 + \ldots + \beta_{nk}\xi_n$$

substitutes accordingly in the first and obtains

$$x_k = \Sigma b_{k\lambda}\beta_{1\lambda} \cdot \xi_1 + \Sigma b_{k\lambda}\beta_{2\lambda} \cdot \xi_2 + \ldots + \Sigma b_{k\lambda}\beta_{n\lambda} \cdot \xi_n.$$

The determinant of this is manifestly an orthogonant on account of Σx^2 and $\Sigma \xi^2$ being each equal to Σy^2, and is by its mode of formation equal to the product of the two given orthogonants. This being noted, ξ_k is added to both sides so that the determinant becomes altered by the addition of 1 to each diagonal element, and according to an assumption of Netto's may be taken as an equivalent for the determinant of

$$x_k + \xi_k = (b_{k1}+\beta_{k1})\,y_1 + (b_{k2}+\beta_{k2})\,y_2 + \ldots + (b_{kn}+\beta_{kn})y_n.$$

If so, and *if the product-orthogonant can be viewed as a Cayleyan orthogonant*, the corollary with which we started can be applied and a pertinent result at once reached. The second assumption, however, is more serious than the first, and is pointedly referred to by Netto himself. Further, he states that Kronecker drew his attention to the wider fact that Cayley's could not be a perfectly general representation, as the orthogonant got from a Cayley's orthogonant by changing the signs of an even number of rows was manifestly not included in Cayley's formula. Thereupon he worked out on the wider basis a substitute for Stieltjes' proposed theorem.

We note for ourselves that, $|\, a_1b_2c_3 \,|$ and $|\, \alpha_1\beta_2\gamma_3 \,|$ being the given orthogonants, we have only to multiply $|\, a_1+\alpha_1 \;\; b_2+\beta_2 \;\; c_3+\gamma_3 \,|$ by 1 in the form $|\, a_1b_2c_3 \,|$ or $|\, \alpha_1\beta_2\gamma_3 \,|$ to see that it is equal to

$$\begin{vmatrix} \Sigma a\alpha+1 & \Sigma a\beta & \Sigma a\gamma \\ \Sigma b\alpha & \Sigma b\beta+1 & \Sigma b\gamma \\ \Sigma c\alpha & \Sigma c\beta & \Sigma c\gamma+1 \end{vmatrix};$$

so that the above-mentioned assumption is justified; * and consequently if the perfect generality of Cayley's representation were granted, Netto's introductory corollary would end the matter. In the second place it must be noted that, since the ω of Netto's first theorem has with Cayley the value 1, it follows that Cayley's ortho-

* In the language of Cayleyan matrices the theorem is that *the determinant of the sum of two orthogonal matrices is equal to the determinant of the sum of their product and unity.*

gonant cannot have -1 for a latent root, and therefore that orthogonants having -1 for a latent root cannot be represented by Cayley's formula.

TARLETON, A. (1887).

[On a new method of obtaining the conditions fulfilled when the harmonic determinant has equal roots. *Proceed. R. Irish Acad.*, (3) i. pp. 10–15.]

If $\Delta_n = 0$ be Lagrange's determinantal equation and Δ_{n-r} be the determinant got from Δ_n by deleting the first r rows and the first r columns, then the theorem here finally proved is that *if the equation have* m *roots equal, the primary minors of* Δ_n, Δ_{n-1}, . . . , Δ_{n-m+2} *all vanish.* The proof is essentially the same as Walecki's of 1882. (See above, p. 285.)

VOSS, A. (1887).

[Ueber bilinearen Formen. *Nachrichten . . . Ges. d. Wiss.* (Göttingen), 1887, pp. 424–433.]

Unexpectedly a half of this paper (§ 2) is concerned with the establishment of Stieltjes' orthogonant theorem of 1884.* The proof given is neither direct nor simple, and therefore loses in value : but, as not unseldom happens in such cases, the theorem (*Hist.*, iii. p. 301) on which it is made to rest is in itself of very considerable interest, its subject being the bordering of an orthogonant with its diagonal elements augmented. For the case where the orthogonant is of the 4th order, $|\,\omega_{11}\omega_{22}\omega_{33}\omega_{44}\,|$ say, and the border is 3-line deep, it is

$$\begin{vmatrix} \omega_{11}+\rho_1 & \omega_{12} & \omega_{13} & \omega_{14} & a_1 & b_1 & c_1 \\ \omega_{21} & \omega_{22}+\rho_2 & \omega_{23} & \omega_{24} & a_2 & b_2 & c_2 \\ \omega_{31} & \omega_{32} & \omega_{33}+\rho_3 & \omega_{34} & a_3 & b_3 & c_3 \\ \omega_{41} & \omega_{42} & \omega_{43} & \omega_{44}+\rho_4 & a_4 & b_4 & c_4 \\ x_1 & x_2 & x_3 & x_4 & . & . & . \\ y_1 & y_2 & y_3 & y_4 & . & . & . \\ z_1 & z_2 & z_3 & z_4 & . & . & . \end{vmatrix}$$

* It turns up also in another paper of Voss' on bilinear forms in *Abhandl. . . . Akad. d. Wiss.* (München), xvii. 2, p. 261.

$$= (-1)^3(\rho_1\rho_2\rho_3\rho_4)^{-1} \begin{vmatrix} \omega_{11}\rho_1^2+\rho_1 & \omega_{21}\rho_2\rho_1 & \omega_{31}\rho_3\rho_1 & \omega_{41}\rho_4\rho_1 & a_1 & b_1 & c_1 \\ \omega_{12}\rho_1\rho_2 & \omega_{22}\rho_2^2+\rho_2 & \omega_{32}\rho_3\rho_2 & \omega_{42}\rho_4\rho_2 & a_2 & b_2 & c_2 \\ \omega_{13}\rho_1\rho_3 & \omega_{23}\rho_2\rho_3 & \omega_{33}\rho_3^2+\rho_3 & \omega_{43}\rho_4\rho_3 & a_3 & b_3 & c_3 \\ \omega_{14}\rho_1\rho_4 & \omega_{24}\rho_2\rho_4 & \omega_{34}\rho_3\rho_4 & \omega_{44}\rho_4^2+\rho_4 & a_4 & b_4 & c_4 \\ x_1 & x_2 & x_3 & x_4 & S_{xa} & S_{xb} & S_{xc} \\ y_1 & y_2 & y_3 & y_4 & S_{ya} & S_{yb} & S_{yc} \\ z_1 & z_2 & z_3 & z_4 & S_{za} & S_{zb} & S_{zc} \end{vmatrix},$$

where

$$S_{xa} = x_1a_1\rho_1^{-1} + x_2a_2\rho_2^{-1} + x_3a_3\rho_3^{-1} + x_4a_4\rho_4^{-1}.$$

No proof is given, but it is not difficult to see that multiplication of the left-hand member by $|\omega_{11}\omega_{22}\omega_{33}\omega_{44}|$, followed by certain operations on rows and columns, will produce the right-hand member.

The case of special interest is that in which each of the ρ's is 1, for then the first 4-line minor on the right-hand side is the conjugate of the corresponding minor on the other side, and the last 3-line minor on the right-hand side is the product of the two bordering arrays. It is this case that Voss makes use of, the additional condition introduced for his purpose being that all the 2-line minors of the first 4-line minor simultaneously vanish. As the determinant involving the S's is then independent of them, the actually-used specialized form of the theorem is *If all the 2-line minors of the first 4-line minor of*

$$\begin{vmatrix} \omega_{11}+1 & \omega_{12} & \omega_{13} & \omega_{14} & a_1 & b_1 & c_1 \\ \omega_{21} & \omega_{22}+1 & \omega_{23} & \omega_{24} & a_2 & b_2 & c_2 \\ \omega_{31} & \omega_{32} & \omega_{33}+1 & \omega_{34} & a_3 & b_3 & c_3 \\ \omega_{41} & \omega_{42} & \omega_{43} & \omega_{44}+1 & a_4 & b_4 & c_4 \\ x_1 & x_2 & x_3 & x_4 & . & . & . \\ y_1 & y_2 & y_3 & y_4 & . & . & . \\ z_1 & z_2 & z_3 & z_4 & . & . & . \end{vmatrix}$$

vanish, the determinant is only altered in sign by changing the first 4-line minor into its conjugate. A separate and simple proof of this would have been a helpful addition to the paper.*

* As regards the seeming predilection for bordered determinants, see below in Chap. XXI. under Stickelberger (1878) and Voss (1889).

MARCHAND, . (1888).

[Discussion de l'équation en *s*. *Nouv. Annales de Math.*, (3) vii. pp. 431–435.]

The object here is to prove the converse of the known theorem regarding the vanishing of minors which follows on the existence of m equal roots in Lagrange's determinantal equation.

RAHUSEN, A. E. (1888).

[Sur quelques propriétés des déterminants, appliquées à une question de géométre à n dimensions. *Annales de l'École Polyt.* (Delft), iv. pp. 104–139.]

Unexpectedly there is found here (pp. 110–121) a contribution to orthogonants which has some special interest because of its mode of approach to the subject. The introductory theorems may be combined in one enunciation as follows: *If* $|\,a_{1n}\,|$ *and* $|\,b_{1n}\,|$ *be such that the product of any two rows of the one is equal to the product of the corresponding two rows of the other, then* (1)

$$|\,a_{1n}\,| \;=\; \pm\,|\,b_{1n}\,|\,;$$

(2) *the critical minors* * *of*

$$|\,(a_{11}-b_{11}).\;\;\ldots\;\;.(a_{nn}-b_{nn})\,|$$

are of even or odd order according as the sign prefixed to $|\,b_{1n}\,|$ *is* $+$ *or* $-$; *and* (3) *the complementaries of the critical minors* * *of*

$$|\,a_{11}+b_{11}.\;\ldots\;.\;a_{nn}+b_{nn}\,|$$

are of similar order in similar circumstances. The proof of (2) is not as simple as could be wished, and the proof of (3) is naturally on the same lines. It is then pointed out that if we specialize by taking for $|\,b_{1n}\,|$ the determinant of unit matrix,

$$\begin{vmatrix} 1 & . & . & \ldots \\ . & 1 & . & \ldots \\ . & . & 1 & \ldots \\ . & . & . & \ldots \end{vmatrix}_n,$$

* The author uses Rouché's form of expression, which may be instructively varied by saying that the *rank* of the determinant in (2) and the *nullity* of the determinant in (3) are even or odd, etc.

the conditions connecting $|\,a_{1n}\,|$, $|\,b_{1n}\,|$ become the conditions for the orthogonance of $|\,a_{1n}\,|$, and the theorems (2), (3) take the form: *If* $|\,a_{1n}\,|$ *be an orthogonant, the critical minors of*

$$\begin{vmatrix} a_{11}-1 & a_{12} & a_{13} & \dots \\ a_{21} & a_{22}-1 & a_{23} & \dots \\ a_{31} & a_{32} & a_{33}-1 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

and the complementaries of the critical minors of

$$\begin{vmatrix} a_{11}+1 & a_{12} & a_{13} & \dots \\ a_{21} & a_{22}+1 & a_{23} & \dots \\ a_{31} & a_{32} & a_{33}+1 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

are of even or odd order according as $|\,a_{1n}\,|$ *is positive or negative.*

LIPSCHITZ, R. (1890).

[Beiträge zu der Theorie der gleichzeitigen Transformation van zwei quadratischen oder bilinearen Formen. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), pp. 485–523.]

Use is here made (pp. 496–497) of a peculiar linear substitution, the determinant of which is an axisymmetric orthogonant, and it is noted that the application of the substitution twice in succession has the effect of the 'identical' substitution.

A skew determinant with univarial diagonal also comes up for consideration (pp. 508, etc.), but without any fresh result, Hovestadt's dissertation (Bonn, 1873) being referred to and his proof regarding the roots reproduced.

KRONECKER, L. (1890).

[Ueber orthogonale Systeme. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), pp. 525–541, 602–607, 691–699, 873–884, 1063–1080; or *Werke*, iii. (1) pp. 371–459.]

Influenced by Lipschitz' paper, and especially by the part dealing with an axisymmetric orthogonal substitution, Kronecker at once set for himself the problem of finding all substitutions of this particular kind. In effect the first form of his result is that *if* $|\,c_{1n}\,|$

be any orthogonant with real elements, then all axisymmetric orthogonants with real elements are obtained by taking

$$\sum_{f=1}^{f=m} c_{fh}c_{fk} - \sum_{g=m+1}^{g=n} c_{gh}c_{gk}$$

for the $(h,k)^{th}$ *element.* In its second form the result in effect is that *if any orthogonal substitution be performed on*

$$y_1^2 + y_2^2 + \ldots + y_m^2 - y_{m+1}^2 - y_{m+2}^2 - \ldots - y_n^2,$$

then the discriminant of the resulting quadric is an axisymmetric orthogonant. He corroborates Lipschitz in the latter's conclusion that *the number of independent elements on which such an orthogonant depends is* $m(n-m)$.

In passing it is worth while to note for ourselves that the orthogonant given by either of these forms of statement is nothing more nor less than the result of multiplying $|\ c_{1n}\ |$ columnwise by $|\ c_{1n}\ |$ with its last $n-m$ rows altered in sign, being consequently in value equal to $(-1)^{n-m} \cdot |\ c_{1n}\ |^2$. A third form of statement, simpler than either of the others, is thus possible : and there is the further advantage associated with it that the orthogonal character of the new determinant follows at once from the theorem regarding the product of two orthogonants.

A section (pp. 603–607) is devoted to a re-examination and restatement of Cayley's construction-theorem, the result, when restricted to real elements, being in effect that *the theorem gives only those orthogonants that have not* -1 *for a latent root.* At the close of the section there is appended the readily-made deduction that *none of Cayley's orthogonants can be axisymmetric* ; and this leads the author to a lengthened investigation (pp. 691–698, 873–875), the main result of which is to establish the fact that Cayley's orthogonant can be made to approximate to an axisymmetric orthogonant, the latter being reached on proceeding to a limit.

Of the remaining sections the only one that directly concerns us is the twelfth (pp. 1063–1068), the opening pages of which serve to show not only that the deficiency in Cayley's orthogonant is unimportant in essence, but that it can be easily made good. We are then led on to the interesting result that *what one such orthogonant cannot give can be obtained by simply multiplying two of them together.*

BRISSE, C. (1890).

[Nouvelle méthode de discussion de l'équation en s. *Nouv. Annales de Math.*, (3) ix. pp. 367–372.]

The interest here is more in connection with the quadric of which Lagrange's determinant is the discriminant than with the determinant itself: and, at any rate, there is nothing fresh regarding the latter.

TABER, H. (1891): METZLER, W. H. (1892).

[On certain properties of symmetric, skew symmetric, and orthogonal matrices. *Proceed. London Math. Soc.*, xxii. pp. 449–469.]

[On certain properties of symmetric, skew symmetric, and orthogonal matrices. *American Journ. of Math.*, xv. pp. 274–282.]

Both these papers employ the terminology and notation of Cayleyan matrices: after all, however, the properties dealt with are properties of determinants, and as such we shall formulate them. The first three may be combined in the one statement that *If a determinant have* k *latent roots each equal to* ξ, *and* $\mathrm{L}(x)$ *be the corresponding latent function, then when the determinant is axisymmetric, orthogonant or zero-axial skew, the nullity-index of* $\mathrm{L}(\xi)$ *is* k. The fourth property, namely, that *the non-vanishing latent roots of a zero-axial skew determinant are pure imaginaries* is a deduction made by Clebsch in 1861 from a theorem of Hermite's (*Hist.*, ii. pp. 449–450): and the companion property, namely, that *the non-vanishing latent roots of an orthogonant* have unity for *their modulus* dates even farther back, having been established by Brioschi in 1854 (*Hist.*, ii. pp. 317–318).

The latter part of the second paper (pp. 278–282) concerns orthogonants that have both $+1$ and -1 for latent roots. First, the assertion is made that such an orthogonant is not included in Cayley's, and then it is sought to show how the want may be supplied. The result reached is simply that Cayley's three-line orthogonant *will* include the special form in question if the signs of its second and third rows be altered. The requisite specialization, in fact, consists in putting the $(2,3)^{\text{th}}$ element of the basic determinant equal to 0.

RADOS, G. (1891).

[Zur Theorie der orthogonalen Substitutionen. *Math. u. naturw. Berichte aus Ungarn,* x. pp. 95–97 ; but originally in Magyar, *Math. és Term. tud. Értésitö,* x. pp. 16–18.]

The theorem here formally stated and proved is in effect that *the compounds of an orthogonant are themselves orthogonants.* The proof is direct and simple, any product of two rows of the compound being shown to be suitably 1 or 0 by using the theorem for the multiplication of two oblong arrays.*

PRYM, F. (1892).

[Ueber orthogonale, involutorische und orthogonal-involutorische Substitutionen. *Abhandl. d. k. Ges. d. Wiss.* (Göttingen), xxxviii. 42 pp.]

The part of this dealing with orthogonal substitution (pp. 3–11) is restricted to the consideration of Cayley's orthogonant, the usefulness of which it seeks to extend. The procedure is based on the fact that an orthogonant remains an orthogonant when any number of its rows are changed in sign,—that is to say, when the rows in order are multiplied by $v_1, v_2, \ldots, v_n$, and

$$v_1^2 = v_2^2 = \ldots = v_n^2 = 1 ;$$

and the result is in effect that if $|\omega_{1n}|$ be Cayley's orthogonant, every one so derived from it is subject to the condition

$$\begin{vmatrix} \omega_{11}+v_1 & \omega_{12} & \omega_{13} & \ldots \\ \omega_{21} & \omega_{22}+v_2 & \omega_{23} & \ldots \\ \omega_{31} & \omega_{32} & \omega_{33}+v_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} \neq 0.$$

When the v's are all positive, the condition agrees with that already found.

'Involutory' substitution comes naturally to be considered along with orthogonal substitution because of an analogy between the two. Thus, while an orthogonal substitution and its reverse are

* It is curious to note the neglect suffered by this paper. Twenty years after its double publication it was unknown in responsible quarters. (See below,

so far alike as to be conjugate, an involutory substitution and its reverse are identical; or, again, while in an orthogonant

$$\mathrm{row}_h \cdot \mathrm{row}_k = \begin{cases} 0 \text{ when } h \neq k \\ 1 \text{ when } h = k, \end{cases}$$

in an 'involutant'

$$\mathrm{row}_h \cdot \mathrm{col}_k = \begin{cases} 0 \text{ when } h \neq k, \\ 1 \text{ when } h = k. \end{cases}$$

Keeping this analogy steadily before him, the author succeeds in finding for Cayley's rule of construction an exact counterpart in the case of the new determinant. Indeed, his working of the two rules differs only in regard to the basic determinant, which in Cayley's is *unit-axial skew* and in the counterpart is *unit-axial axisymmetric with a zero in the* $(\mathrm{h}, \mathrm{k})^{th}$ *place when* $\mathrm{v_h}=\mathrm{v_k}$. Thus, the order being the third and $v_1 = 1$, $v_2 = -1$, $v_3 = -1$, the basic determinant and its adjugate are

$$\begin{vmatrix} 1 & \alpha & \beta \\ \alpha & 1 & . \\ \beta & . & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & -\alpha & -\beta \\ -\alpha & 1-\beta^2 & \alpha\beta \\ -\beta & \alpha\beta & 1-\alpha^2 \end{vmatrix},$$

and the involutant

$$\begin{vmatrix} \frac{2}{\mathrm{B}} - 1 & \frac{-2\alpha}{\mathrm{B}} & \frac{-2\beta}{\mathrm{B}} \\ \frac{2\alpha}{\mathrm{B}} & \frac{2(\beta^2-1)}{\mathrm{B}} + 1 & \frac{-2\alpha\beta}{\mathrm{B}} \\ \frac{2\beta}{\mathrm{B}} & \frac{-2\alpha\beta}{\mathrm{B}} & \frac{2(\alpha^2-1)}{\mathrm{B}} + 1 \end{vmatrix},$$

where B stands for $1-\alpha^2-\beta^2$. Following on this is a detailed consideration of cases, the greater portion of the paper (pp. 16–38) being so occupied.

Lastly, substitutions that are both orthogonal and involutory are taken up and shortly discussed (pp. 38–42). The determinant of such substitutions is readily seen to be axisymmetric: its most general expression is nevertheless far from simple, the statement of the theorem, which deals with the mode of constructing it, occupying a full page and a half. (Cf. Lipschitz and Kronecker above, pp. 295–296.)

IGEL, B. (1892).

[Zur Theorie der Determinanten. *Monatshefte f. Math. u. Phys.*, iii. pp. 55–68.]

The second section (pp. 60–64) of this paper concerns the determinant

$$\begin{vmatrix} a_{11}-b_{11} & a_{12} & a_{13} & -b_{21} & . & . & -b_{31} & . & . \\ a_{21} & a_{22}-b_{11} & a_{23} & . & -b_{21} & . & . & -b_{31} & . \\ a_{31} & a_{32} & a_{33}-b_{11} & . & . & -b_{21} & . & . & -b_{31} \\ -b_{12} & . & . & a_{11}-b_{22} & a_{12} & a_{13} & -b_{32} & . & . \\ . & -b_{12} & . & a_{21} & a_{22}-b_{22} & a_{23} & . & -b_{32} & . \\ . & . & -b_{12} & a_{31} & a_{32} & a_{33}-b_{22} & . & . & -b_{32} \\ -b_{13} & . & . & -b_{23} & . & . & a_{11}-b_{33} & a_{12} & a_{13} \\ . & -b_{13} & . & . & -b_{23} & . & a_{21} & a_{22}-b_{33} & a_{23} \\ . & . & -b_{13} & . & . & -b_{23} & a_{31} & a_{32} & a_{33}-b_{33} \end{vmatrix},$$

or D say, which may be recognized as Sylvester's and Cayley's eliminant of the nine linear equations combined in the single matrical equation

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

(See below in Chap. XXII.) On the assumption that $D = 0$ and that the cofactors of the elements of its first row are equal to the cofactors of the elements of its first column, the train of reasoning followed is considered sufficient to establish "the orthogonal character of the x's and—what is the same thing—of the primary minors of D."

No reference is made to D being the eliminant of the two Lagrangian cubics

$$\begin{vmatrix} a_{11}-x & a_{12} & a_{13} \\ a_{21} & a_{22}-x & a_{23} \\ a_{31} & a_{32} & a_{33}-x \end{vmatrix} = 0 = \begin{vmatrix} b_{11}-x & b_{12} & b_{13} \\ b_{21} & b_{22}-x & b_{23} \\ b_{31} & b_{32} & b_{33}-x \end{vmatrix};$$

indeed it is clear that, whether Igel knew of Sylvester's work of 1884 on matrices or not, any connection of it with his own had not entered his mind.

SCHOUTE, P. H. (1892).

[Question 11731. *Educ. Times*, xlv. p. 489; lxiii. p. 380: *Math. from Educ. Times*, (2) xix. pp. 52–54.]

The proposition here is a mere variant of Brioschi's (*Hist.*, ii. pp. 317–318), to the effect that *the real latent roots of an orthogonant must be either* 1 *or* −1. Denoting by $f(x)$ the orthogonant with each of its diagonal elements diminished by x, the solver (Muir) forms the equation

$$f(x) \cdot f(-x) = 0,$$

whose roots are squares of the roots of the given equation, and finding that the product-determinant is skew with $1-x^2$ in each place of the diagonal, he is able to show that for even orders there are no real roots at all and for odd orders only one real root whose square is 1.

The possible loss of generality in using a Cayleyan orthogonant, as Brioschi does, is referred to in our account of Netto's paper of 1886. (See above, pp. 290–292.)

BANG, A. S. (1893).

[Om en Trediegradsligning. *Nyt Tidss. f. Math.*, iv. B, pp. 57–60; ix. B, pp. 94–96.]

The result here given is that *the roots of the equation*

$$\begin{vmatrix} a-x & b & c \\ d & e-x & f \\ g & h & k-x \end{vmatrix} = 0$$

are all real if $bfg = cdh$ *and* $\frac{bf}{c}, \frac{fg}{d}, \frac{gb}{h}$ *agree in sign*,—a generalization of the case where the determinant is axisymmetric. Another curious deduction that may be made from his fundamental equality is that *A three-line axisymmetric determinant is expressible in terms of its non-diagonal elements and their complementary minors.*

TABER, H. (1893, 1894).

[On orthogonal substitution. *Papers published by American Math. Soc.*, i. pp. 395–400.]

[On orthogonal substitutions that can be expressed as a function of a single alternate (or skew symmetric) linear substitution. *American Journ. of Math.*, xvi. pp. 123–130.]

[On orthogonal substitutions. *Bull. New-York Math. Soc.*, iii. pp. 251–259.]

These are again papers that employ only the algebra of Cayleyan matrices and that nevertheless deal with a subject viewable as purely determinantal. As before, therefore, appropriate note must be taken of them. They start with a recognition of the shortcomings of Cayley's orthogonant, stating now definitely that it fails in but one respect, namely, *the inability to represent an orthogonant having* -1 *for a latent root*. The most interesting point connected with the results is the part played in them by the square of the defaulting orthogonant itself. Dividing orthogonants into those that are the square of an orthogonant and those that are not, the author succeeds in assigning to the first class (1) all 2-line and 3-line positive-unit orthogonants, (2) all n-line positive-unit orthogonants with real elements, (3) Cayley's orthogonant, (4) all positive-unit orthogonants that are axisymmetric: and his most noteworthy theorem is that all those thus assigned are representable by the square of Cayley's orthogonant. The other properties of orthogonants of this class, and properties of orthogonants of the second class, are given in a summary at the end (pp. 258–9) of the last of the three papers.

METZLER, W. H. (1893).

[Compound determinants. *American Journ. of Math.*, xvi. pp. 131–150.]

At the close of this paper (pp. 149–150) there are under the heading 'Matrices' two theorems on orthogonants. They are both, however, rediscoveries, the one being Rados' of 1891 and the other Siacci's second theorem of 1872; but the fresh point of view is interesting. (*Hist.*, iv. p. 298; iii. p. 292.)

NETTO, E. (1894).

[Zur Theorie der orthogonalen Determinanten. *Acta Math.*, xix. pp. 105–114.]

Netto here resumes the subject of his paper of 1886, and, having first shown that when $| c_{1n} |$, or O say, is an orthogonant, the determinant

$$\begin{vmatrix} c_{11}+1 & c_{12} & \dots \\ c_{21} & c_{22}+1 & \dots \\ \dots & \dots & \dots \end{vmatrix}_n, \text{ or Q say,}$$

includes and is included in Stieltjes' determinant, proceeds to prove that *when* O $= +1$ *the vanishing of* Q *entails the vanishing of all its primary minors.* To this end he shows by ordinary multiplication that whether O is positive or negative unity,

$$QO = Q, \quad \begin{cases} Q_{\kappa\kappa}O = Q - Q_{\kappa\kappa}, \\ Q_{\kappa\bar{\lambda}}O = Q_{\lambda\kappa}, \end{cases}$$

the second of which recalls Siacci's fourth theorem of 1872, and is proof that when Q vanishes, any *coaxial* minor, $Q_{\kappa\kappa}$, vanishes also. For the case where the minor is non-coaxial, $Q_{\kappa\lambda}$, he takes the general identity

$$Q_{\kappa\kappa}Q_{\lambda\lambda} - Q_{\kappa\lambda}Q_{\lambda\kappa} = QQ_{\kappa\kappa,\lambda\lambda},$$

and knowing that $Q_{\kappa\kappa}$ and $Q_{\lambda\lambda}$ have Q for a factor, he sees that $-Q_{\kappa\lambda}Q_{\lambda\kappa}$, *i.e.* O $(Q_{\kappa\lambda})^2$, is in the same condition. He then passes to the establishment of a companion theorem, namely, that *when* O $= -1$, Q *necessarily vanishes, and, if in addition the primary minors of* Q *vanish, the secondary minors must vanish also.* In like manner he indicates briefly that *when* O $= +1$ *the vanishing of* Q *and its secondary minors entails the vanishing of the tertiary minors* : and from the three results thus reached he confidently draws the conclusion that *According as* O *is* $+1$ *or* -1 *the first set of minors of* Q *that do not wholly vanish must be of the order* n$-$2m *or* n$-$2m$-$1.

TABER, H. (1896).

[On a two-fold generalization of Stieltjes' theorem. *Proceed. London Math. Soc.*, xxvii. pp. 613–621.]

There are here given four theorems having a close family likeness, the third of them being that referred to in the title. The first may

be formulated thus: *If* -1 *be added to each of the diagonal elements of an odd power of a positive unit orthogonant, and the resulting determinant be such that its minors of the order* 2k *are all zero, then its minors of the order* $2k-1$ *are all zero also*: and where in this

$$-1, \quad \text{positive}, \quad 2k, \quad 2k-1,$$

appear, the others have respectively

$$-1, \quad \text{negative}, \quad 2k+1, \quad 2k,$$
$$+1, \quad \text{positive}, \quad 2k, \quad 2k+1,$$
$$+1, \quad \text{negative}, \quad 2k-1, \quad 2k.$$

The proofs are mainly dependent on the fact that an odd power of a zero-axial skew determinant is a determinant of the same kind, and the fact that when in a zero-axial skew determinant all the minors of order $2k$ vanish, so also must all the minors of order $2k-1$. Combining the third and fourth theorems and taking the case where the odd power is the first we have a result reached by Voss in 1877, namely, *If* $|\,a_{1n}\,|$ *be an orthogonant, and the minors of order* $n-r$ *in*

$$\begin{vmatrix} a_{11}+1 & a_{12} & \cdots \\ a_{21} & a_{22}+1 & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

are all zero but not all those of order $n-r-1$, *then* r *is odd or even according as* $|\,a_{1n}\,|$ *equals* $+1$ *or* -1. It is pointed out, however, that, notwithstanding an assertion of Voss', the like combination and specialization of the other two theorems hold only when n is even.

MUIR, T. (1896, 1897).

[On Lagrange's determinantal equation. *Philos. Magazine*, xliii. pp. 220–226.]

[On Sylvester's proof of the reality of the roots of Lagrange's determinantal equation. *American Journ. of Math.*, xix. pp. 312–318.]

Working with quaternions Tait arrived anew at Bang's result of 1893, finding* that "the roots of

* Tait, P. G. On the linear and vector function. *Proceed. Roy. Soc. Edinburgh*, xxi. pp. 160–164, 310–312.

$$\begin{vmatrix} A-x & b & c \\ d & E-x & f \\ g & h & I-x \end{vmatrix} = 0$$

are always all real, provided the single condition

$$cdh = bfg$$

be satisfied." Muir's investigation is a consequence of this, his mode of procedure being to multiply the rows of the given general determinant in order by $\mu_1, \mu_2, \ldots$ and the columns in order by $\mu_1^{-1}, \mu_2^{-1}, \ldots$, and then ascertain the conditions necessary for axisymmetry in the resulting new form of the determinant. The theorem reached is: *The* n^{thic} *equation*

$$\begin{vmatrix} 11-x & 12 & 13 & \ldots. \\ 21 & 22-x & 23 & \ldots. \\ 31 & 32 & 33-x & \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0$$

will have all its roots real if in the case of every pair h, k *of the indices* 2, 3, . . . , n *we have*

$$1h \cdot hk \cdot k1 = h1 \cdot kh \cdot 1k,$$

and

$$1h \cdot h1 = +.$$

These conditions, it is incidentally shown, ensure that in the case of every triad of the indices 1, 2, . . . , *n* we shall have

$$lh \cdot hk \cdot kl = hl \cdot kh \cdot lk,$$

and

$$hk \cdot kh = +.$$

The second paper establishes the same result by means of Sylvester's method of 1852 (*Hist.*, ii. pp. 310–311): and thereafter draws attention to properties of the determinant $| 11 \cdot 22 \ldots . nn |$, whose elements are so conditioned.

PASCAL, E. (1897).

[I Determinanti: . . . . viii+330 pp. Milano.]

Pascal extends (pp. 204–227) Brioschi's theorem regarding the latent roots of an orthogonant to the case where the value of the orthogonant is −1, showing that when the order is odd the solitary

real root is -1, and when the order is even there are two real roots $+1$, -1. He also gives corresponding extensions of Siacci's third and fourth theorems, and provides a companion to Siacci's second, namely, *If to each element of an orthogonant of value* ± 1 *there be added the corresponding element of an orthogonant of value* -1, *the determinant so formed vanishes.*

These minor advances, too, it must be noted, are accompanied by better and shorter proofs, of the kind used by Netto and exemplified by us under Siacci. (*Hist.*, iii. pp. 291–294.)

ELFRINKHOF, L. VAN (1897).

[Eene eigenschap van de orthogonale substitutie van de vierde orde. *Handel. Nederlandsch Natuur-en-Geneesk. Congress* (Delft), pp. 237–240.]

The first point of interest here is that Cayley's four-line orthogonant remains an orthogonant when an additional variable is taken and inserted in all places necessary for making the elements homogeneous. If ω be taken for the said variable, this means (*Hist.*, ii. p. 306) that the determinant

$$\begin{vmatrix}
\begin{pmatrix} f^2+g^2+h^2-\theta^2 \\ a^2-b^2-c^2+\omega^2 \end{pmatrix} & 2(f\theta+a\omega+bh-cg) & 2(g\theta+b\omega+cf-ah) & 2(h\theta+c\omega+ag-bf) \\
-f\theta-a\omega+bh-cg) & \begin{pmatrix} -a^2+b^2+c^2+\omega^2 \\ +f^2-g^2-h^2-\theta^2 \end{pmatrix} & 2(-c\theta-h\omega+fg-ab) & 2(b\theta+g\omega+hf-ca) \\
-gh-b\omega+cf-ah) & 2(c\theta+h\omega+fg-ab) & \begin{pmatrix} a^2-b^2+c^2+\omega^2 \\ -f^2+g^2-h^2-\theta^2 \end{pmatrix} & 2(-a\theta-f\omega+gh-bc) \\
-h\theta-c\omega+ag-bf) & 2(-b\theta-g\omega+hf-ca) & 2(a\theta+f\omega+gh-bc) & \begin{pmatrix} a^2+b^2-c^2+\omega^2 \\ -f^2-g^2+h^2-\theta^2 \end{pmatrix}
\end{vmatrix}$$

after each of its elements has been divided by Δ is a positive unit orthogonant

$$\begin{aligned}
&\text{if} \quad \omega\theta = af+bg+ch, \\
&\text{and} \quad \Delta = a^2+b^2+c^2+\omega^2+f^2+g^2+h^2+\theta^2, \\
&\textit{i.e.} \quad = \begin{vmatrix} \omega & a & b & c \\ -a & \omega & -h & g \\ -b & h & \omega & f \\ -c & -g & f & \omega \end{vmatrix} \div \omega^2.
\end{aligned}$$

The truth of the extension the author seems to consider self-evident,

The next and main point of interest is that the orthogonant in the extended form as just given is resolvable into

$$\begin{vmatrix} \frac{\omega-\theta}{\sqrt{\Delta}} & \frac{-a-f}{\sqrt{\Delta}} & \frac{-b-g}{\sqrt{\Delta}} & \frac{-c-h}{\sqrt{\Delta}} \\ \frac{a+f}{\sqrt{\Delta}} & \frac{\omega-\theta}{\Delta\sqrt{}} & \frac{-c-h}{\sqrt{\Delta}} & \frac{b+g}{\sqrt{\Delta}} \\ \frac{b+g}{\sqrt{\Delta}} & \frac{c+h}{\sqrt{\Delta}} & \frac{\omega-\theta}{\sqrt{\Delta}} & \frac{-a-f}{\sqrt{\Delta}} \\ \frac{c+h}{\sqrt{\Delta}} & \frac{-b-g}{\sqrt{\Delta}} & \frac{a+f}{\sqrt{\Delta}} & \frac{\omega-\theta}{\sqrt{\Delta}} \end{vmatrix} \cdot \begin{vmatrix} \frac{\omega+\theta}{\sqrt{\Delta}} & \frac{f-a}{\sqrt{\Delta}} & \frac{g-b}{\sqrt{\Delta}} & \frac{h-c}{\sqrt{\Delta}} \\ \frac{a-f}{\sqrt{\Delta}} & \frac{\omega+\theta}{\sqrt{\Delta}} & \frac{c-h}{\sqrt{\Delta}} & \frac{g-b}{\sqrt{\Delta}} \\ \frac{b-g}{\sqrt{\Delta}} & \frac{h-c}{\sqrt{\Delta}} & \frac{\omega+\theta}{\sqrt{\Delta}} & \frac{a-f}{\sqrt{\Delta}} \\ \frac{c-h}{\sqrt{\Delta}} & \frac{b-g}{\sqrt{\Delta}} & \frac{f-a}{\sqrt{\Delta}} & \frac{\omega+\theta}{\sqrt{\Delta}} \end{vmatrix},$$

and, what is still more interesting, that each of the two new determinants is itself a positive unit orthogonant. These facts are readily verifiable by applying the multiplication-theorem and the definition of an orthogonant. It may be noted, too, that the two factors are almost quite alike in form, both belonging to the type of four-line orthogonant which is skew and has only four distinct elements (*Hist.*, ii. pp. 287–288). The result thus reached is consequently seen to be that *Cayley's four-line orthogonant* (or, if we please, the extended form of the same) *is resolvable into two four-line orthogonants of Souillart's type.*

The author's introduction of the additional variable ω, as appears from our above modification of his procedure, is not at all necessary for the establishment of his result. The same is true of the set of substitutions which he makes for $a, b, c, \omega, f, g, h, \theta$.

JAHNKE, E. (1897, 1910).

[Ueber einen Zusammenhang zwischen den Elementen orthogonaler Neuner und Sechzehner systeme. *Crelle's Journ.*, cxviii. pp. 224–233 ; *Intermédiaire des Math.*, v. p. 224 ; xvii. p. 198.]

A partial rediscovery of Rados' theorem of 1891 is here drawn attention to. The main subject of the paper is a development of results reached by F. Caspary in his studies on the theta-functions.* The nature of the connection referred to in the title is formulated in two theorems.

* In particular see *Comptes rendus* ... (Paris), civ. (1887), pp. 490–493.

RADOS, G. (1898).

[Ueber die Bedingungsgleichungen zwischen den Coefficienten der orthogonalen Substitutionen. *Math. u. naturw. Berichte aus Ungarn*, xvi. pp. 236–240: originally in *Math. és Termesz. Értesitö*, xvi. pp. 123–127.]

The object here is to establish the consistency and independence of the set of equations in question. The former property, consistency, is held to be vouched for by the known existence of any particular solution of the set, for example, the solution whose determinant has the matrix unity. To prove the other property, mutual independence, use is made of the theorem that *if* $\phi_1, \phi_2, \ldots, \phi_n$ *be functions of* $x_1, x_2, \ldots, x_{n+r}$, *the* ϕ*'s will be mutually independent if the Jacobian-like array* (*Hist.*, iii. p. 260)

$$\left\| \begin{matrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_{n+r}} \\ \cdot & \cdot & \cdots & \cdot \\ \frac{\partial \phi_n}{\partial x_1} & \frac{\partial \phi_n}{\partial x_2} & \cdots & \frac{\partial \phi_n}{\partial x_{n+r}} \end{matrix} \right\|$$

be non-evanescent. Thus, if the order be the 3rd and the conditioning equations consequently be

$$\left.\begin{aligned} \omega_{11}^2 + \omega_{12}^2 + \omega_{13}^2 &= 1 \\ \omega_{11}\omega_{21} + \omega_{12}\omega_{22} + \omega_{13}\omega_{23} &= 0 \\ \omega_{11}\omega_{31} + \omega_{12}\omega_{32} + \omega_{13}\omega_{33} &= 0 \\ \omega_{21}^2 + \omega_{22}^2 + \omega_{23}^2 &= 1 \\ \omega_{21}\omega_{31} + \omega_{22}\omega_{32} + \omega_{23}\omega_{33} &= 0 \\ \omega_{31}^2 + \omega_{32}^2 + \omega_{33}^2 &= 1 \end{aligned}\right\}$$

the array to be tested is

$$\left\| \begin{matrix} 2\omega_{11} & 2\omega_{12} & 2\omega_{13} & \cdot & \cdot & \cdot & & & \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{11} & \omega_{12} & \omega_{13} & \cdot & \cdot & \cdot \\ \omega_{31} & \omega_{32} & \omega_{33} & \cdot & \cdot & \cdot & \omega_{11} & \omega_{12} & \omega_{13} \\ \cdot & \cdot & \cdot & 2\omega_{21} & 2\omega_{22} & 2\omega_{23} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \omega_{31} & \omega_{32} & \omega_{33} & \omega_{21} & \omega_{22} & \omega_{23} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 2\omega_{31} & 2\omega_{32} & 2\omega_{33} \end{matrix} \right\|,$$

and what we have to show is that all its six-line determinants do not vanish. Clearly this will be done if we can show that the sum

of the squares of the said determinants does not vanish, and this again if we can show that the so-called square of the array does not vanish. Now the square of the array is readily seen to be

$$\begin{vmatrix} 4 & . & . & . & . & . \\ . & 2 & . & . & . & . \\ . & . & 2 & . & . & . \\ . & . & . & 4 & . & . \\ . & . & . & . & 2 & . \\ . & . & . & . & . & 4 \end{vmatrix}, \textit{ i.e. } 2^2 \cdot 2^3 \cdot 2^4, \textit{ i.e. } 2^9 :$$

consequently the desired end is reached.

METZLER, W. H. (1898).

[On the roots of a determinantal equation. *American Journ. of Math.*, xxi. pp. 367–368.]

The theorem established here is a companion to Muir's of 1896–97, being in effect that *the latent roots of* | 11·22· . . . ·nn | *are all pure imaginary if in the case of every pair* h, k *of the indices* 2, 3, . . . , n *we have*

$$1h \cdot hk \cdot k1 = -h1 \cdot kh \cdot 1k,$$

$$1h \cdot h1 = -,$$

and $$11 = 22 = \ldots = 0.$$

ELLIOTT, E. B. (1899).

[A simple proof of the reality of the roots of discriminating determinant equations, and of kindred facts. *Quart. Journ. of Math.*, xxxi. pp. 233–240.]

Such an equation as Lagrange's determinantal equation is here recommended to be viewed in its character as a test for a set of linear equations being consistent. In the second case considered, namely, that dealt with by Routh in 1884, it is found in this way that the equation can only be satisfied by an imaginary value of x when either (1) it is an identity satisfied by all values, or (2) the quadrics of which $| a_{1n} |$ and $| b_{1n} |$ are the discriminants can assume both signs for real values of the variables. With this may be compared Cayley's result of 1846 (*Hist.*, ii. pp. 112–113). Two

other cases treated in the same fashion are brought forward for the first time. If we denote the matrix of Routh's case by

$$\begin{pmatrix} \\ | a_{1n} | \end{pmatrix} + x \begin{pmatrix} \\ | b_{1n} | \end{pmatrix},$$

$| a_{1n} |$ and $| b_{1n} |$ being axisymmetric, then the matrices of the third and fourth cases are

$$x \begin{pmatrix} \\ | a_{1n} | \end{pmatrix} - \begin{pmatrix} \\ | c_{1n} | \end{pmatrix},$$

$$\begin{pmatrix} \\ | a_{1n} | \end{pmatrix} - x \begin{pmatrix} \\ | c_{1n} | \end{pmatrix},$$

where $| c_{1n} |$ is zero-axial skew. In the third case it is shown that, *provided the quadric of which* $| a_{1n} |$ *is the discriminant be always* > 0 *or always* < 0 *for real values of its variables, the roots are all pure imaginaries when* n *is even, and are pure imaginaries accompanied by one or more zeros when* n *is odd.* In the fourth case the result is similar, provided the quadric cannot change sign for real values of its variables.

Bibliographical Note.

As in the preceding volume (pp. 306–308) we append the titles of writings that are wholly or mainly applicational, that is to say, which deal with the subject of orthogonal transformation without giving any special attention to the determinant forms made use of :

1880. Gundelfinger, S. Ueber die Transformation einer quadratischen Form in eine Summe von Quadraten. *Crelle's Journ.*, xci. pp. 221–237.

1880. Laurent, H. Sur la réduction des polynomes du second degré homogènes à des sommes de carrés. *Nouv. Annales de Math.*, (2) xix. pp. 12–27.

1882. Walecki, . Équation en S de degré m et décomposition d'une forme quadratique en carrés. *Nouv. Annales de Math.*, (3) i. pp. 401–409, 556–560.

1884. Köhler, . Sur la décomposition des polynomes homogènes du second degré en somme de carrés. *Journ. de math. spéc.*, viii. pp. 185–189, 209–214.

1886. Benoît, . Note sur lu décomposition d'une forme quadratique à m variables en une somme de $m-n$ carrés. *Nouv. Annales de Math.*, (3) v. pp. 30–36.

1888. Studnička, F. J. Neue Transformation einer homogenen quadratischen Form von n Variabeln in die Summe von n Quadraten. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), 1888, pp. 256–265.

1889. Sylvester, J. J. A new proof that a general quadric may be reduced to its canonical form by means of a real orthogonal substitution. *Messenger of Math.*, (2) xix. pp. 1–5.

1890. Lipschitz, R. Beiträge zu der Theorie der gleichseitigen Transformation von zwei quadratischen oder bilinearen Formen. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), 1890, pp. 485–523.

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1895. Taber, H. On those orthogonal substitutions that can be generated by the repetition of an infinitesimal orthogonal substitution. *Proceed. London Math. Soc.*, xxvi. pp. 364–376.

1896. Kneser, A. Bemerkungen zu der ausnahmlosen Auflösung des Problems, eine quadratischen Form. . . . *Archiv d. Math. u. Phys.*, (2) xv. pp. 225–231.

CHAPTER XIII.

PERSYMMETRIC DETERMINANTS, FROM 1881 TO 1899.

THE work done on persymmetric determinants, which for the preceding twenty-year period had shown a falling off, again gives evidence of growth, the number of writings here reported on being at least a half more than for the period 1860–1879.

HEUN, K. (1881): KRONECKER, L. (1881): HEUN, K. (1881).

[Neue Darstellung der Kugelfunctionen und der verwandten Functionen durch Determinanten. *Nachrichten . . . Ges. d. Wiss.* (Göttingen), pp. 104–119.]

[Auszug aus einem Briefe . . . *Nachrichten . . . Ges. d. Wiss.* (Göttingen), 1881, pp. 271–279; or *Werke*, ii. pp. 103–112.]

[Die Kugelfunctionen und Lamé'schen Functionen als Determinanten. Dissert. 32 pp. Göttingen.]

The earlier part of the dissertation is an orderly exposition of known results. The two forms of determinant considered are those used by Sylvester in 1851 (*Hist.*, ii. pp. 334–335), namely,

$$\begin{vmatrix} a_1x-a_2 & a_2x-a_3 & \dots & a_nx-a_{n+1} \\ a_2x-a_3 & a_3x-a_4 & \dots & a_{n+1}x-a_{n+2} \\ \dots & \dots & \dots & \dots \\ a_nx-a_{n+1} & a_{n+1}x-a_{n+2} & \dots & a_{2n+1}x-a_{2n} \end{vmatrix}, \quad \text{or } P_n, \text{ say,}$$

and

$$\begin{vmatrix} 1 & a_1 & a_2 & \dots & a_n \\ x & a_2 & a_3 & \dots & a_{n+1} \\ x^2 & a_3 & a_4 & \dots & a_{n+2} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_{n+1},$$

the sign-factor connecting them being $(-1)^n$. First there is established the linear homogeneous relation which connects P_n, P_{n-1}, P_{n-2}, and which in substance is due to Jacobi (*Crelle's Journ.*, xv. p. 123). With Heun it takes the form

$$P_n - \left\{\frac{A\beta - B\alpha}{\alpha^2} + \frac{A}{\alpha}x\right\} P_{n-1} + \frac{A^2}{\alpha^2} P_{n-2} = 0,$$

where A, $-B$ and α, $-\beta$ are the coefficients of the two highest powers of x in P_n and P_{n-1} respectively. The proof consists in dividing the descending series for P_n by that for P_{n-1} and then suitably transforming the remainder: it suffers from the unattractiveness of the latter part. After such a trinomial recurrence-formula the natural step is to a continued fraction, F say, of the form

$$\cfrac{-a_1^2}{-a_2 + a_1x - \cfrac{q_1^2}{p_1 + q_1x - \cfrac{q_2^2}{p_2 + q_2x - \ddots}}}$$

whose n^{th} convergent has the denominator P_n. The corresponding numerator Q_n, we are told, is

$$\begin{vmatrix} . & . & a_1 & \dots & a_n \\ . & a_1 & a_2 & \dots & a_{n+1} \\ a_1 & a_2 & a_3 & \dots & a_{n+2} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & a_n & a_{n+1} & \dots & a_{2n} \\ 1 & x & x^2 & \dots & . \end{vmatrix},$$

and each of the equations

$$P_n(x) = 0, \quad Q_n(x) = 0,$$

as is readily seen from the recurrence-formula, has all its roots real and different.

After drawing attention to the known instances, besides Jacobi's, of the natural occurrence of $P_n(x)$, the first step is taken, doubtless under Heine's guidance, to bring it into touch with Lamé's functions, the specialization consisting in connecting the a's by a recurrence-formula

$$a_r = \lambda_{r,1}a_{r-1} + \lambda_{r,2}a_{r-2} + \dots + \lambda_{r,p}a_{r-p},$$

and so allowing the fraction F to be continued infinitely and to be represented by the series

$$\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\ldots.$$

It is then shown that such a formula is satisfied by integrals of the type

$$\int_{\zeta_{s-1}}^{\zeta_s}\frac{z^{r-1}\partial z}{\sqrt{\psi(z)}},$$

where $\psi(z) \equiv z(z-\xi_1)(z-\xi_2)\ldots(z-\xi_p)$;

and that then

$$\mathrm{F} = c_1\int_0^{\zeta_1}\frac{\partial z}{(x-z)\sqrt{\psi(z)}}+c_2\int_{\zeta_1}^{\zeta_2}\frac{\partial z}{(x-z)\sqrt{\psi(z)}}+\ldots+c_p\int_{\zeta_{p-1}}^{\zeta_p}\frac{\partial z}{(x-z)\sqrt{\psi(z)}}.$$

The final step of importance is the formulating (p. 19) of the differential equation of the second order which $\mathrm{P}_n(x)$ must satisfy in order to coalesce with Lamé's functions. From this the passage is readily made (p. 22) to spherical harmonics ('Kugelfunctionen') and related functions, a few examples being given where the integrals representing the a's are easily evaluated. Thus Laplace's function of the $(2n)^{\text{th}}$ degree in x, which Glaisher had expressed as a recurrent in 1876, is found equal to

$$\begin{vmatrix} x^2-\frac{1}{3} & \frac{1}{3}x^2-\frac{1}{5} & \frac{1}{5}x^2-\frac{1}{7} & \ldots. \\ \frac{1}{3}x^2-\frac{1}{5} & \frac{1}{5}x^2-\frac{1}{7} & \frac{1}{7}x^2-\frac{1}{9} & \ldots. \\ \frac{1}{5}x^2-\frac{1}{7} & \frac{1}{7}x^2-\frac{1}{9} & \frac{1}{9}x^2-\frac{1}{11} & \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n \div \frac{1\cdot2\cdot3\ldots2n}{1\cdot3\cdot5\ldots(4n-1)}\cdot\mathrm{C},$$

where C is the persymmetric determinant of the n^{th} order whose elements are

$$1,\ \frac{1}{3},\ \frac{1}{5},\ \ldots.,\ \frac{1}{4n-3}.$$

In his previous shorter communication Heun had already expressed the n^{th} 'Kugelfunction' as an arithmetic multiple of

$$\begin{vmatrix} x & \frac{1}{3} & \frac{1}{5}x & \frac{1}{5} & \ldots. \\ \frac{1}{3} & \frac{1}{3}x & \frac{1}{5} & \frac{1}{5}x & \ldots. \\ \frac{1}{3}x & \frac{1}{3} & \frac{1}{5}x & \frac{1}{7} & \ldots. \\ \frac{1}{5} & \frac{1}{5}x & \frac{1}{7} & \frac{1}{7}x & \ldots. \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n,$$

also its differential-coefficients in similar fashion, and, as might be expected,

$$\cos 2\theta = 2^2 \begin{vmatrix} \cos\theta & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\cos\theta \end{vmatrix}, \quad \cos 3\theta = 2^6 \begin{vmatrix} \cos\theta & \frac{1}{2} & \frac{1}{2}\cos\theta \\ \frac{1}{2} & \frac{1}{2}\cos\theta & \frac{1\cdot 3}{2\cdot 4} \\ \frac{1}{2}\cos\theta & \frac{1\cdot 3}{2\cdot 4} & \frac{1\cdot 3}{2\cdot 4}\cos\theta \end{vmatrix}, \ \ldots$$

Kronecker's letter, suggested by Heun's communication, tells the story of his own acquaintance with persymmetric determinants, emphasizing and illustrating the fact of their persistent connection with the problem of finding three integral functions

$\mathrm{F}(x)$ of degree μ, $\Phi(x)$ of degree $\nu - n'$, $\Psi(x)$ of degree ν,

so that with two given integral functions

$f_1(x)$ of degree n, $f_2(x)$ of degree $n - n'$ $(< n)$

there may exist the relation

$$\mathrm{F}(x) = f_2(x) \cdot \Psi(x) - f_1(x) \cdot \Phi(x)$$

and $\mu + \nu < n$.

KAPTEYN, W. (1881).

[Note sur une classe de fonctions symétriques. *Archiv d. Math. u. Phys.*, lxvii. pp. 102–104.]

Involved in this is a result reached by Muir about the same date, namely,

$$\begin{vmatrix} a_1 & a_2 & a_3 & \ldots. & a_n \\ \mathrm{U}a_n & a_1 & a_2 & \ldots. & a_{n-1} \\ \mathrm{U}a_{n-1} & \mathrm{U}a_n & a_1 & \ldots. & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathrm{U}a_2 & \mathrm{U}a_3 & \mathrm{U}a_4 & \ldots. & a_1 \end{vmatrix} = (a_1 + \rho_1 a_2 + \rho_1^2 a_3 + \ldots + \rho_1^{n-1} a_n) \cdot (a_1 + \rho_2 a_2 + \rho_2^2 a_3 + \ldots + \rho_2^{n-1} a_n) \cdot \ldots \cdot (a_1 + \rho_n a_2 + \rho_n^2 a_3 + \ldots + \rho_n^{n-1} a_n),$$

where $\rho_1, \rho_2, \ldots$ are the n^{th} roots of U. With Muir * it takes the form

$$(-1)^{\frac{1}{4}4(4+1)} \mathrm{P}(b\mathrm{U}, c\mathrm{U}, d\mathrm{U}, e\mathrm{U}, a, b, c, d, e) = \mathrm{C}(a, b\rho, c\rho^2, d\rho^3, e\rho^4),$$

* Text-book (1881), p. 194, ex. 38. Here also are found the equality

$$\mathrm{P}(a_1 - a_2, a_2 - a_3, \ldots, a_m - a_1, a_1 - a_2, \ldots, a_{m-3} - a_{m-2}) = \mathrm{P}(a_3 - a_4, a_4 - a_5, \ldots, a_m - a_1, a_1 - a_2, \ldots, a_{m-1} - a_m),$$

and the evaluation of $\mathrm{P}(a_1, \ldots, a_{2n-1})$ when $a_r = r^m + c_1 r^{m-1} + \ldots + c_m$.

and is established with the greatest ease by multiplying the columns on the right-hand side by $\rho^4, \rho^3, \rho^2, \rho, 1$ respectively and then dividing the rows in similar fashion. He recalls the previously familiar cases where $U = 1$ and $U = -1$.

CHRYSTAL, G. (1881).

[On a special class of Sturmians. *Transac. R. Soc. Edinburgh*, xxx. pp. 161–165.]

Chrystal's result is avowedly a slight extension of Joachimsthal's of 1854 (*Hist.*, ii. pp. 169–171). It is that *if* s_p *stand for*

$$a_1^p + a_2^p + \ldots + a_n^p,$$

and

$$S_n(x) \text{ for } (-1)^n \begin{vmatrix} 1 & s_p & s_{p+1} & \ldots & s_{p+n-1} \\ x & s_{p+1} & s_{p+2} & \ldots & s_{p+n} \\ x^2 & s_{p+2} & s_{p+3} & \ldots & s_{p+n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^n & s_{p+n} & s_{p+n+1} & \ldots & s_{p+2n-1} \end{vmatrix},$$

i.e. for

$$a_1^p a_2^p \ldots a_n^p \cdot \xi(a_1, a_2, \ldots, a_n) \cdot (x-a_1)(x-a_2) \ldots (x-a_n),$$

then $S_n(x), S_{n-1}(x), \ldots, S_1(x), S_0(x)$ *form a Sturmian series.* In his definition of a Sturmian series he includes, besides the requirement that the member preceding and the member following a vanishing member shall have opposite signs, the further requirement that the first two members shall have opposite signs when the first is about to vanish. To show that $S_n(x), \ldots, S_0(x)$ satisfies the former requirement he puts $S_n(x)$ in its purely persymmetric form, and uses a property which accompanies such a transformation, the fuller theorem being

$$\begin{vmatrix} 1 & x & x^2 & x^3 & \ldots \\ b_1 & b_2 & b_3 & b_4 & \ldots \\ b_2 & b_3 & b_4 & b_5 & \ldots \\ b_3 & b_4 & b_5 & b_6 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} b_2-xb_1 & b_3-xb_2 & b_4-xb_3 & \ldots \\ b_3-xb_2 & b_4-xb_3 & b_5-xb_4 & \ldots \\ b_4-xb_3 & b_5-xb_4 & b_6-xb_5 & \ldots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

and the first r*-line coaxial minor on the left is equal to the first* $(r-1)$*-line coaxial minor on the right.*

KRONECKER, L. (1881).

[Zur Theorie der Elimination einer Variabeln aus zwei algebraischen Gleichungen. *Monatsb. . . . Akad. d. Wiss.* (Berlin), pp. 535–600; or *Werke*, ii. pp. 115–192. Abstract in *Bull. des Sci. Math.*, (2) x. (2) pp. 47–58.]

From a general theorem, which we have already quoted, Kronecker makes the deduction (pp. 559–560): *If in the array*

$$\begin{array}{cccccc} c_0 & c_1 & \dots & c_{n-1} & f_{0n} & f_{0,n+1} \ \dots \\ c_1 & c_2 & \dots & c_n & f_{1n} & f_{1,n+1} \ \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_n & c_{n+1} & \dots & c_{2n-1} & f_{nn} & f_{n,n+1} \ \dots \end{array},$$

where each column after the n^{th} *is an aggregate of multiples of the first* n *columns, there be one* m*-line minor that does not vanish, while the minors of higher order all vanish, then no linear relation can exist between the first* m *rows nor between the first* m *columns of the given array.* From this it at once follows that the first m-line minor of all, namely, the persymmetric determinant $P(c_0, c_1, \dots, c_{2m-2})$, cannot vanish; and as a consequence it must either be the non-vanishing minor mentioned in the data or another so far like it. The finding of m for any given set of c's is thus facilitated.

It is next shown that if m still denote the nullity-index of the first n columns of the array

$$\begin{array}{cccccc} c_0 & c_1 & \dots & c_{n-1} & c_n & \dots \\ c_1 & c_2 & \dots & c_n & c_{n+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ c_n & c_{n+1} & \dots & c_{2n-1} & c_{2n} & \dots \end{array}$$

the c's are characterized (1) by being the subject of a recurrence-formula of order * m but not of lower order, (2) by the property that each column of the array can be expressed as a linear function

* When a series of at least $2m$ quantities is such that every member of it is expressible as one and the same linear function of the m members immediately preceding, it is said by Kronecker to be subject to a linear recurrence-formula of the m^{th} order. The slightly different English usage is followed in *Hist.*, iii. pp. 318–9,

of the first m columns but not of the first $m-1$ columns, (3) by the conditions

$$P(c_0, c_1, \ldots, c_{2m-2}) \gtrless 0, \quad \begin{Vmatrix} c_0 & c_1 & \ldots & c_{m-1} & c_m & \ldots \\ c_1 & c_2 & \ldots & c_m & c_{m+1} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_m & c_{m+1} & \ldots & c_{2m} & c_{2m+1} & \ldots \end{Vmatrix} = 0,$$

and (4) by the conditions

$$P(c_0, c_1, \ldots, c_{2m-2}) \gtrless 0,$$

$$P(c_0, c_1, \ldots, c_{2m}) = 0 = P(c_0, c_1, \ldots, c_{2m+2}) = \ldots.$$

Following the persymmetric determinant $P(c_0, c_1, \ldots, c_{2m-2})$ there naturally come the two others,

$$P(0, c_0, xc_0 - c_1, xc_1 - c_2, \ldots, xc_{2m-2} - c_{2m-1}),$$

$$P(xc_0 - c_1, xc_1 - c_2, \ldots, xc_{2m-2} - c_{2m-1}).$$

Calling these C_m, D_m Kronecker defines them by the identity

$$\begin{vmatrix} U & c & \ldots. & c_{m-1} \\ c_0 & c_0x - c_1 & \ldots. & xc_{m-1} - c_m \\ c_1 & c_1x - c_2 & \ldots. & xc_m - c_{m+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_{m-1} & c_{m-1}x - c_m & \ldots. & xc_{2m-2} - c_{2m-1} \end{vmatrix} \equiv UD_m - C_{m1};$$

and then by resolving into factors the two-line minor whose elements are those at the corners of the adjugate of $-C_m$, he readily obtains

$$C_mD_{m-1} - C_{m-1}D_m = \{P(c_0, c_1, \ldots, c_{2m-2})\}^2.$$

Next, he transforms the determinant $UD_m - C_m$ into

$$\begin{vmatrix} c_0 & c_1 & \ldots. & c_{m-1} & U \\ c_1 & c_2 & \ldots. & c_m & -c_0 + xU \\ c_2 & c_3 & \ldots. & c_{m+1} & -c_1 - c_0x + x^2U \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ c_m & c_{m+1} & \ldots. & c_{2m-1} & -c_{m-1} - c_{m-2}x - \ldots - c_0x^{m-1} + x^mU \end{vmatrix},$$

and putting

$$U = c_0x^{-1} + c_1x^{-2} + \ldots. \text{ ad inf.}$$

formulates in two ways the necessary and sufficient conditions for the determinant vanishing identically, and therefore for the existence of the equality

$$C_m = D_m \cdot (c_0x^{-1} + c_1x^{-2} + \ldots \text{ ad. inf.}).$$

Since C_m/D_m is independent of c_{2m}, c_{2m+1}, . . . , it is clear that the conditions must at least involve material for the determination of these coefficients in terms of c_0, c_1, . . . , c_{2m-1}.

The last result which specially concerns us is now reached (§ 10, p. 568), namely, *If* f(x) *be of lower degree than* g(x), and

$$\frac{f(x)}{g(x)} \equiv c_0 x^{-1} + c_1 x^{-2} + \ldots,$$

the necessary and sufficient conditions for f(x) *and* g(x) *having a common factor of the* $(n-m)^{th}$ *degree is the vanishing of the first* n−m *determinants of the array*

$$\begin{array}{ccccccc} c_0 & c_1 & \ldots & c_m & c_{m+1} & \ldots \\ c_1 & c_2 & \ldots & c_{m+1} & c_{m+2} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ c_m & c_{m+1} & \ldots & c_{2m} & c_{2m+1} & \ldots, \end{array}$$

with the additional condition

$$P(c_0, c_1, \ldots, c_{2m-2}) \gtrless 0$$

if the said common factor is to be the highest.

MUIR, T. (1881).

[On a property of persymmetric determinants. *Messenger of Math.*, xi. pp. 65–67.]

What is given here is a generalization and fresh proof of Hankel's theorem. The determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_2 & a_3 & a_4 & \ldots & a_{n+1} \\ a_3 & a_4 & a_5 & \ldots & a_{n+2} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_n & a_{n+1} & a_{n+2} & \ldots & a_{2n-1} \end{vmatrix}$$

is multiplied by 1 in the form

$$\begin{vmatrix} 1 & . & . & \ldots & . \\ m & 1 & . & \ldots & . \\ m^2 & 2m & 1 & \ldots & . \\ m^3 & 3m^2 & 3m & \ldots & . \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m^{n-1} & (n-1)_1 m^{n-2} & (n-1)_2 m^{n-3} & \ldots & 1 \end{vmatrix}$$

and the product obtained, $|\,V_{1n}\,|$ say, by the same, the result being denoted by E_{1n}. By reason of this,

$$\begin{aligned}
E_{rs} &= m^{r-1}\gamma_{s1} + (r-1)_1 m^{r-2}\gamma_{s2} + (r-1)_2 m^{r-3}\gamma_{s3} + \dots \\
&= \quad m^{r-1}\{m^{s-1}a_1 + (s-1)_1 m^{s-2}a_2 + (s-1)_2 m^{s-3}a_3 + \dots\} \\
&\quad + (r-1)_1 m^{r-2}\{m^{s-1}a_2 + (s-1)_1 m^{s-2}a_3 + (s-1)_2 m^{s-3}a_4 + \dots\} \\
&\quad + (r-1)_2 m^{r-3}\{m^{s-1}a_3 + (s-1)_1 m^{s-2}a_4 + (s-1)_2 m^{s-3}a_5 + \dots\} \\
&\quad + \dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&= m^{r+s-2}a_1 + \{(s-1)_1 + (r-1)_1\}m^{r+s-3}a_2 \\
&\qquad + \{(s-1)_2 + (s-1)_1(r-1)_1 + (r-1)_2\}m^{r+s-4}a_3 \\
&\qquad + \dots\dots\dots\dots\dots\dots\dots \\
&= m^{r+2-2}a_1 + (r+s-2)_1 m^{r+s-3}a_2 + (r+s-2)_2 m^{r+s-4}a_3 + \dots \\
&= (a_1, a_2, a_3, \dots \text{◊} m+1)^{r+s-2}.
\end{aligned}$$

It therefore follows that the element E_{rs} does not alter for values of r and s whose sum remains constant, that is to say, $|\,E_{1n}\,|$ is persymmetric. We consequently have the theorem: *The persymmetric determinant of* $a_1, a_2, \dots, a_{2n-1}$ *is equal to the persymmetric determinant of*

$a_1, (a_1, a_2 \text{◊} m, 1), (a_1, a_2, a_3 \text{◊} m, 1)^2, \dots, (a_1, a_2, \dots, a_{2n-1} \text{◊} m, 1)^{2n-2}$.

The case where m is -1 is that at first referred to.

MUIR, T. (1885).

[Question 8016. *Educ. Times*, xxxviii. p. 69, p. 209: *Math. from Educ. Times*, xliii. p. 103.]

The equality here intimated is

$$\begin{vmatrix} 1 & \frac{1}{4}\Sigma a & \frac{1}{6}\Sigma ab \\ \frac{1}{4}\Sigma a & \frac{1}{6}\Sigma ab & \frac{1}{4}\Sigma abc \\ \frac{1}{6}\Sigma ab & \frac{1}{4}\Sigma abc & abcd \end{vmatrix} = \begin{matrix} \frac{1}{2}\left[\frac{1}{6}\Sigma ab - \frac{1}{2}(ab+cd)\right] \\ \cdot\left[\frac{1}{6}\Sigma ab - \frac{1}{2}(ac+bd)\right] \\ \cdot\left[\frac{1}{6}\Sigma ab - \frac{1}{2}(ad+bc)\right]. \end{matrix}$$

The proof elicited is merely a verification, and does not help towards ascertaining whether there exists a corresponding equality for the case of a four-line determinant with six variables and resolvable into four cubic factors.

WARD, P. C. (1885): MUIR, T. (1886).

[On the rationalization of $a^{\frac{1}{n}}+b^{\frac{1}{n}}+c^{\frac{1}{n}}=0$. *Messenger of Math.*, xv. pp. 42–44.]

[Questions 8380, 8459. *Educ. Times*, xxxviii. p. 411; xxxix. p. 71: *Math. from Educ. Times*, xlvii. p. 107.]

[Question 8408. *Educ. Times*, xxxix. p. 34.]

In substance Ward's procedure is to remove one of the left-hand terms to the right-hand side, raise both sides to the n^{th} power, and then dialytically eliminate the radicals. For example, when n is 3, we put

$$a^{\frac{1}{3}}+b^{\frac{1}{3}} = -c^{\frac{1}{3}};$$

which by cubing gives us

$$a+b+c+3a\left(\frac{b}{a}\right)^{\frac{1}{3}}+3a\left(\frac{b}{a}\right)^{\frac{2}{3}} = 0;$$

whence
$$(a+b+c)\left(\frac{b}{a}\right)^{\frac{1}{3}}+3a\left(\frac{b}{a}\right)^{\frac{2}{3}}+3b = 0,$$

and
$$(a+b+c)\left(\frac{b}{a}\right)^{\frac{2}{3}}+3b+3b\left(\frac{b}{a}\right)^{\frac{1}{3}} = 0;$$

so that on eliminating $(b/a)^{\frac{1}{3}}$, $(b/a)^{\frac{2}{3}}$ we have

$$\begin{vmatrix} a+b+c & 3a & 3a \\ 3b & a+b+c & 3a \\ 3b & 3b & a+b+c \end{vmatrix} = 0.$$

Similarly when n is 4 there is found

$$\begin{vmatrix} a+b-c & 4a & 6a & 4a \\ 4b & a+b-c & 4a & 6a \\ 6b & 4b & a+b-c & 4a \\ 4b & 6b & 4b & a+b-c \end{vmatrix} = 0.$$

The determinant thus obtained may be viewed as the eliminant of

$$\left.\begin{aligned} a(x+1)^n-(-1)^n c &= 0 \\ x^n &= b/a \end{aligned}\right\},$$

and as being related to the circulant. Hence its resolvability into

linear factors, which however we know of otherwise,* it being, for the fifth order, a case of

$$\mathrm{P}(\beta h,\ \gamma h,\ \delta h,\ \epsilon h,\ \alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon)\ ;$$

hence also the use to which it can be put in the solution of equations.

The final expansion must be symmetrical with respect to a, b, c ; and there thus arise pairs of equivalent determinants.

Muir's pair is of a different kind, namely,

$$\begin{vmatrix} a+b-c & 4a & 6a & 4a \\ 4b & a+b-c & 4a & 6a \\ 6b & 4b & a+b-c & 4a \\ 4b & 6b & 4b & a+b-c \end{vmatrix} = \begin{vmatrix} \Sigma_1 & \Sigma_2 & \Sigma_3 & . \\ -4 & \Sigma_1 & \Sigma_2 & \Sigma_3 \\ . & -16 & \Sigma_1 & \Sigma_2 \\ . & . & -4 & \Sigma_1 \end{vmatrix}$$

where $\Sigma_1 \equiv a+b+c$, $\Sigma_2 \equiv ab+bc+ca$, $\Sigma_3 \equiv abc$. No proof of it, other than an unsuggestive verification, was ever brought forward.

SCHRADER, W. (1887).

(See under this heading in Chap. XXII.)

SEGAR, H. W. (1892).

[Question 11462. *Educ. Times*, xlv. p. 153.]

Here we have the interesting proposition that *if we multiply the elements of the last row of the persymmetric determinant*

$$\begin{vmatrix} 1! & 2! & 3! & \dots & n! \\ 2! & 3! & 4! & \dots & (n+1)! \\ . & . & . & . & . \\ n! & (n+1)! & (n+2)! & \dots & (2n-1)! \end{vmatrix}$$

by n, n+1, n+2, . . . , 2n−1 *respectively, the determinant is thereby multiplied by* n^2+n-1.

In default of a proof we may remark that as a first proposition of the kind it would be better to make the row-multipliers 1, 2, 3, . . . , n, as the multiplier of the determinant would then be n^2, and to obtain Segar's result we should only have to add the original

* See above under Kapteyn, W., pp. 315–316.

determinant with its last row multiplied by $n-1$. As a further hint we may point out that the value of the original determinant is

$$1!\,2!\,3!\,\ldots\,n!\cdot 1!\,2!\,3!\,\ldots\,(n-1)!,$$

a result which is readily got by removing the factors $1!, 2!, 3!, \ldots$ from the rows and the factors $1, 1!, 2!, 3!, \ldots$ from the columns, and so leaving the well-known unit determinant

$$\begin{vmatrix} 1 & 2 & 3 & 4 & \ldots \\ 1 & 3 & 6 & 10 & \ldots \\ 1 & 4 & 10 & 20 & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}.$$

SCHOUTE, P. H. (1892): ESCHERICH, G. v. (1892).

[Calcul d'un déterminant. *Journ. de Math. Spéc.*, (4) i. pp. 54–56.]

[Bestimmung einer Determinante. *Monatshefte f. Math. u. Phys.*, iii. pp. 19–20.]

Already dealt with by Scott in 1879 (*Hist.*, iii. p. 325).

SEGAR, H. W. (1892).

[The deduction of certain determinants from others of indeterminate form. *Messenger of Math.*, xxii. pp. 57–67.]

As arithmetical examples of a very general theorem in alternants Segar gives

$$\begin{vmatrix} \frac{1}{(r-1)!} & \frac{1}{r!} & . & . & \ldots \\ \frac{1}{(r-2)!} & \frac{2}{(r-1)!} & \frac{1}{r!} & . & \ldots \\ \frac{1}{(r-3)!} & \frac{3}{(r-2)!} & \frac{3}{(r-1)!} & \frac{1}{r!} & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}_n = \frac{(r+n-1)!}{(r!)^n\,.\,(r-1)!},$$

$$\mathrm{P}(1!,\ 2!,\ 3!,\ \ldots)_n = n!!(n-1)!!,$$

$$\mathrm{P}\left(\frac{1}{1!},\ \frac{1}{2!},\ \frac{1}{3!},\ \ldots\right)_n = \frac{n!!(n-1)!!}{(2n)!!}.$$

The second of these we have just proved independently.

BRUNN, H. (1892).

[Ein Satz über orthosymmetrische und verwandte Determinanten aus den fundamentalen symmetrischen Functionen. *Zeitschrift f. Math. u. Phys.*, xxxvii. pp. 291–297.]

Brunn's theorem is to the effect that *if the simple symmetric functions of* a, b, c, . . . *be denoted by* $\Sigma_1, \Sigma_2, \ldots$, *then any determinant*

$$\begin{vmatrix} \Sigma_r & \Sigma_s & \Sigma_t & \ldots. \\ \Sigma_{r'} & \Sigma_{s'} & \Sigma_{t'} & \ldots. \\ \Sigma_{r''} & \Sigma_{s''} & \Sigma_{t''} & \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

in which the suffixes diminish from right to left and from top to bottom is necessarily positive, so long as a, b, . . . *are positive.* Much pains is taken with the proof. For each Σ, Σ_h say, there is substituted

$$\Sigma'_h + a\Sigma'_{h-1},$$

where Σ' differs from Σ in referring to the elements with a left out, and the determinant expanded in ascending powers of a. Thus, the simple persymmetric determinant

$$\begin{vmatrix} \Sigma_4 & \Sigma_5 & \Sigma_6 & \Sigma_7 \\ \Sigma_3 & \Sigma_4 & \Sigma_5 & \Sigma_6 \\ \Sigma_2 & \Sigma_3 & \Sigma_4 & \Sigma_5 \\ \Sigma_1 & \Sigma_2 & \Sigma_3 & \Sigma_4 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma_4'+a\Sigma_3' & \Sigma_5'+a\Sigma_4' & \Sigma_6'+a\Sigma_5' & \Sigma_7'+a\Sigma_6' \\ \Sigma_3'+a\Sigma_2' & \Sigma_4'+a\Sigma_3' & \Sigma_5'+a\Sigma_4' & \Sigma_6'+a\Sigma_5' \\ \Sigma_2'+a\Sigma_1' & \Sigma_3'+a\Sigma_2' & \Sigma_4'+a\Sigma_3' & \Sigma_5'+a\Sigma_4' \\ \Sigma_1'+a & \Sigma_2'+a\Sigma_1' & \Sigma_3'+a\Sigma_2' & \Sigma_4'+a\Sigma_3' \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma_4' & \Sigma_5' & \Sigma_6' & \Sigma_7' \\ \Sigma_3' & \Sigma_4' & \Sigma_5' & \Sigma_6' \\ \Sigma_2' & \Sigma_3' & \Sigma_4' & \Sigma_5' \\ \Sigma_1' & \Sigma_2' & \Sigma_3' & \Sigma_4' \end{vmatrix} + a \begin{vmatrix} \Sigma_3' & \Sigma_5' & \Sigma_6' & \Sigma_7' \\ \Sigma_2' & \Sigma_4' & \Sigma_5' & \Sigma_6' \\ \Sigma_1' & \Sigma_3' & \Sigma_4' & \Sigma_5' \\ 1 & \Sigma_2' & \Sigma_3' & \Sigma_4' \end{vmatrix} + a^2 \begin{vmatrix} \Sigma_3' & \Sigma_4' & \Sigma_6' & \Sigma_7' \\ \Sigma_2' & \Sigma_3' & \Sigma_5' & \Sigma_6' \\ \Sigma_1' & \Sigma_2' & \Sigma_4' & \Sigma_5' \\ 1 & \Sigma_1' & \Sigma_3' & \Sigma_4' \end{vmatrix}$$

$$+ a^3 \begin{vmatrix} \Sigma_3' & \Sigma_4' & \Sigma_5' & \Sigma_7' \\ \Sigma_2' & \Sigma_3' & \Sigma_4' & \Sigma_6' \\ \Sigma_1' & \Sigma_2' & \Sigma_3' & \Sigma_5' \\ 1 & \Sigma_1' & \Sigma_2' & \Sigma_4' \end{vmatrix} + a^4 \begin{vmatrix} \Sigma_3' & \Sigma_4' & \Sigma_5' & \Sigma_6' \\ \Sigma_2' & \Sigma_3' & \Sigma_4' & \Sigma_5' \\ \Sigma_1' & \Sigma_2' & \Sigma_3' & \Sigma_4' \\ 1 & \Sigma_1' & \Sigma_2' & \Sigma_3' \end{vmatrix}.$$

Here, because of the simplicity of the example, there is only one determinant attached to each power of a, but even these are no longer all persymmetric, agreeing merely in having each of them a diminishing series of suffixes from right to left and from top to bottom. It is clear, however, that if they can be proved positive, the whole will be positive. We are then directed to treat each one of them in the same fashion as the original and to continue thereafter the like process, with the assurance that ultimately we shall have no determinants but those with zero or unit elements, and that these elements will be so distributed that the value of each such final determinant will be 0 or 1.

It is well to recall that note has already been taken of simple special cases where the final expansion is definitely known; for example, the persymmetric recurrent

$$\begin{vmatrix} \Sigma a & \Sigma ab & \Sigma abc & abcd \\ 1 & \Sigma a & \Sigma ab & \Sigma abc \\ . & 1 & \Sigma a & \Sigma ab \\ . & . & 1 & \Sigma a \end{vmatrix},$$

which is equal to

$$\Sigma a^4 + \Sigma a^3 b + \Sigma a^2 bc + \Sigma a^2 b^2 + abcd$$

or Wronski's function $\aleph_4$ (*Hist.*, ii. pp. 216–217).

FROBENIUS, G. (1894).

[Ueber das Trägheitsgesetz der quadratischen Formen. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), pp. 241–256, 407–431; or *Crelle's Journ.*, cxiv. pp. 187–230.]

In the sections of this which concern quadrics whose discriminant is persymmetric, the most interesting portion from the point of view of determinants is that in which is established Jacobi's recurrence-formula above referred to under Heun * (1881). The proof, which has the merit of being the first that is purely determinantal, consists essentially in noting the two known relations

$$\begin{aligned} a\mathrm{P}_n - \mathrm{B}\mathrm{P}_{n-1} - \mathrm{A}\xi &= 0, \\ \mathrm{A}\mathrm{P}_{n-2} - \beta\mathrm{P}_{n-1} + a\xi &= 0, \end{aligned}$$

* Frobenius does not mention Heun but refers to Hattendorf, whose name in this connection we have omitted to give (see *Hist.*, iii. p. 325).

and eliminating the two primary minors ξ, ζ by means of the new relation

$$\xi - \zeta = x\mathrm{P}_{n-1}.$$

A generalization is also effected, the P's in the new formula being no longer consecutive.

NETTO, E. (1895).

[Zur Theorie der Resultanten. *Crelle's Journ.*, cxvi. pp. 33–49.]

The title of this rather important paper is not particularly suitable. Only the first of the four fresh theorems contained in it has any bearing on the subject of the resultant of two binary quantics. The others, though seeming to concern a particular persymmetric determinant, are really not so restricted, but deal with properties of persymmetric determinants in general.

The first theorem, with a necessary sign-factor inserted, is that *if* $c_0, c_1, c_2, \ldots$ *be defined by the identical equation*

$$\frac{b_0x^{n-1} + b_1x^{n-2} + \ldots + b_{n-1}}{a_0x^n + a_1x^{n-1} + \ldots + a_n} \equiv c_0x^{-1} + c_1x^{-2} + \ldots,$$

then the persymmetric determinant

$$\mathrm{P}(c_0, c_1, c_2, \ldots, c_{2m}) = \frac{(-1)^{\frac{1}{2}m(m+1)}}{a_0^{2m+1}} \begin{vmatrix} a_0 & a_1 & a_2 & \ldots & a_{2m} \\ . & a_0 & a_1 & \ldots & a_{2m-1} \\ . & . & a_0 & \ldots & a_{2m-2} \\ . & . & . & \ldots & . \\ . & . & . & \ldots & a_{m+1} \\ b_0 & b_1 & b_2 & \ldots & b_{2m} \\ . & b_0 & b_1 & \ldots & b_{2m-1} \\ . & . & . & \ldots & . \\ . & . & . & \ldots & b_m \end{vmatrix}.$$

More than one proof is referred to, but probably the simplest and most natural consists in expressing the b's on the right in terms of the a's and c's, namely,

$$b_0 = a_0c_0, \quad b_1 = a_0c_1 + a_1c_0, \quad b_2 = a_0c_2 + a_1c_1 + a_2c_0, \ldots,$$

and then making the evidently possible simplifications.

The second theorem is in effect that *if* $\overline{rs}$ *denote the determinant got from the persymmetric* m*-by-*(m+2) *array*

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_4 & a_5 & a_6 & a_7 & a_8 & a_9 \end{matrix}$$

by deleting the r^{th} *and* s^{th} *columns, then the adjugate of the* (m+1)*-line persymmetric determinant*

$$P(a_1, a_2, a_3, \ldots, a_9)$$

is

$$\begin{vmatrix} \overline{12} & \overline{13} & \overline{14} & \overline{15} & \overline{16} \\ \overline{13} & \overline{14}+\overline{23} & \overline{15}+\overline{24} & \overline{16}+\overline{25} & \overline{26} \\ \overline{14} & \overline{15}+\overline{24} & \overline{16}+\overline{25}+\overline{34} & \overline{26}+\overline{35} & \overline{36} \\ \overline{15} & \overline{16}+\overline{25} & \overline{26}+\overline{35} & \overline{36}+\overline{45} & \overline{46} \\ \overline{16} & \overline{26} & \overline{36} & \overline{46} & \overline{56} \end{vmatrix}.$$

The adjugate by definition is evidently simpler than this, each of its elements being a single four-line determinant; but the author justly considers the result to be nevertheless notable, because of its curious external resemblance to Bezout's condensed eliminant of

$$\left.\begin{aligned} a_1x^n + a_2x^{n-1} + \ldots + a_n &= 0 \\ b_1x^n + b_2x^{n-1} + \ldots + b_n &= 0 \end{aligned}\right\},$$

namely, in the case where n is 6,

$$\begin{vmatrix} |a_1b_2| & |a_1b_3| & |a_1b_4| & |a_1b_5| & |a_1b_6| \\ |a_1b_3| & |a_1b_4|+|a_2b_3| & |a_1b_5|+|a_2b_4| & |a_1b_6|+|a_2b_5| & |a_2b_6| \\ |a_1b_4| & |a_1b_5|+|a_2b_4| & |a_1b_6|+|a_2b_5|+|a_3b_4| & |a_2b_6|+|a_3b_5| & |a_3b_6| \\ |a_1b_5| & |a_1b_6|+|a_2b_5| & |a_2b_6|+|a_3b_5| & |a_3b_6|+|a_4b_5| & |a_4b_6| \\ |a_1b_6| & |a_2b_6| & |a_3b_6| & |a_4b_6| & |a_5b_6| \end{vmatrix}$$

The nature of the proof will be understood from the fact that part of it consists in showing that the central element of the adjugate

$$\begin{vmatrix} a_1 & a_2 & a_4 & a_5 \\ a_2 & a_3 & a_5 & a_6 \\ a_4 & a_5 & a_7 & a_8 \\ a_5 & a_6 & a_8 & a_9 \end{vmatrix} = \overline{16} + \overline{25} + \overline{34},$$

—an equality belonging to a type which we have seen studied at intervals since 1880.

The third theorem concerns the same array as the second, and, like the second, is an instance of what may almost be called ' mimicry of form.' It is that

$$\overline{\alpha\beta}\cdot\overline{\gamma\delta} - \overline{\alpha\gamma}\cdot\overline{\beta\delta} + \overline{\alpha\delta}\cdot\overline{\beta\gamma} = 0,$$

where α, β, γ, δ are column-numbers in order of magnitude.

The fourth theorem concerns the vanishing of a persymmetric determinant, but it had for many years been known to hold for a determinant of a more general type (*Hist.*, iii. p. 113).

The last two pages of the paper proper are devoted to a purely determinantal proof of the persymmetry of Bezout's eliminant; and this is followed by two interesting postscripts, one establishing the recurrence-formula of Jacobi's persymmetric determinant, and the other connected with the highest-common-factor of two binary quantics. The proof in the former postscript depends on equating Laplace-expansions of two forms of a determinant, and in respect of neatness and brevity is preferable to that of Frobenius of the year before (see p. 325 above).

LÉVY, L. (1895).

[Questions 1686, 1687. *Nouv. Annales de Math.*, (3) xv. pp. 33*–34*; (4) xvii. pp. 238–9.]

It is here recalled that the equation

$$\frac{\partial^n x^n (x-1)^n}{\partial x^n} = 0$$

is equivalent to

$$\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots & \frac{1}{n+3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix} = 0,$$

and proof is sought of a more general result, namely, where the differential-coefficient is the $(2n-k)^{\text{th}}$, the last row of the determinant is

$$1 \quad x \quad x^2 \quad \dots \quad x^k,$$

and the elements of the persymmetric array are

$$\frac{1}{(n-k+1)(n-k+2)\dots(2n-2k+1)}, \quad \frac{1}{(n-k+2)(n-k+3)\dots(2n-2k+2)}, \dots.$$

NETTO, E. (1896).

[Zur Theorie der Resultanten. *Crelle's Journ.*, cxvii. pp. 57–71.]

Almost the whole of this so-called appendix to the author's paper of the preceding year is occupied with a single theorem, namely, *If in the μ^{th} member of the series*

$$\begin{vmatrix} c_0 & 1 \\ c_1 & x \end{vmatrix}, \quad \begin{vmatrix} c_0 & c_1 & 1 \\ c_1 & c_2 & x \\ c_2 & c_3 & x^2 \end{vmatrix}, \quad \begin{vmatrix} c_0 & c_1 & c_2 & 1 \\ c_1 & c_2 & c_3 & x \\ c_2 & c_3 & c_4 & x^2 \\ c_3 & c_4 & c_5 & x^3 \end{vmatrix}, \dots.$$

the cofactor of the highest power of x *do not vanish, and the next member with the like property be the ν^{th} $(\nu > \mu+1)$, then the intermediate members all vanish identically except the last of them, and this differs from the μ^{th} merely by a multiplier independent of* x.

As the vanishing to be established is independent of any value of x, it is clear that the theorem is really one concerning the minors of a persymmetric array.

CAZZANIGA, T. (1897).

[Relazioni fra i minori di un determinante di Hankel. *Rendic. . . . R. Ist. Lombardo* (Milano), (2) xxxi. pp. 610–614.]

The theorem here established is that *In any persymmetric determinant the minor obtained by deleting the 1st and rth rows and 1st and sth columns is equal to the sum of two other secondary minors of the determinant, namely, the minor obtained by deleting the 1st and $(r+1)^{th}$ rows and the 1st and $(s-1)^{th}$ columns, and the minor obtained by deleting the 1st and 2nd rows and $(s-1)^{th}$ and rth columns.* The

author, having obtained the case where $r = s = 3$ from Pascal, reached his generalization by using an equality of Netto's.

We may note that a convenient notation for the theorem here is

$$\begin{vmatrix} 1 & r \\ 1 & s \end{vmatrix} = \begin{vmatrix} 1 & r+1 \\ 1 & s-1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ s-1 & r \end{vmatrix},$$

the minor complementary to

$$\begin{vmatrix} h, & k, & l, & \dots \\ p, & q, & r, & \dots \end{vmatrix} \text{ being thus } \begin{vmatrix} h, & k, & l, & \dots \\ p, & q, & r, & \dots \end{vmatrix}.$$

NANSON, E. J. (1898).

[Questions 13957, 14108. *Educ. Times*, li. p. 389; lii. p. 93.]

The first result here is that *if the persymmetric determinants* $\mathrm{P}(a_1, a_2, a_3)$, $\mathrm{P}(a_1, a_2, a_3, a_4, a_5)$, ..., $\mathrm{P}(a_1, a_2, \dots, a_{2n-1})$ *vanish, then* $a_1, a_2, \dots, a_{n+1}$ *are in equirational progression.* In default of a proof we may point out that if the assertion hold true when n is 3,—in other words, if $a_4 = ra_3 = r^2a_2 = r^3a_1$,—then the vanishing of

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \\ a_4 & a_5 & a_6 & a_7 \end{vmatrix}$$

implies that

$$-a_1(a_5 - r^4a_1) = 0.$$

The second result concerns the vanishing of minors of

$$\mathrm{P}(a_1, a_2, \dots, a_{2n-1})$$

under the same conditions.

STUDNIČKA, F. J. (1898).

[Nové poučky a některých determinantech zvláštních. *Věstník České Akad.* ..., vii. pp. 477–493.]

The second part of this (pp. 486–488) deals lightly with the persymmetric determinant whose elements are

$$a^m, (a+1)^m, \dots, (a+2n-2)^m,$$

a case of one of Hankel's (*Hist.*, iii. p. 314).

BAUR, L. (1898).

[Ueber die verschiedenen Wurzeln einer algebraischen Gleichung und der Ordnungen. *Math. Annalen*, lii. pp. 113–119.]

This paper, which gives no really new property of persymmetric determinants, is noted because of its exceptionally full information regarding the usefulness of

$$\mathrm{P}(s_0, s_1, \ldots, s_{2n-2})$$

in dealing with questions concerning the character of the roots of an equation of the n^{th} degree—the number of different roots, the equation of these roots, their multiplicities, the number of complex root-pairs, and so forth (*Hist.*, ii. pp. 162–164, 330–331, etc.).

MUIR, T. (1899).

[Note on a persymmetric eliminant. *Proceed. R. Soc. Edinburgh*, xxii. pp. 543–546.]

Schoute having used

$$\begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = 0$$

as an equation resulting from the elimination of the p's and λ's from the set of five equations

$$\left.\begin{aligned} s_0 &= p_1\lambda_1{}^0 + p_2\lambda_2{}^0 \\ s_1 &= p_1\lambda_1{}^1 + p_2\lambda_2{}^1 \\ &\cdots\cdots\cdots \\ s_4 &= p_1\lambda_1{}^4 + p_2\lambda_2{}^4 \end{aligned}\right\}.$$

Muir generalizes the result, and shows in two ways that there is superfluous provision for the evanescence of the determinant; the three-line instance, for example, being equal to

$$\begin{vmatrix} 1 & 1 & . \\ \lambda_1 & \lambda_2 & . \\ \lambda_1{}^2 & \lambda_2{}^2 & . \end{vmatrix} . \begin{vmatrix} p_1 & p_2 & . \\ p_1\lambda_1 & p_2\lambda_2 & . \\ p_1\lambda_1{}^2 & p_2\lambda_2{}^2 & . \end{vmatrix}.$$

ŠTUDNIČKA, F. J. (1899).

(See under this heading in Chap. IX.)

CHAPTER XIV.

BIGRADIENTS, FROM 1877 TO 1899.

THE number of writings to be reported on here is practically the same as for the preceding twenty-year period.

VENTÉJOL, . (1877): BIEHLER, C. (1880).

[Sur la transformation du déterminant de M. Sylvester en celui de Cauchy. *Nouv. Annales de Math.*, (2) xix. pp. 110–115: Ventéjol's reply, pp. 188–192.]

The existence of Ventéjol's lithographed note on the subject is here brought to light through a reference to it by Biehler and a consequent letter of reclamation by Ventéjol himself. Ventéjol takes the case of the equations

$$a_0x^4 + \ldots + a_4 = 0, \qquad b_0x^4 + \ldots + b_4 = 0,$$

and Biehler the case of

$$a_0x^m + \ldots + a_m = 0, \qquad b_0x^p + \ldots + b_p = 0.$$

Neither of them makes reference to Trudi (1862).

PAIGE, C. LE (1880).

[Sur l'élimination. *Comptes rendus . . . Acad. des Sci.* (Paris), xc. pp. 1210–1212.]

The subject here is essentially the same as that of Mansion's paper of 1879. The procedure, however, is different, the multiplication of Sylvester's eliminant by itself being made to give the square of Bezout's. Thus, multiplying

$$\begin{vmatrix} . & . & a_0 & a_1 & a_2 & a_3 \\ . & -a_0 & -a_1 & -a_2 & -a_3 & . \\ a_0 & a_1 & a_2 & a_3 & . & . \\ b_0 & b_1 & b_2 & b_3 & . & . \\ . & -b_0 & -b_1 & -b_2 & -b_3 & . \\ . & . & b_0 & b_1 & b_2 & b_3 \end{vmatrix}$$

columnwise by

$$\begin{vmatrix} . & . & b_3 & b_2 & b_1 & b_0 \\ . & -b_3 & -b_2 & -b_1 & -b_0 & . \\ b_3 & b_2 & b_1 & b_0 & . & . \\ -a_3 & -a_2 & -a_1 & -a_0 & . & . \\ . & a_3 & a_2 & a_1 & a_0 & . \\ . & . & -a_3 & -a_2 & -a_1 & -a_0 \end{vmatrix},$$

he obtains

$$\begin{vmatrix} |a_0b_3| & |a_0b_2| & |a_0b_1| & . & . & . \\ |a_1b_3| & |a_0b_3|+|a_1b_2| & |a_0b_2| & . & . & . \\ |a_2b_3| & |a_1b_3| & |a_0b_3| & . & . & . \\ . & . & . & |a_3b_0| & |a_2b_0| & |a_1b_0| \\ . & . & . & |a_3b_1| & |a_2b_1|+|a_3b_0| & |a_2b_0| \\ . & . & . & |a_3b_2| & |a_3b_1| & |a_3b_0| \end{vmatrix}.$$

Similarly he shows that the central secondary minor of Sylvester's eliminant, as above written, is equal to a primary minor of the other.

It is not unimportant for us to note that as a consequence of Le Paige's process we have

Bezout's eliminant of $\begin{cases} a_0x^3 + a_1x^2 + a_2x + a_3 \\ b_0x^3 + b_1x^2 + b_2x + b_3 \end{cases}$

$$= \begin{Vmatrix} . & . & a_0 & b_0 & . & . \\ . & -a_0 & a_1 & b_1 & -b_0 & . \\ a_0 & -a_1 & a_2 & b_2 & -b_1 & b_0 \\ a_1 & -a_2 & a_3 & b_3 & -b_2 & b_1 \\ a_2 & -a_3 & . & . & -b_3 & b_2 \\ a_3 & . & . & . & . & b_3 \end{Vmatrix} \cdot \begin{Vmatrix} . & . & b_3 & -a_3 & . & . \\ . & -b_3 & b_2 & -a_2 & a_3 & . \\ b_3 & -b_2 & b_1 & -a_1 & a_2 & -a_3 \\ b_2 & -b_1 & b_0 & -a_0 & a_1 & -a_2 \\ b_1 & -b_0 & . & . & a_0 & -a_1 \\ b_0 & . & . & . & . & a_0 \end{Vmatrix},$$

the second product, however, being only a variant of the first.

FARKAS, J. (1880).

[Question 600. *Nouv. Corresp. Math.*, vi. p. 526.]

The determinant here to be considered is

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_1 & a_2+a_0 & a_3 & a_4 & a_5 & a_6 & a_7 & . \\ a_2 & a_3+a_1 & a_4+a_0 & a_5 & a_6 & a_7 & . & . \\ a_3 & a_4+a_2 & a_5+a_1 & a_6+a_0 & a_7 & . & . & . \\ a_4 & a_5+a_3 & a_6+a_2 & a_7+a_1 & a_0 & . & . & . \\ a_5 & a_6+a_4 & a_7+a_3 & a_2 & a_1 & a_0 & . & . \\ a_6 & a_7+a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & . \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{vmatrix},$$

whose six-line central array is seen to be the sum of the two arrays

$$\begin{matrix} a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_3 & a_4 & a_5 & a_6 & a_7 & . \\ a_4 & a_5 & a_6 & a_7 & . & . \\ a_5 & a_6 & a_7 & . & . & . \\ a_6 & a_7 & . & . & . & . \\ a_7 & . & . & . & . & . \end{matrix}, \qquad \begin{matrix} a_0 & . & . & . & . & . \\ a_1 & a_0 & . & . & . & . \\ a_2 & a_1 & a_0 & . & . & . \\ a_3 & a_2 & a_1 & a_0 & . & . \\ a_4 & a_3 & a_2 & a_1 & a_0 & . \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{matrix},$$

and the property set for proof is the divisibility * of the given determinant by the determinant of the difference of the two said arrays.

In the absence of a proof we ourselves may point out that by adding to the first column the sum of all the others we can remove the factor

$$a_0 + a_1 + a_2 + \ldots + a_7;$$

that then by diminishing each row, beginning with the last, by the row immediately preceding we can replace the determinant by one of the 7th order; and that in the third place by increasing each column, beginning with the first, by the sum of all that follow it we reach a determinant whose array is the difference of two triangular arrays, namely,

* The quotient is also sought, and a generalisation of the result.

$$\begin{vmatrix} a_0-a_1 & -a_2 & -a_3 & -a_4 & -a_5 & -a_6 & -a_7 \\ a_1-a_2 & a_0-a_3 & -a_4 & -a_5 & -a_6 & -a_7 & . \\ a_2-a_3 & a_1-a_4 & a_0-a_5 & -a_6 & -a_7 & . & . \\ a_3-a_4 & a_2-a_5 & a_1-a_6 & a_0-a_7 & . & . & . \\ a_4-a_5 & a_3-a_6 & a_2-a_7 & a_1 & a_0 & . & . \\ a_5-a_6 & a_4-a_7 & a_3 & a_2 & a_1 & a_0 & . \\ a_6-a_7 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{vmatrix}.$$

Treating this now in a closely resembling fashion of three steps we can remove the factor

$$a_0 - a_1 + a_2 - \ldots - a_7,$$

and obtain the six-line cofactor sought.

It is worth noting that by putting $a_6 = a_7 = 0$ we obtain the next lower case of the theorem, and similarly in general: also that the determinant may be viewed as a quasi-eliminant (dialytic) of the equations

$$\left.\begin{aligned} a_0x^7 + a_1x^6 + \ldots + a_7 &= 0 \\ x^{7+m} &= x^{7-m} \end{aligned}\right\}.$$

FARKAS, J. (1881).

[Question 3. *Mathesis*, i. pp. 12–13; viii. pp. 195–202.]

The theorem here set for proof and generalization is that *the equation of the binary products of the roots of the equation*

$$a + bx + cx^2 + dx^3 = 0$$

is, when the squares are admitted,

$$\begin{vmatrix} y^3 & . & . & . & . & -1 \\ . & y^2 & . & . & -1 & . \\ . & . & y & -1 & . & . \\ . & . & a & b & c & d \\ . & a & b & c & d & . \\ a & b & c & d & . & . \end{vmatrix} = 0,$$

and, when they are not, is

$$\begin{vmatrix} z^2 & . & . & . & 1 \\ . & z & . & 1 & . \\ . & . & 1 & . & . \\ . & a & b & c & d \\ a & b & c & d & . \end{vmatrix} = 0.$$

The proofs given are little better than verifications, and are consequently not instructive as to the origin of the determinants or to the legitimacy of the origin. The six-line determinant is a kind of dialytic eliminant of

$$\left.\begin{aligned} a + bx + cx^2 + dx^3 &= 0 \\ y &= x^2 \end{aligned}\right\},$$

the first three equations used being

$$y^3 - x^6 = 0, \quad y^2x - x^5 = 0, \quad yx^2 = x^4;$$

and no explanation is attempted as to why this should include among the roots of the resulting equation the binary products other than the squares.

CROCCHI, L. (1882.)

[Sopra la corrispondenza tra i coefficienti di un' equazione algebrica e le funzioni simmetriche complete. *Giornale di Mat.*, xx. pp. 301–320.]

Crocchi here develops at considerable length the subject of his paper of 1879; but so far as determinants are concerned there is only one fresh result, namely, an expression for the square of the difference-product in terms of aleph functions, in analogy of course with an expression in terms of the simple combinatory symmetric functions. Unfortunately, both expressions seem to be incorrect, a mischance which in the case of the latter is unexpected, because of a correct form having been given by Brioschi in 1854 and taken over by Baltzer in 1857. The former, too, which is new, is obtained exactly as Brioschi obtained the other (*Hist.*, ii. pp. 346–347), namely, from the persymmetric determinant of the s's. In the case where the number of variables is 4, the squared difference-product

$$= \mathrm{P}\begin{pmatrix} s_1 & & & s_5 \\ s_0 & s_2 & \dots & s_6 \end{pmatrix} = \begin{vmatrix} 1 & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . \\ . & . & 1 & . & . & . & . \\ . & . & . & s_0 & s_1 & s_2 & s_3 \\ . & . & s_0 & s_1 & s_2 & s_3 & s_4 \\ . & s_0 & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_0 & s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \end{vmatrix},$$

and the 7-line determinant here, after we perform the operations

$$\begin{aligned} \mathrm{col}_7 + \aleph_1 \mathrm{col}_6 + \aleph_2 \mathrm{col}_5 + \ldots + \aleph_6 \mathrm{col}_1, \\ \mathrm{col}_6 + \aleph_1 \mathrm{col}_5 + \ldots + \aleph_5 \mathrm{col}_1, \\ \ldots\ldots\ldots\ldots \\ \mathrm{col}_2 + \aleph_1 \mathrm{col}_1, \end{aligned}$$

and use the equalities (*Hist.*, ii. p. 217)

$$\begin{aligned} s_6 + \aleph_1 s_5 + \aleph_2 s_4 + \ldots + \aleph_6 s_0 &= 10\aleph_6, \\ s_5 + \aleph_1 s_4 + \ldots + \aleph_5 s_0 &= 9\aleph_5, \\ \ldots\ldots\ldots\ldots\ldots & \\ s_1 + \aleph_1 s_0 &= 5\aleph_1, \end{aligned}$$

becomes

$$\begin{vmatrix} 1 & \aleph_1 & \aleph_2 & \aleph_3 & \aleph_4 & \aleph_5 & \aleph_6 \\ . & 1 & \aleph_1 & \aleph_2 & \aleph_3 & \aleph_4 & \aleph_5 \\ . & . & 1 & \aleph_1 & \aleph_2 & \aleph_3 & \aleph_4 \\ . & . & . & 4 & 5\aleph_1 & 6\aleph_2 & 7\aleph_3 \\ . & . & 4 & 5\aleph_1 & 6\aleph_2 & 7\aleph_3 & 8\aleph_4 \\ . & 4 & 5\aleph_1 & 6\aleph_2 & 7\aleph_3 & 8\aleph_4 & 9\aleph_5 \\ 4 & 5\aleph_1 & 6\aleph_2 & 7\aleph_3 & 8\aleph_4 & 9\aleph_5 & 10\aleph_6 \end{vmatrix}.$$

The last row of this can evidently be changed into

$$. \quad \aleph_1 \quad 2\aleph_2 \quad 3\aleph_3 \quad 4\aleph_4 \quad 5\aleph_5 \quad 6\aleph_6$$

and the order reduced to the 6th, which being done, the determinant is found to be different from Crocchi's in the latter having for its last row

$$5\aleph_1, \quad 6\aleph_2, \quad 7\aleph_3, \quad \ldots\ldots, \quad 10\aleph_6.$$

REUSCHLE, C. (1884).

[Zur Resultantenbildung. *Zeitschrift f. Math. u. Phys.*, xxx. pp. 106–110, 304.]

The subject here is again the two determinant forms of eliminant, and the transformation of the one into the other. Save for special clearness it is not noteworthy.

GORDAN, P. (1885).

[Vorlesungen über Invariantentheorie. xi+201 pp. Leipzig.]

In dealing with the subject of the resultant of two binary quantics (pp. 131–166), the form on which Gordan relies is the bigradient. For some reason Euler's product of differences is not made available until later. From the point of view of determinants this is a matter of considerable interest, as the establishment of each property of the resultant is thereby made the solution of a problem in determinants. On this account we note that the properties are

$$\begin{aligned}
\text{Res}\,(f, g) &= \text{Res}\left\{x^n g\left(\frac{1}{x}\right), x^m f\left(\frac{1}{x}\right)\right\},\\
\text{Res}\left\{f(\lambda x), g(\lambda x)\right\} &= \lambda^{mn} \cdot \text{Res}\,(f, g),\\
\text{Res}\,(f+gh, g) &= \text{Res}\,(f, g),\\
\text{Res}\left\{f(x+u), g(x+u)\right\} &= \text{Res}\,(f, g),\\
\text{Res}\,(f, gh) &= \text{Res}\,(f, g) \cdot \text{Res}\,(f, h).
\end{aligned}$$

In these f is of the m^{th} degree and g of the n^{th}, m being $> n$; and in the third the degree of $gh <$ the degree of f. The last property is illustrated by the case where

$$\begin{aligned}
f &\equiv a_0x^3 + a_1x^2 + a_2x + a_3,\\
g &\equiv b_0x^2 + b_1x \;\; + b_2,\\
h &\equiv c_0x \;\; + c_1,
\end{aligned}$$

and where therefore the equality to be proved is

$$\begin{vmatrix}
a_0 & a_1 & a_2 & a_3 & . & .\\
. & a_0 & a_1 & a_2 & a_3 & .\\
. & . & a_0 & a_1 & a_2 & a_3\\
b_0c_0 & b_1c_0+b_0c_1 & b_2c_0+b_1c_1 & b_2c_1 & . & .\\
. & b_0c_0 & b_1c_0+b_0c_1 & b_2c_0+b_1c_1 & b_2c_1 & .\\
. & . & b_0c_0 & b_1c_0+b_0c_1 & b_2c_0+b_1c_1 & b_2c_1
\end{vmatrix}$$

$$= \begin{vmatrix}
a_0 & a_1 & a_2 & a_3 & .\\
. & a_0 & a_1 & a_2 & a_3\\
b_0 & b_1 & b_2 & . & .\\
. & b_0 & b_1 & b_2 & .\\
. & . & b_0 & b_1 & b_2
\end{vmatrix} \cdot \begin{vmatrix}
a_0 & a_1 & a_2 & a_3\\
c_0 & c_1 & . & .\\
. & c_0 & c_1 & .\\
. & . & c_0 & c_1
\end{vmatrix}.$$

The proof is somewhat far-fetched and lengthy, being dependent on the viewing of the bigradient eliminant as a Jacobian, and the transformation of

$$\frac{\partial(x^2f,\ xf,\ f,\ x^2gh,\ xgh,\ gh)}{\partial(x^5,\ x^4,\ x^3,\ x^2,\ x,\ x^0)} \quad \text{into} \quad \frac{\partial(xfh,\ fh,\ f,\ x^2gh,\ xgh,\ gh)}{\partial(x^4h,\ x^3h,\ x^2h,\ xh,\ x^0h,\ x^0)}.$$

In connection therewith it may be noted as a curious fact that the six-line determinant involved is readily found to be the product of two 5-by-7 arrays.

HEAL, W. E. (1886).

[Expression of the coefficients of Sturm's functions as determinants. *Annals of Math.*, ii. pp. 85–89.]

Previous papers dealing more comprehensively with the same subject might have been referred to, for example, Heilermann's of 1852.

POMEY, E. (1888).

[Sur le plus grand commun diviseur de deux polynômes entiers. *Nouv. Annales de Math.*, (3) vii. pp. 66–90, 407–427.]

In the first paper here the subject is treated with surprising fullness, there being carefully enunciated and proved four lemmas and four theorems in which use is made of the bigradient, and the corresponding four lemmas and four theorems depending on the use of the Bezout-Cauchy eliminant. Even this, however, is not considered to be sufficient, for the second paper covers exactly the same ground, being meant apparently as an amended edition of the first.

LORIA, G. (1888).

[Zur Eliminationstheorie. *Zeitschrift f. Math. u. Phys.*, xxxiii. pp. 357–358.]

The author properly draws attention to the usefulness of Sylvester's dialytic eliminant when certain coefficients of the original equations are functions of y and there is a question as to the degree in y of the resultant.

TYLER, H. W. (1891).

[Beziehungen zwischen der Sylvester'schen und der Bézout'schen Determinante. *Sitzungsb. d. phys.-med. Soc.* (Erlangen), xxiii. pp. 33–128: also separately as an Erlangen Dissert., 96 pp. München.]

It is only the last two chapters (pp. 93–125) of this that directly discuss the relations referred to in the title: the other three are little more than a fresh and clear exposition of those parts of the theory of determinants that are required in the discussion. Thus, the first (pp. 33–56) is a chapter on linear equations: the second (pp. 56–68) deals with the resultant of two binary n^{thics}: and the third (pp. 68–92) with a close examination of the arrays of the two forms of the said resultant. To these chapters belong 21 of the 26 results formally enunciated in the dissertation. Probably the most interesting (no. xvii. p. 75) is one which we have seen foreshadowed under Le Paige (1880), and is to the effect that *Bezout's eliminant is expressible as the result of multiplying any* n *consecutive columns of Sylvester's eliminant by a transformation of the other* n *columns, the multiplication being of course column-by-column.* For example, when n is 3 and the columns chosen are the 1st, 2nd, 3rd, we have

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ . & a_0 & a_1 \\ . & . & a_0 \\ b_0 & b_1 & b_2 \\ . & b_0 & b_1 \\ . & . & b_0 \end{vmatrix} \cdot \begin{vmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & . \\ b_3 & . & . \\ -a_1 & -a_2 & -a_3 \\ -a_2 & -a_3 & . \\ -a_3 & . & . \end{vmatrix} = \begin{vmatrix} |a_0b_1| & |a_0b_2| & |a_0b_3| \\ |a_0b_2| & |a_0b_3| + |a_1b_2| & |a_1b_3| \\ |a_0b_3| & |a_1b_3| & |a_2b_3| \end{vmatrix}.$$

Here and elsewhere, we may note, the author would have profited by using the umbral notation: thus, Sylvester's eliminant being denoted by $\left|\begin{matrix}123456\\123456\end{matrix}\right|$, the product just obtained and its three fellows * are

* Strictly speaking, the fourth is only a variant of the first. If it be considered different, then there are two others, namely,

$$\left|\begin{matrix}123456\\156\end{matrix}\right| \cdot \left|\begin{matrix}46\bar{5}\bar{1}32\\234\end{matrix}\right|,$$

$$\left|\begin{matrix}123456\\126\end{matrix}\right| \cdot \left|\begin{matrix}54\bar{6}\bar{2}13\\345\end{matrix}\right|,$$

the number being either n or $2n$, and not $n+1$.

$$\left|\begin{matrix}123456\\123\end{matrix}\right| \cdot \left|\begin{matrix}654\bar{3}\bar{2}\bar{1}\\456\end{matrix}\right|,$$

$$\left|\begin{matrix}123456\\234\end{matrix}\right| \cdot \left|\begin{matrix}\bar{4}651\bar{3}\bar{2}\\156\end{matrix}\right|,$$

$$\left|\begin{matrix}123456\\345\end{matrix}\right| \cdot \left|\begin{matrix}\bar{5}\bar{4}621\bar{3}\\126\end{matrix}\right|,$$

$$\left|\begin{matrix}123456\\456\end{matrix}\right| \cdot \left|\begin{matrix}\bar{6}\bar{5}\bar{4}321\\123\end{matrix}\right|,$$

where change of signs in a row is indicated by an overhead —. By this means also the law of formation of the products could have been readily formulated.

Of the three results in the fourth chapter the first part of the less familiar is to the effect that *only* 2n−1 *primary minors of Bezout's eliminant are different, and to each one of them can be found an equal among the secondary minors of Sylvester's eliminant*: the converse part is not so general. As for the fifth chapter, the relations given in it, namely, between the secondary minors of Bezout's determinant and the quaternary minors of Sylvester's, are in themselves much less simple, and therefore require lengthier statement than, unfortunately, we are able to give here.

BIERMANN, O. (1891): HASKELL, M. W. (1892).

[Ueber die Resultante ganzer Functionen. *Monatshefte f. Math.*, ii. pp. 143–146.]

[Note on resultants. *Bull. New-York Math. Soc.*, i. pp. 223–224.]

These both deal with Gordan's equality of 1885,

$$\text{Res}\,(f+gh, g) \;=\; \text{Res}\,(f, g).$$

The first, as the result of determinantal transformations, scrupulously prefixes to the right-hand member the factor

$$(-1)^{n(n+q-m)} \cdot {b_0}^{n+q-m},$$

where q is the degree of h, $m < n$, and b_0 is the coefficient of x^n in g. The second insists on the removal of Gordan's restriction regarding the degree of gh.

WAELSCH, E. (1891).

[Ein Satz über die Resultante algebraischer Gleichungen und seine geometrische Anwendung. *Monatshefte f. Math. u. Phys.*, ii. pp. 421–428.]

The resultant in question is that of two binary quantics, and the theorem regarding it concerns the determinants of the two-row array of its first differential-coefficients with respect to the coefficients of the quantics, namely, that each of these determinants contains the resultant as a factor. This known theorem * (Salmon's *Modern Higher Algebra* of 1866, pp. 76–77) it is sought to amplify by showing how to calculate expeditiously the cofactors, but the author's interest in the geometrical side of the matter makes his success less than it might have been; the subject seems well worth further study.

MANSION, P. (1893).

[Sur l'éliminant de deux équations algèbriques. *Annales . . . Soc. Sc.* (Bruxelles), xviii. 1re partie, pp. 5–8.]

Still another mode of transforming Sylvester's bigradient into Euler's product of differences (*Hist.*, ii. pp. 369–370, 374–375), the others specially referred to by the author being Bruno's (1859, p. 27) and Gordan's (1885, p. 179).

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* According to Salmon we know from Richelot's theorem (*Hist.*, iii. p. 328) that when R = 0,

$$\omega^k = \frac{\partial R}{\partial a_p} : \frac{\partial R}{\partial a_{p-k}} = \frac{\partial R}{\partial a_q} : \frac{\partial R}{\partial a_{q-k}} = \frac{\partial R}{\partial b_q} : \frac{\partial R}{\partial b_{q-k}},$$

and consequently that

$$\frac{\partial R}{\partial a_p} \cdot \frac{\partial R}{\partial a^{q-k}} = \frac{\partial R}{\partial a_{p-k}} \cdot \frac{\partial R}{\partial a_q} \quad \text{and} \quad \frac{\partial R}{\partial a_p} \cdot \frac{\partial R}{\partial b_{q-k}} = \frac{\partial R}{\partial a_{p-k}} \cdot \frac{\partial R}{\partial b_q}.$$

It thus follows that

$$\frac{\partial R}{\partial a_p} \cdot \frac{\partial R}{\partial a^{q-k}} - \frac{\partial R}{\partial a_{p-k}} \cdot \frac{\partial R}{\partial a_q} \quad \text{and} \quad \frac{\partial R}{\partial a_p} \cdot \frac{\partial R}{\partial b_{q-k}} - \frac{\partial R}{\partial a_{p-k}} \cdot \frac{\partial R}{\partial b_q}$$

both contain R as a factor. He adds that in the former case the cofactor is

$$\frac{\partial^2 R}{\partial a_p \partial a_{q-k}} - \frac{\partial^2 R}{\partial a_{p-k} \partial a_q},$$

but that in the latter the corresponding statement does not hold.

GORDAN, P. (1893, 1894).

[Ueber die Sylvester'sche Resultante. *Verhandl. d. Ges. deutscher Naturf. u. Aerzte*, ii. 1, p. 4.]

[Ueber die Resultante. *Math. Annalen*, xlv. pp. 405–409.]

The first of these notes concerns the familiar resultant

$$\begin{vmatrix} (a,\ b,\ c,\ d)_3 \\ (p,\ q,\ r,\ s)_3 \end{vmatrix}$$

of two binary cubics: the second is more important, the subject being the Laplace expansion of

$$\begin{vmatrix} (a_0,\ a_1,\ \ldots,\ a_m)_n \\ (b_0,\ b_1,\ \ldots,\ b_n)_m \end{vmatrix}$$

in terms of the n-line minors of the a-array and the m-line minors of the b-array, and the object being to facilitate the evaluation on the lines originally followed by Cayley (*Hist.*, ii. pp. 366–8).

TAYLOR, W. W. (1895): MUIR, T. (1896).

[Evaluation of a certain dialytic determinant. *Proceed. London Math. Soc.*, xxvii. pp. 60–66.]

[On the eliminant of $f(x) = 0, f(1/x) = 0$. *Proceed. R. Soc. Edinburgh*, xxi. pp. 360–368.]

Both of these papers concern the factors of the dialytic eliminant of such a pair of equations as

$$\left.\begin{aligned} ax^6 + bx^5 + \ldots + g &= 0 \\ gx^6 + fx^5 + \ldots + a &= 0 \end{aligned}\right\}.$$

The first author removes the linear factors at the outset, and then by a rather lengthy procedure transforms the resulting ten-line determinant into

$$\begin{vmatrix} a & b & c & d & e-g \\ . & a & b & c-g & d-f \\ . & . & a-g & b-f & c-e \\ . & -g & -f & a-e & b-d \\ -g & -f & -e & -d & a-c \end{vmatrix}^2.$$

The second author points out that the given eliminant being

centro-symmetric resolves itself into two six-line determinants at once, namely,

$$\begin{vmatrix} a & b & c & d & e & f+g \\ . & a & b & c & d+g & e+f \\ . & . & a & b+g & c+f & d+e \\ . & . & g & a+f & b+e & c+d \\ . & g & f & e & a+d & b+c \\ g & f & e & d & c & a+b \end{vmatrix}, \quad \begin{vmatrix} a & b & c & d & e & f-g \\ . & a & b & c & d-g & e-f \\ . & . & a & b-g & c-f & d-e \\ . & . & -g & a-f & b-e & c-d \\ . & -g & -f & -e & a-d & b-c \\ -g & -f & -e & -d & c & a-b \end{vmatrix};$$

that $a+b+c+\ldots+g$ is removable as a factor from the one, and $a-b+c-\ldots+g$ from the other; and that the cofactor in each is readily transformable into the same five-line determinant

$$\begin{vmatrix} a & b & c & d & e-g \\ . & a & b & c-g & d-f \\ . & . & a-g & b-f & c-e \\ . & -g & -f & a-e & b-d \\ -g & -f & -c & -d & a-c \end{vmatrix},$$

which thus appears squared in the final result.

Muir, however, proceeds further. First he draws attention to the fact that the second six-line determinant is the same function of $a, -b, c, -d, e, -f, g$ as the first is of a, b, c, d, e, f, g: and that therefore the resolvability of the second is included in that of the first. Also, the actual fact of resolvability is considered so curious as to deserve separate formulation, namely, *if* $\#$ *be used to indicate addition in the case of arrays, then*

$$\begin{vmatrix} a & b & c & d & e & f \\ . & a & b & c & d & e \\ . & . & a & b & c & d \\ . & . & . & a & b & c \\ . & . & . & . & a & b \\ . & . & . & . & . & a \end{vmatrix} \# \begin{vmatrix} . & . & . & . & . & g \\ . & . & . & . & g & f \\ . & . & . & g & f & e \\ . & . & g & f & e & d \\ . & g & f & e & d & c \\ g & f & e & d & c & b \end{vmatrix}$$

$$= (a+b+c+\ldots+g) \cdot \begin{vmatrix} a & b & c & d & e & . & . & . & . & -g \\ . & a & b & c & d & . & . & . & -g & -f \\ . & . & a & b & c & \# & . & -g & -f & -e \\ . & . & . & a & b & . & -g & -f & -e & -d \\ . & . & . & . & a & -g & -f & -e & -d & -c \end{vmatrix},$$

and it holds for any number of letters, a simple example being

$$\begin{vmatrix} a & b & c & d & & . & . & . & e \\ . & a & b & c & & . & . & e & d \\ . & . & a & b & \# & . & e & d & c \\ . & . & . & a & & e & d & c & b \end{vmatrix} = (a+b+c+d+e) \begin{vmatrix} a & b & c & & . & . & -e \\ . & a & b & \# & . & -e & -d \\ . & . & a & & -e & -d & -c \end{vmatrix}.$$

The existence of similar results is then enquired into; and it is found that if we denote the kind of determinant which appears on the right in these examples by the functional symbol χ, we have a fresh series of identities exemplified by

$$\begin{vmatrix} a & b & c & d & & . & . & . & d \\ . & a & b & c & & . & . & d & c \\ . & . & a & b & \# & . & d & c & b \\ . & . & . & a & & d & c & b & a \end{vmatrix} = 2(a+b+c+d)(a-b+c-d) \cdot \chi_2,$$

and another series exemplified by

$$\begin{vmatrix} a & b & c & d & e & & . & . & . & . & d \\ . & a & b & c & d & & . & . & . & d & c \\ . & . & a & b & c & \# & . & . & d & c & b \\ . & . & . & a & b & & . & d & c & b & a \\ . & . & . & . & a & & d & c & b & a & . \end{vmatrix} = -ed(a+b+c+d) \cdot \chi_2,$$

where χ_2 or χ $(abcd)$ stands for

$$\begin{vmatrix} a & b & & . & -d \\ . & a & \# & -d & -c \end{vmatrix}.$$

NETTO, E. (1896).

[Zur Theorie der Resultanten. *Crelle's Journ.*, cxvii. pp. 57–71.]

The last four pages here are occupied with the theorem that *If* S *be the bigradient eliminant of two binary quantics*, $S_{\alpha\beta}$ *denote the minor obtained from* S *by deleting the* α^{th} *row and* β^{th} *column, then*

$$S_{\alpha\mu}S_{\beta\nu} - S_{\alpha\rho}S_{\beta\sigma}$$

is divisible by S *so long as* $\mu+\nu = \rho+\sigma$. The proof opens with the case where

$$\beta = 1, \quad \rho = \nu - 1, \quad \sigma = \mu + 1,$$

and consists in constructing a single determinant equal to the difference in question, and then so transforming it as to put in evidence the desired factor. In the next place this first result is gradually widened with the help of the known case where $\rho, \sigma = \nu, \mu$: but the process is considerably lengthened by the necessity which it involves of establishing the irresolvability of S. At the close the bearing of the theorem on Salmon's mentioned above (p. 342) is briefly dealt with.

GOURSAT, E. (1896–1897).

[Leçons sur l'Integration des Equations aux dérivées partielles du second order à deux variables indépendantes. i. p. 25; ii. p. 298. Paris.]

Incidentally there here appear two evaluated bigradients involving differentials. As, however, they represent results which are perfectly true independently of all differentiation, we shall state them quite differently. In the notation of *Hist.*, iii. pp. 330–331, they are

$$\begin{vmatrix} (a, b, c)_r \\ (x^r, rx^{r-1}y, \ldots, y^r)_2 \end{vmatrix} = \begin{vmatrix} (a, b, c)_1 \\ (x, y)_2 \end{vmatrix}^r,$$

$$\begin{vmatrix} (a_1, a_2, \ldots, a_r)_2 \\ (x^2, 2xy, y^2)_{r-1} \end{vmatrix} = \begin{vmatrix} (a_1, a_2, \ldots, a_r)_1 \\ (x, y)_{r-1} \end{vmatrix}^2;$$

for example, when r is 3,

$$\begin{vmatrix} a & b & c & . & . \\ . & a & b & c & . \\ . & . & a & b & c \\ x^3 & 3x^2y & 3xy^2 & y^3 & . \\ . & x^3 & 3x^2y & 3xy^2 & y^3 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & . \\ . & x & y \end{vmatrix}^3,$$

$$\begin{vmatrix} a & b & c & . \\ . & a & b & c \\ x^2 & 2xy & y^2 & . \\ . & x^2 & 2xy & y^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ x & y & . \\ . & x & y \end{vmatrix}^2.$$

No proof is given, but if the determinants be viewed as eliminants a way is at once seen to be open.

ROE, E. D. (1898).

[Die Entwickelung der Sylvester'schen Determinante nach Normal-Form. Dissert. (?) vi+52 pp. Leipzig.]

This is a simple study of the final expansion of a bigradient, the classification and properties of its terms, and all possible aids towards the calculation of it in any individual case. Although as an exposition it is clear and conspicuously full, probably its real value lies in giving at the end the final expansion of the eliminant $R_{5,1}$, when the coefficients of the given equations are

$$a_0, \quad -a_1, \quad a_2, \quad -a_3, \quad a_4, \quad -a_5$$
$$b_0, \quad b_1, \quad b_2, \quad b_3, \quad b_4.$$

The terms are arranged in five groups, the basis of classification being connected with the two pairs of coefficients

$$a_0, b_0, \quad a_5, b_4.$$

To the 5th group belong those terms which contain both of both pairs,—terms of so-called 'normal-form': to the 2nd and 3rd groups belong those containing both of one pair and one of the other pair: to the 1st those containing only one of each and being 'completely reducible': and to the 4th those containing only one of each and not 'completely reducible.' All the groups except the 5th consist of terms reducible to terms of determinants of lower order, so that 'normal' is evidently meant to be equivalent to 'irreducible'; indeed the writer himself opposes 'normal' to 'reducible' speaking, elsewhere than in the title, of the development as being "according to normal and reducible forms."

STUDNIČKA, F. J. (1898).

[Objasněni Borchardtovy poučky o jakosti kořenů rovnic algebraických. *Časopis pro pěstování math. a fys.*, xxvii. pp. 237–246.]

This is a simple exposition of the use of a Sturmian series. Nothing fresh occurs in connection with the determinants employed (*Hist.*, ii. pp. 330–331, 346–347). As usual with Studnička the illustrative examples make the paper specially helpful to students.

VAN VLECK, E. B. (1899).

[On the determination of a series of Sturm's functions by the calculation of a single determinant. *Annals of Math.*, (2) i. pp. 1–13.]

What falls to be noted here is an expeditious way of arriving at the set of Sturmian remainders when the two originating functions

$$a_0x^n + a_1x^{n-1} + \ldots, \qquad b_0x^n + b_1x^{n-1} + \ldots$$

have arithmetical coefficients. The process consists in the so-called transformation of the dialytic eliminant

$$\begin{vmatrix} a_0 & a_1 & a_2 & \ldots \\ b_0 & b_1 & b_2 & \ldots \\ . & a_0 & a_1 & \ldots \\ . & b_0 & b_1 & \ldots \\ . & . & . & . \end{vmatrix}$$

so as to have nothing but zeros on the left of the diagonal, the first row being utilized along with the second to produce the new second, the new second along with the third to produce the new third, and so on. It is then affirmed that the even rows in the resulting determinant furnish the coefficients of the successive Sturmian remainders. When the second function is the derivate of the first, the process commences of course with the alteration of the third row; for example, the function being

$$x^6 + x^5 - x^4 - x^3 + x^2 - x + 1,$$

the altered determinant is

$$\begin{vmatrix} 1 & 1 & -1 & -1 & 1 & -1 & 1 & . & . & \ldots \\ . & 6 & 5 & -4 & -3 & 2 & -1 & . & . & \ldots \\ . & . & 1 & -2 & -3 & 4 & -5 & 6 & . & \ldots \\ . & . & . & 17 & 14 & -27 & 32 & -37 & . & \ldots \\ . & . & . & . & -8 & -4 & 6 & -8 & 17 & \ldots \\ . & . & . & . & . & 44 & -114 & 120 & -7 & \ldots \\ . & . & . & . & . & . & . & . & . & . \end{vmatrix}.$$

In thus finding the Sturmian remainders

$$\begin{aligned} 17x^4 + 14x^3 - 27x^2 + 32x \ &- 37, \\ - 44x^3 + 114x^2 - 120x + \ &7, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

the author rightly claims considerable saving of labour.

We should prefer, however, to free the matter from connection with the dialytic eliminant, substituting the proposition: *If from the given pair of rows of arithmetical coefficients a third row be formed whose values are* $|a_0b_1|$, $|a_0b_2|$, . . . , $|a_0b_n|$ *next a fourth row formed similarly from the second and third, and so on, then the elements of the* $(2r+1)^{th}$ *row are the coefficients of the* r^{th} *Sturmian remainder.* It would perhaps be still better to substitute a theorem in pure bigradients, affirming that *the elements of the* 4^{th}, 5^{th}, . . . *rows are*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ . & b_1 & b_2 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & a_4 \\ . & b_1 & b_3 \\ b_1 & b_2 & b_4 \end{vmatrix}, \quad \begin{vmatrix} a_1 & a_2 & a_5 \\ . & b_1 & b_4 \\ b_1 & b_2 & b_5 \end{vmatrix}, \quad \ldots .$$

$$b_1 \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ . & a_1 & a_2 & a_3 \\ . & b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix}, \qquad b_1 \begin{vmatrix} b_1 & b_2 & b_3 & b_5 \\ . & a_1 & a_2 & a_4 \\ . & b_1 & b_2 & b_4 \\ b_1 & b_2 & b_3 & b_5 \end{vmatrix}, \quad \ldots .$$

. . . . . .

This would have applications to other subjects than Sturmian remainders.

Bibliographical List.

As before we append herewith a list of papers which have an indirect bearing on bigradients through being occupied with the resultant of two binary quantics, or with their highest-common-factor, or with their common roots. To students of these special subjects it need scarcely be added that the papers are of all degrees of importance:

1880. Biehler, C. Sur un procédé d'élimination. *Nouv. Annales de Math.*, (2) xix. pp. 202–206.

1881. Hioux, V. Racines communes à deux équations algèbriques entières. *Annales de l'éc. norm.*, (2) x. pp. 383 390.

1882. Biehler, C. Sur l'elimination. *Nouv. Annales de Math.*, (3) i. pp. 529–542.

1884. Amigues, E. Elimination par la méthode de Bezout perfectionnée par Cauchy. *Journ. de Math. Spéc.*, (2) viii. pp. 101–104.

1884. Stephanos, C. Sur la théorie des formes binaires et l'élimination. *Annales de l'École Norm.*, (3) i. pp. 329–389.

1886. Voss, A. Ueber einen Fundamentalsatz aus der Theorie der algebraischen Functionen. *Math. Annalen*, xxvii. pp. 527–536.

1887. Schendel, L. Verschiedene Darstellungen der Resultante zweier binären Formen. *Zeitschrift f. Math. u. Phys.*, xxxiii. pp. 1–13, 65–77.

1887. Biehler, C. Sur l'élimination par le méthode d'Euler. *Nouv. Annales de Math.*, (3) vi. pp. 67–75.

1888. Pomey, E. Sur le plus grand commun diviseur de deux polynômes entiers. *Nouv. Annales de Math.*, (3) vii. pp. 66–90, 407–427.

1888. Netto, E. Anwendung der Modulsysteme auf eine elementare algebraische Frage. *Crelle's Journ.*, civ. pp. 321–340.

1889. Lucas, E. Sur le plus grand commun diviseur algébrique. *Journ. de Math. Spéc.*, (3) iii. pp. 25–27, 49–50.

1889. Kretkowski, W. Beitrag zur Eliminationstheorie. *Prace mat. fiz.* (Warschau), ii. pp. 21–32.

1889. Netto, E. Ueber den grössten gemeinsamen Teiler zweier ganzer Functionen. *Mitteil. d. math. Ges.* (Hamburg), ii. pp. 36–43.

1890. Netto, E. Ueber den gemeinsamen Teiler zweier ganzen Functionen einer Veränderlichen. *Crelle's Journ.*, cvi. pp. 81–88.

1893. Garbieri, G. Sulla teoria della eliminazione fra due equazioni. *Atti . . . Accad. Gioenia*, (4) vi. pp. 1–9.

1895. Lüroth, J. Kurze Ableitung der Bedingungen, dass zwei algebraische Gleichungen mehrere Wurzeln gemein haben. *Zeitschrift f. Math. u. Phys.*, xl. pp. 247–251.

1895. Andoyer, H. Sur la division algébrique appliquée aux polynômes homogènes. *Journ. (de Liouville) de Math.* (5) i. pp. 61–90.

1895. Netto, E. Ueber die Structur der Resultanten binärer Formen. *Nachrichten . . . Ges. d. Wiss.*, 1895, pp. 209–210.

1895. Humbert, E. Note sur le résultant de deux équations entières. *Revue de Math. Spéc.*, iii. (5e année), pp. 201–202.

1895. Nöther, M. Ueber den gemeinsamen Factor zweier binärer Formen. *Sitzungsb. d. phys.-med. Soc.* (Erlangen), xxvii. pp. 110–115.

1896. Hadamard, J. Mémoire sur l'élimination. *Acta Math.*, xx. pp. 201–238.

1897. Longchamps, G. de. Sur le résultant de deux formes binaires. *Journ. de Math. Spéc.*, xxi. pp. 11–12.

1898. Szüts, N. v. Ueber die Bildung der Resultante und des grössten gemeinsamen Theiler zweier ganzer rationaler Functionen einer Variabeln. *Monatshefte f. Math. u. Phys.*, ix. pp. 34–42.

1898. Gordan, P. Sur le résultant de deux équations. *Comptes rendus . . . Acad. des Sci.* (Paris), cxxvii. pp. 539–541.

1898. Hadamard, J. Sur les conditions de décomposition des formes. *Bull. Soc. Math. de France*, xxvii. pp. 34–47.

CHAPTER XV.

HESSIANS, FROM 1880 TO 1897.

HERE, as in the associated case of Jacobians, there is no increase of progress to report, the number of writings to be dealt with having fallen off by a third, exactly as in that case.

SCOTT, R. F. (1880).

[A Treatise on the Theory of Determinants. . . . xi+251 pp. Cambridge.]

There are here recorded three examples of evaluated Hessians. First, it is fully shown (p. 143) that if

$$u \equiv x_1^2 x_2^2 + \ldots + x_1^2 x_n^2 + x_2^2 x_3^2 + \ldots + x_{n-1}^2 x_n^2,$$

and $\sigma \equiv \frac{1}{3}(x_1^2 + \ldots + x_n^2)$, then

$$\mathrm{H}(u) = 6^n (\sigma - x_1^2)(\sigma - x_2^2) \ldots (\sigma - x_n^2)\left\{1 + \frac{2}{3}\sum \frac{x_1^2}{\sigma - x_1^2}\right\}.$$

Next, it is noted (p. 231) that if $u_1, u_2, \ldots, u_n$ be linear functions of $x_1, x_2, \ldots, x_n$, then

$$\mathrm{H}(\log u_1 u_2 \ldots u_n) = (-1)^n \left\{\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)}\right\}^2 (u_1 u_2 \ldots u_n)^{-2};$$

and that if

$$v_n \equiv (x+y+z)^n + (x-y-z)^n + (-x+y-z)^n + (x+y-z)^n,$$

then, save for an arithmetical factor,

$$\mathrm{H}(v_n) = v_{2-n}(x^4+y^4+z^4-2x^2y^2-2y^2z^2-2z^2x^2)^{n-2}.$$

BAUER, G. (1883).

[Die Hesse'sche Determinante der Hesse'schen Fläche einer Fläche dritter Ordnung. *Denkschr. . . . Akad. d. Wiss.* (München), xiv. (3), pp. 77–90.]

There is here extended to surfaces a theorem of Hesse's regarding a cubic curve, and what we have to note is the property of $H\{H(u_{33})\}$ analogous to that given in Hesse's original paper of 1844 (*Hist.*, ii. p. 377).

VOSS, A. (1887).

[Zur Theorie der Hesse'sche Determinante. *Math. Annalen*, xxx. pp. 418–424.]

Here, as in Bauer's paper of 1883, the subject is the Hessian of the Hessian: and notwithstanding the title the interest is even more markedly geometrical.

GERBALDI, F. (1889): SCHOUTE, P. H. (1889).

[Un teorema sull' Hessiana d'una forma binaria. *Rendic. del Circolo Mat.* (Palermo), iii. pp. 22–26.]

[Sur un théorème relatif à l'Hessienne d'une forme binaire. *Rendic. del Circolo Mat.* (Palermo), iii. pp. 160–164.]

The main theorem here we have already given under Sylvester's name (1879): but Gerbaldi adds that if the Hessian have only imaginary roots and assume only positive values, then the roots of the quantic are also all imaginary.*

McMAHON, J. (1889): FRANKLIN, F. (1890).

[On the expression of the Hessian of a binary quantic in terms of the roots. *Annals of Math.*, v. pp. 17–18.]

[On the Hessian of a product of linear functions. *Annals of Math.*, v. pp. 103–105.]

The result stated and proved in the first of these papers might have been viewed as already known (see, for example, *Hist.*,

* Another paper on an already known result in the Theory of Quantics is GERBALDI, F. Sull' Hessiano del prodotto di due forme ternarie. *Rendic. del Circolo Mat.* (Palermo), iii. pp. 60–66.

iii. p. 371). Franklin's generalization is of a different degree of importance, being that

$$H(u_1u_2 \ldots u_n) = (-1)^{k-1} \cdot (n-1) \cdot (u_1u_2 \ldots u_n)^{k-2} \sum \{ | a_{1k} | \cdot u_{k+1} \ldots u_n \}^2,$$

where $u_r \equiv a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rk}x_k$.

PASCH, M. (1890).

[Ueber bilineare Formen . . . *Math. Annalen*, xxxviii. pp. 24–49.]

Incidentally, and not by a determinantal transformation, there is here (p. 41) reached the fact that *the Hessian of* $| a_1b_2c_3 |$ *with respect to the elements is* $-2 | a_1b_2c_3 |^3$. The author also notes (pp. 46–47) the less important fact that the Hessian of a three-line axisymmetric determinant Δ is $-16\Delta^2$, quoting in connection therewith the name of A. Breuer (1888), and not the names of Sylvester (1841) and Cayley (1856) (*Hist.*, i. p. 244 ; ii. pp. 143–144).

MUIR, T. (1893).

[Question 11814. *Educ. Times*, xlvi. p. 37 : *Math. from Educ. Times*, (2) xiv. pp. 38–39.]

A simple identity connected with the Hessian of a binary cubic.

PAINLEVÉ, P. (1894).

[Note sur une identité entre certains déterminants. *Bull. Soc. Math. de France*, xxii. pp. 116–119.]

The theorem here concerns two determinants got by performing two different sets of operations on a homogeneous function of x, x_1, x_2, . . . , x_n of the m^{th} degree. The first operation consists in forming the Hessian and then putting in it $x = 1$. The second consists in putting $x = 1$ in the given function at the outset and forming with the resulting function ϕ the n-line determinant whose $(r, s)^{\text{th}}$ element is

$$m\phi \frac{\partial^2 \phi}{\partial x_r \partial x_s} - (m-1) \frac{\partial \phi}{\partial x_r} \frac{\partial \phi}{\partial x_s}.$$

The ratio of the former result to the latter is

$$(m\phi)^{n-1} / (m-1).$$

NANSON, E. J. (1895).

[Hessian of an implicit function. *Messenger of Math.*, xxv. pp. 137–138.]

The result here established is that *if* $\phi(u, x_1, x_2, \ldots, x_n) = 0$, *the Hessian of* u *with respect to the* x*'s is*

$$\frac{(-1)^{n+1}}{\phi_0^{n+2}} \begin{vmatrix} \phi_{00} & \phi_{01} & \cdots & \phi_{0n} & \phi_0 \\ \phi_{10} & \phi_{11} & \cdots & \phi_{1n} & \phi_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{n0} & \phi_{n1} & \cdots & \phi_{nn} & \phi_n \\ \phi_0 & \phi_1 & \cdots & \phi_n & \cdot \end{vmatrix},$$

where the suffixes 0, 1, 2, . . . , n *denote differentiation with respect to* $u, x_1, x_2, \ldots, x_n$ *respectively*, and where therefore the determinant is the bordered Hessian of ϕ.

We note for ourselves that, when ϕ is a homogeneous integral function, Brioschi's theorem regarding the bordered Hessian (*Hist.*, ii. pp. 396–397) applies, and gives an interesting corollary.*

REUSCHLE, C. (1897).

[Konstituententheorie, eine neue, principielle und genetische Methode zur Invariantentheorie. *Verhandl. d. internat. math. Kongresses* (Zürich), pp. 122–140.]

'Konstituenten' are entities that arise in connection with any quantic by differentiation: thus, u being a quantic of the m^{th} degree in $x_1, x_2, \ldots$, its h^{th} 'primary constituent,' its $(h, k)^{\text{th}}$ 'secondary constituent,' and so forth, are

$$\frac{1}{m}\frac{\partial u}{\partial x_h}, \quad \frac{1}{m(m-1)}\frac{\partial^2 u}{\partial x_h \partial x_k}, \quad \ldots\ldots$$

respectively. As is not unnatural, the determinant with the secondary constituents for elements comes to the surface in the course of work, and some space is given to the consideration of its adjugate.

* Immediately following on this paper in the *Messenger* Nanson has another dealing with the change of independent variables in the Hessian.

CHAPTER XVI.

CIRCULANTS, FROM 1880 TO 1899.

THE period under review shows a marked increase in the study of circulants, the number of writings being almost double the number for the preceding period. The segregation of those on 'block' circulants, which was tentatively introduced in Vol. III., is here definitely adopted.

SCOTT, R. F. (1880).

[Note on a determinant theorem of Mr. Glaisher's. *Quart. Journ. of Math.*, xvii. pp. 129–132.]

Taking C(a, b, c, d, e, f) as an example of an even-ordered circulant, and performing the operations

$$\begin{array}{lll} \mathrm{col}_1 + \mathrm{col}_4, & \mathrm{col}_2 + \mathrm{col}_5, & \mathrm{col}_3 + \mathrm{col}_6, \\ \mathrm{row}_3 - \mathrm{row}_1, & \mathrm{row}_4 - \mathrm{row}_2, & \mathrm{row}_5 - \mathrm{row}_3, \end{array}$$

Scott finds it equal to

$$\begin{vmatrix} a+d & b+e & c+f \\ c+f & a+d & b+e \\ b+e & c+f & a+d \end{vmatrix} \cdot \begin{vmatrix} a-d & b-e & c-f \\ f-c & a-d & b-e \\ e-b & f-c & a-d \end{vmatrix}.$$

The form of the second factor leads him to draw attention to what he calls a cyclically skew determinant, namely,

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ -a_n & a_1 & a_2 & \dots & a_{n-1} \\ -a_{n-1} & -a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ -a_2 & -a_3 & -a_4 & \dots & a_1 \end{vmatrix},$$

of which an equivalent is

$$\prod_{r=1}^{r=n}\left(a_1+a_2\omega_r+a_3\omega_r{}^2+\ldots+a_n\omega_r{}^{n-1}\right),$$

where ω_r is any root of the equation

$$x^n+1=0.$$

His theorem is thus to the effect that *a circulant of the* $(2m)^{th}$ *order is expressible as the product of a circulant and a skew circulant each of the* m^{th} *order*; and he gives an alternative proof of it dependent on the ordinary factorial expressions for the determinants in question. It is noted also that on account of the relation between the roots of the equations

$$x^{2n+1}+1=0, \qquad x^{2n+1}-1=0,$$

the circulant of

$$a_1,\ -a_2,\ a_3,\ -a_4,\ \ldots,\ -a_{2n},\ a_{2n+1}$$

is equal to the skew circulant of

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_{2n},\ a_{2n+1},$$

or, say, to

$$C'(a_1,\ a_2,\ \ldots,\ a_{2n+1}).$$

GLAISHER, J. W. L. (1880).

[On some algebraical expressions which are unaltered by certain substitutions. *Messenger of Math.*, x. pp. 60–63.]

In effect the main result of the paper is that if s stands for

$$a_1+a_2+\ldots+a_n,$$

then

$$C(s-a_1,\ s-a_2,\ \ldots,\ s-a_n)=(-1)^{n-1}(n-1)C(a_1,\ a_2,\ \ldots,\ a_n),$$

and consequently that

$$\frac{C(s-a_1,\ s-a_2,\ \ldots,\ s-a_n)}{(s-a_1)+(s-a_2)+\ldots+(s-a_n)}=(-1)^{n-1}\frac{C(a_1,\ a_2,\ \ldots,\ a_n)}{a_1+a_2+\ldots+a_n},$$

a simple example of the latter being

$$\begin{vmatrix}1 & c+a & a+b\\ 1 & b+c & c+a\\ 1 & a+b & b+c\end{vmatrix}=(-1)^2\begin{vmatrix}1 & b & c\\ 1 & a & b\\ 1 & c & a\end{vmatrix},$$

or, in non-determinant notation,

$$(b+c)^2 + (c+a)^2 + (a+b)^2 - (c+a)(a+b) - (a+b)(b+c) - (b+c)(c+a) = a^2 + b^2 + c^2 - bc - ca - ab.$$

The result of making one or two other readily suggested substitutions for $a_1, a_2, \ldots, a_n$ in $C(a_1, a_2, \ldots, a)$ is also noted.

WEIHRAUCH, K. (1880).

[Ueber doppelt-orthosymmetrische Determinanten. *Zeitschrift f. Math. u. Phys.*, xxvi. pp. 64–70.]

Weihrauch establishes Spottiswoode's property by using Fürstenau's theorem of 1879; that is to say, he multiplies each element in the $(r, s)^{\text{th}}$ place by ω^{r-s}, where ω is any one of the n^{th} roots of unity, and then takes the sum of the rows.

More interesting is the fresh proposition that *the circulant of* $a_1, a_2, \ldots, a_n$, *is equal to the sum of the coefficients of the equation whose roots are the* n^{th} *powers of the roots of the equation*

$$a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0.$$

By way of proof it is noted that in the case where n is 4 we have

$$C(a, b, c, d) = (a + b\omega_1 + c\omega_1^2 + d\omega_1^3)(a + b\omega_2 + c\omega_2^2 + d\omega_2^3)(a + b\omega_3 + c\omega_3^2 + d\omega_3^3)(a + b\omega_4 + c\omega_4^2 + d\omega_4^3),$$

so that, if ξ_1, ξ_2, ξ_3 be the roots of the equation

$$ax^3 + bx^2 + cx + d = 0,$$

there results

$$\begin{aligned} C(a, b, c, d) &= a(1-\omega_1\xi_1)(1-\omega_1\xi_2)(1-\omega_1\xi_3) \\ &\quad \cdot a(1-\omega_2\xi_1)(1-\omega_2\xi_2)(1-\omega_2\xi_3) \\ &\quad \cdot a(1-\omega_3\xi_1)(1-\omega_3\xi_2)(1-\omega_3\xi_3) \\ &\quad \cdot a(1-\omega_4\xi_1)(1-\omega_4\xi_2)(1-\omega_4\xi_3) \\ &= a^4 \cdot (1-\omega_1\xi_1)(1-\omega_2\xi_1)(1-\omega_3\xi_1)(1-\omega_4\xi_1) \\ &\quad \cdot (1-\omega_1\xi_2)(1-\omega_2\xi_2)(1-\omega_3\xi_2)(1-\omega_4\xi_2) \\ &\quad \cdot (1-\omega_1\xi_3)(1-\omega_2\xi_3)(1-\omega_3\xi_3)(1-\omega_4\xi_3) \\ &= a^4(1-\xi_1^4)(1-\xi_2^4)(1-\xi_3^4), \end{aligned}$$

as desired. A more general result is: *The sum of the coefficients of*

the equation whose roots are the q^{th} *powers of the roots of the equation*

$$a_0x^p + a_1x^{p-1} + \ldots + a_p = 0$$

is

$$C(a_0+a_q+a_{2q}+\ldots,\ a_1+a_{q+1}+a_{2q+1}+\ldots,\ a_2+a_{q+2}+a_{2q+2}+\ldots,\ \ldots).$$

Instead of this, however, Weihrauch in effect proposes to substitute a wider proposition, namely, that *the equation whose roots are the* 4^{th} *powers of the roots of the equation* $ax^3 + bx^2 + cx + d = 0$ *is*

$$-\begin{vmatrix} a & b & c & d \\ b & c & d & ay \\ c & d & ay & by \\ d & ay & by & cy \end{vmatrix} = 0,$$

the previous result being thence obtained on putting $y = 1$. To establish this $y^{\frac{1}{4}}$ is substituted for x in the given equation, and rationalization is effected in Cayley's manner of 1852 (*Hist.*, ii. pp. 124–126), the full set of equations being

$$\left.\begin{aligned} ay^{\frac{3}{4}} + by^{\frac{2}{4}} + cy^{\frac{1}{4}} + d &= 0 \\ ay + by^{\frac{3}{4}} + cy^{\frac{2}{4}} + dy^{\frac{1}{4}} &= 0 \\ ay \cdot y^{\frac{1}{4}} + by + cy^{\frac{3}{4}} + dy^{\frac{2}{4}} &= 0 \\ ay \cdot y^{\frac{2}{4}} + by \cdot y^{\frac{1}{4}} + cy + dy^{\frac{3}{4}} &= 0 \end{aligned}\right\}$$

and the eliminands $y^{\frac{3}{4}}$, $y^{\frac{2}{4}}$, $y^{\frac{1}{4}}$.

It is also noted that the same procedure is effective when the roots of the new equation are to be the p^{th} powers of the roots of the given equation. Thus, p being 3, we substitute $y^{\frac{1}{3}}$ for x, and multiply twice by $y^{\frac{1}{3}}$, the set of equations now being

$$\left.\begin{aligned} ay + by^{\frac{2}{3}} + cy^{\frac{1}{3}} + d &= 0 \\ ay \cdot y^{\frac{1}{3}} + by + cy^{\frac{2}{3}} + dy^{\frac{1}{3}} &= 0 \\ ay \cdot y^{\frac{2}{3}} + by \cdot y^{\frac{1}{3}} + cy + dy^{\frac{2}{3}} &= 0 \end{aligned}\right\}$$

and the desired equation

$$-\begin{vmatrix} b & c & d+ay \\ c & d+ay & by \\ d+ay & by & cy \end{vmatrix} = 0.$$

Here, it may be remarked, the sum of the coefficients is still a circulant, namely, $C(d+a, b, c)$.

WEIHRAUCH, K. (1880).

[Werth einiger doppelt-orthosymmetrischer Determinanten. *Zeitschrift f. Math. u. Phys.*, xxvi. pp. 132–133.]

Three of the four results are Scott's of the year 1878, and the remaining one, namely,

$$C\{1, 3, 6, \ldots, \tfrac{1}{2}n(n+1)\} = (-1)^{n-1}\frac{n^{n-2}(n+1)(n+2)}{3\cdot 2^{n+1}}\{(n+3)^n - (n+1)^n\},$$

resembles Scott's third.

CLARIANA Y RICART, L. (1880).

[Aplicacion de las determinantes á la resolucion de las ecuaciones de cuarto grado. *Crónica cientifica* . . . (Barcelona), iii. pp. 425–429.]

The determinant used is the 4-line circulant, the author thus forestalling by a year or two the proposal of Muir and others (see below, p. 370).

SCHAPIRA, H. (1881).

[Elements of a Theory of general Cofunctions, and their Applications. (In Russian.) Vol. I. (1) i. 228 pp. Odessa.]*

The first chapter (pp. 109–124) of the second section of this book is devoted to an account of circulants in general; and in it there is one fresh result that falls to be noted, namely, *If* $\alpha, \beta, \gamma, \ldots$ *be the linear factors of the* n*-line circulant* C(a, b, c, . . .), *then*

$$C(\alpha, \beta, \gamma, \ldots) = n^n \cdot abc \ldots .$$

The reason for including such a chapter lies in the fact that in the investigation of 'cofunctions' circulants are found to be of special utility: the whole of the second chapter, indeed (pp. 125–144), is occupied with showing how symmetrical functions of cofunctions can be represented as circulants. The properties of other determinants, such as Jacobians and alternants, are also utilized.

* This is but the first portion of an elaborately planned work, which failed to materialize according to the original design. A second instalment (viii + 224 pp.) of the first volume did not appear until 1892.

SCOTT, R. F. (1881).

[Mathematical notes. *Messenger of Math.*, x. pp. 142–149.]

There is here established (pp. 148–149) the curious theorem

$$\mathrm{C}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)\mathrm{C}(x_1, x_2, \ldots, x_n) = n^n,$$

recalling a previous theorem of his (*Hist.*, iii. p. 380). Also it is noted that the determinant got by bordering $\mathrm{C}(x_1, x_2, \ldots, x_n)$ symmetrically with 0, 1, 1, . . . , 1 is equal to

$$-\frac{n\mathrm{C}(x_1, x_2, \ldots, x_n)}{x_1 + x_2 + \ldots + x_n}.$$

MUIR, T. (1881).

[On new and recently discovered properties of certain symmetric determinants. *Quart. Journ. of Math.*, xviii. pp. 166–177.]

To start with, Glaisher's theorem regarding a circulant of the $(2n)^{\text{th}}$ order is shown to be readily obtained by multiplying together two slightly altered forms of the circulant. The first form is got by advancing the odd-numbered rows in order to the first n places, and then altering the signs in the even-numbered columns, the result being

$$(-1)^{\frac{1}{2}(3+1)3} \cdot \mathrm{C}(a, b, c, d, e, f) = \begin{vmatrix} a & -b & c & -d & e & -f \\ e & -f & a & -b & c & -d \\ c & -d & e & -f & a & -b \\ f & -a & b & -c & d & -e \\ d & -e & f & -a & b & -c \\ b & -c & d & -e & f & -a \end{vmatrix}.$$

The second form is got from the first by deleting the negative signs, reversing the order of the rows and then the order of the columns, the result being

$$(-1)^{\frac{1}{2}(3-1)3} \cdot \mathrm{C}(a, b, c, d, e, f) = \begin{vmatrix} a & f & e & d & c & b \\ c & b & a & f & e & d \\ e & d & c & b & a & f \\ b & a & f & e & d & c \\ d & c & b & a & f & e \\ f & e & d & c & b & a \end{vmatrix}.$$

From these two results by multiplication there is obtained

$$(-1)^3\{\mathrm{C}(a, b, c, d, e, f)\}^2 = \begin{vmatrix} \mathrm{P} & \mathrm{R} & \mathrm{Q} & . & . & . \\ \mathrm{Q} & \mathrm{P} & \mathrm{R} & . & . & . \\ \mathrm{R} & \mathrm{Q} & \mathrm{P} & . & . & . \\ . & . & . & -\mathrm{P} & -\mathrm{R} & -\mathrm{Q} \\ . & . & . & -\mathrm{Q} & -\mathrm{P} & -\mathrm{R} \\ . & . & . & -\mathrm{R} & -\mathrm{Q} & -\mathrm{P} \end{vmatrix},$$

and therefore

$$\mathrm{C}(a, b, c, d, e, f) = \mathrm{C}(\mathrm{P}, \mathrm{Q}, \mathrm{R}),$$

where

$$\begin{aligned} \mathrm{P} &= a^2 - bf + ce - d^2 + ec - fb = a^2 + 2ce - d^2 - 2bf, \\ \mathrm{Q} &= ae - bd + c^2 - db + ea - f^2 = c^2 + 2ea - f^2 - 2db, \\ \mathrm{R} &= ac - b^2 + ca - df + e^2 - fd = e^2 + 2ac - b^2 - 2fd. \end{aligned}$$

In the second place, Scott's theorem regarding a circulant of the $(2n)^{\text{th}}$ order is shown to be immediately deducible from Zehfuss' theorem of 1862, the reason being that by reversing the order of the last n rows, and thereafter the order of the last n columns, the circulant becomes centrosymmetric. For example,

$$\mathrm{C}(a, b, c, d, e, f) = \begin{vmatrix} a & b & c & f & e & d \\ f & a & b & e & d & c \\ e & f & a & d & c & b \\ b & c & d & a & f & e \\ c & d & e & b & a & f \\ d & e & f & c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a+d & b+e & c+f \\ c+f & a+d & b+e \\ b+e & c+f & a+d \end{vmatrix} \cdot \begin{vmatrix} a-d & b-e & c-f \\ f-c & a-d & b-e \\ e-b & f-c & a-d \end{vmatrix}$$

$$= \mathrm{C}(a+d, b+e, c+f) \cdot \mathrm{C}'(a-d, b-e, c-f).$$

It is next found that when n is odd there is no difficulty in deducing Glaisher's theorem from Scott's. We have only to change the columns of $\mathrm{C}(a+d, b+e, c+f)$ into rows, alter the signs of the second row and second column of $\mathrm{C}'(a-d, b-e, c-f)$, and then

perform row-by-row multiplication, there being a series of identities like

$$(a+d,\ c+f,\ b+e\,|\,a-d,\ e-b,\ c-f) \ = \ a^2 + 2ce - d^2 - 2bf$$

to be relied on.

Lastly, attention is drawn to the fact that *there is a second way of expressing a circulant of the* $(4n+2)^{th}$ *order as a circulant of the* $(2n+1)^{th}$ *order*, namely, by resolving it in accordance with Scott's theorem, substituting for the skew circulant an ordinary circulant, and then performing row-by-row multiplication. For example,

$$\mathrm{C}(a,\ b,\ c,\ d,\ e,\ f) \ = \ \begin{vmatrix} a+f & b+e & c+d \\ c+d & a+f & b+e \\ b+e & c+d & a+f \end{vmatrix} \cdot \begin{vmatrix} a-f & e-b & c-d \\ c-d & a-f & e-b \\ e-b & c-d & a-f \end{vmatrix},$$

$$= \begin{vmatrix} \mathrm{L} & \mathrm{M} & \mathrm{N} \\ \mathrm{N} & \mathrm{L} & \mathrm{M} \\ \mathrm{M} & \mathrm{N} & \mathrm{L} \end{vmatrix},$$

where

$$\mathrm{L} \ = \ a^2 + e^2 + c^2 - b^2 - d^2 - f^2 \ = \ \mathring{\Sigma}a^2 - \mathring{\Sigma}b^2, \text{ say},$$
$$\mathrm{M} \ = \ \mathring{\Sigma}ae - \mathring{\Sigma}bd + \mathring{\Sigma}ab - \mathring{\Sigma}ad,$$
$$\mathrm{N} \ = \ \mathring{\Sigma}ae - \mathring{\Sigma}bd - \mathring{\Sigma}ab + \mathring{\Sigma}ad,$$

the $\mathring{\Sigma}$ implying summation of the terms got by changing

$$a,\ e,\ c \quad \text{into} \quad e,\ c,\ a,$$

and

$$b,\ d,\ f \quad \text{into} \quad d,\ f,\ b.$$

MUIR, T. (1881).

[On the resolution of a certain determinant into quadratic factors. *Messenger of Math.*, xi. pp. 105–108.]

The theorem here established is that *the determinant whose every element is the sum of the corresponding elements of the circulants*

$$\mathrm{C}(a_1,\ a_2,\ \ldots,\ a_n), \qquad (-1)^{\frac{1}{2}(n-1)(n-2)} \cdot \mathrm{C}(b_1,\ b_2,\ \ldots,\ b_n)$$

is divisible by

$$(a_1+\omega a_2+\ldots+\omega^{n-1}a_n)(a_1+\omega^{-1}a_2+\ldots+\omega^{-n+1}a_n)$$
$$-(b_1+\omega b_2+\ldots+\omega^{n-1}b_n)(b_1+\omega^{-1}b_2+\ldots+\omega^{-n+1}b_n),$$

ω *being one of the imaginary* n^{th} *roots of unity.*

When n is 5 the determinant in question is*

$$\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 & a_4+b_4 & a_5+b_5 \\ a_5+b_2 & a_1+b_3 & a_2+b_4 & a_3+b_5 & a_4+b_1 \\ a_4+b_3 & a_5+b_4 & a_1+b_5 & a_2+b_1 & a_3+b_2 \\ a_3+b_4 & a_4+b_5 & a_5+b_1 & a_1+b_2 & a_2+b_3 \\ a_2+b_5 & a_3+b_1 & a_4+b_2 & a_5+b_3 & a_1+b_4 \end{vmatrix},$$

and may be viewed as the eliminant of a set of linear homogeneous equations in x_1, x_2, x_3, x_4, x_5. Using the multipliers $\omega^0, \omega^1, \omega^2, \omega^3, \omega^4$ with these equations and performing addition, we obtain

$$\frac{a_1+\omega^{-1}a_2+\ldots+\omega^{-4}a_5}{b_1+\omega b_2+\ldots+\omega_4 b_5} = -\frac{x_1+\omega x_5+\omega^2 x_4+\omega^3 x_3+\omega^4 x_2}{x_1+\omega x_2+\omega^2 x_3+\omega^3 x_4+\omega^4 x_5},$$

and therefore, on putting ω^{-1} for ω,

$$\frac{a_1+\omega a_2+\ldots+\omega^4 a_5}{b_1+\omega^{-1}b_2+\ldots+\omega^{-4}b_5} = -\frac{x_1+\omega^4 x_5+\omega^3 x_4+\omega^2 x_0+\omega x_2}{x_1+\omega^4 x_2+\omega^3 x_3+\omega^2 x_4+\omega x_5},$$

from which two equalities there results by multiplication

$$\frac{a_1+\omega^{-1}a_2+\ldots+\omega^{-4}a_5}{b_1+\omega b_2+\ldots+\omega^4 b_5} \cdot \frac{a_1+\omega a_2+\ldots+\omega^4 a_5}{b_1+\omega^{-1}b_2+\ldots+\omega^{-4}b_5} = 1.$$

This being independent of the x's furnishes a factor of the eliminant, and the factor thus reached is seen to be the quadratic factor specified in the enunciation of the theorem.

Note is taken that when n is odd there is the linear factor

$$a_1 + a_2 + \ldots + a_n + b_1 + b_2 + \ldots + b_n,$$

and that when n is even there is in addition to this the factor

$$a_1 - a_2 + \ldots - a_n + b_1 - b_2 + \ldots - b_n.$$

Further, the multiplications indicated in the quadratic factor are actually performed, and ω made to disappear through substitutions like $2\cos\frac{2}{5}\pi$ for $\omega+\omega^{-1}$.

* The determinant may be described otherwise, and perhaps better, as having its array equal to the sum of two circulant arrays one of which is symmetric with respect to the secondary diagonal and the other with respect to the primary diagonal.

MUIR, T. (1881).

[On circulants of odd order. *Quart. Journ. of Math.*, xviii. pp. 261–265.]

The main theorem here established is that *the cofactor of*

$$a_1 + a_2 + \ldots + a_{2n+1}$$

in $C(a_1, a_2, \ldots, a_{2n+1})$ *is expressible as a symmetric determinant of the* n^{th} *order whose distinct elements are the* $\frac{1}{2}n(n+1)$ *differences of the cyclic sums*

$$\overset{\circ}{\Sigma}a_1^2, \quad \overset{\circ}{\Sigma}a_1a_2, \quad \overset{\circ}{\Sigma}a_1a_3, \quad \ldots, \quad \overset{\circ}{\Sigma}a_1a_n.$$

Calling these cyclic sums A_1, A_2, A_3, A_4, in the case where n is 3, and multiplying $C(a_1, a_2, \ldots, a_7)$ by itself in the form obtained by diminishing each row by the immediately preceding row, we find

$$(C_7)^2 = \begin{vmatrix} A_1 & A_2 & A_3 & A_4 & A_4 & A_3 & A_2 \\ A_2-A_1 & A_1-A_2 & A_2-A_3 & A_3-A_4 & . & A_4-A_3 & A_3-A_2 \\ A_3-A_2 & A_2-A_1 & A_1-A_2 & A_2-A_3 & A_3-A_4 & . & A_4-A_3 \\ A_4-A_3 & A_3-A_2 & A_2-A_1 & A_1-A_2 & A_2-A_3 & A_3-A_4 & . \\ . & A_4-A_3 & A_3-A_2 & A_2-A_1 & A_1-A_2 & A_2-A_3 & A_3-A_4 \\ A_3-A_4 & . & A_4-A_3 & A_3-A_2 & A_2-A_1 & A_1-A_2 & A_2-A_3 \\ A_2-A_3 & A_3-A_4 & . & A_4-A_3 & A_3-A_2 & A_2-A_1 & A_1-A_2 \end{vmatrix}.$$

Here the increasing of the first column by all the others shows that

$$A_1 + 2A_2 + 2A_3 + 2A_4, \quad i.e. \quad (a_1+a_2+\ldots+a_7)^2$$

is a factor, and the full result takes the form

$$(C_7)^2 = (a_1+a_2+\ldots+a_7)^2 \begin{vmatrix} \alpha & \beta & \gamma & . & -\gamma & -\beta \\ -\alpha & \alpha & \beta & \gamma & . & -\gamma \\ -\beta & -\alpha & \alpha & \beta & \gamma & . \\ -\gamma & -\beta & -\alpha & \alpha & \beta & \gamma \\ . & -\gamma & -\beta & -\alpha & \alpha & \beta \\ \gamma & . & -\gamma & -\beta & -\alpha & \alpha \end{vmatrix}$$

if for shortness' sake we write

$$\alpha, \beta, \gamma \quad \text{for} \quad A_1 - A_2, \quad A_2 - A_3, \quad A_3 - A_4.$$

Multiplying this new determinant by

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & . \\ . & . & 1 & 1 & . & . \\ . & . & . & 1 & . & . \\ . & . & . & . & 1 & . \\ . & . & . & . & . & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & . & . & . & . & . \\ . & 1 & . & . & . & . \\ . & . & 1 & . & . & . \\ . & . & . & 1 & . & . \\ . & . & 1 & 1 & 1 & . \\ . & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

in succession we obtain

$$(C_7)^2 = (\Sigma a)^2 \begin{vmatrix} \alpha & \beta & \gamma & . & . & . \\ \beta & \alpha+\beta+\gamma & \beta+\gamma & . & . & . \\ \gamma & \beta+\gamma & \alpha+\beta & . & . & . \\ . & \gamma & \beta & \alpha & \beta & \gamma \\ -\gamma & . & \gamma & \beta & \alpha+\beta+\gamma & \beta+\gamma \\ -\beta & -\gamma & . & \gamma & \beta+\gamma & \alpha+\beta \end{vmatrix},$$

and thence

$$C_7 = \Sigma a \cdot \begin{vmatrix} A_1-A_2 & A_2-A_3 & A_3-A_4 \\ A_2-A_3 & A_1-A_4 & A_2-A_4 \\ A_3-A_4 & A_2-A_4 & A_1-A_3 \end{vmatrix},$$

as desired.

The latter part of the proof is seen to turn on the transformability of a certain six-line persymmetric determinant into the square of a three-line axisymmetric determinant, and this property is stated to hold for a persymmetric determinant of the $(2n)^{\text{th}}$ order whose distinct elements are

$$-a_2, \ldots, -a_n, \ 0, \ a_n, \ldots, a_1, \ -a_1, \ldots, -a_n, \ 0, \ a_n, \ldots, a_3,$$

the elements of the n-line axisymmetric determinant being of the form

$$a_h + a_{h+1} + \ldots + a_{h+k-1}.$$

TORELLI, G. (1882).

[Sui determinanti circolanti. *Rendic. . . . Accad. delle Sci. Fis. e Mat.* (Napoli), pp. 3–11.]

This paper, following avowedly on those of Glaisher and Scott and Muir, contains important generalizations.

The first theorem is to the effect that *any circulant of the* $(rs)^{th}$ *order is expressible as a circulant of the* s^{th} *order, and each of the elements of the latter is an aggregate of* r*-line minors,* r^{s-1} *in number, of the former.* Torelli's proof, which is the natural extension of Glaisher's, is stated in perfectly general terms. We shall, for the sake of greater clearness, apply it only to the case where $rs = 15$, the elements of the circulant being

$$a_1,\ a_2,\ a_3, \ldots, a_{15}.$$

The fifteenth roots of 1 are the three third roots of each of the five fifth roots of 1, so that if γ_1, γ_2, γ_3 be the third roots of 1 and $\epsilon_1, \epsilon_2, \ldots, \epsilon_5$ the fifth roots of 1, the fifteenth roots of 1 may be represented by

$$\gamma_1\epsilon_1^{\frac{1}{3}},\ \gamma_2\epsilon_1^{\frac{1}{3}},\ \gamma_3\epsilon_1^{\frac{1}{3}};\quad \gamma_1\epsilon_2^{\frac{1}{3}},\ \gamma_2\epsilon_2^{\frac{1}{3}},\ \gamma_3\epsilon_3^{\frac{1}{3}};\quad \ldots;\quad \gamma_1\epsilon_5^{\frac{1}{3}},\ \gamma_2\epsilon_5^{\frac{1}{3}},\ \gamma_3\epsilon_5^{\frac{1}{3}},$$

and the first of the fifteen linear factors of the given circulant by

$$a_1 + a_2(\gamma_1\epsilon_1^{\frac{1}{3}}) + a_3(\gamma_1\epsilon_1^{\frac{1}{3}})^2 + a_4(\gamma_1\epsilon_1^{\frac{1}{3}})^3 + \ldots + a_{15}(\gamma_1\epsilon_1^{\frac{1}{3}})^{14}.$$

But this, by attending to the powers of $\gamma_1\epsilon_2^{\frac{1}{3}}$ and rearranging accordingly is equal to

$$(a_1+a_4\epsilon_1+a_7\epsilon_1^2+a_{10}\epsilon_1^3+a_{13}\epsilon_1^4) + (a_2+a_5\epsilon_1+\ldots+a_{14}\epsilon_1^4)\gamma_1\epsilon_1^{\frac{1}{3}} + (a_3+a_6\epsilon_1+\ldots+a_{15}\epsilon_1^4)\gamma_1^2\epsilon_1^{\frac{2}{3}},$$

and as the second and third factors differ from it merely in having γ_2, γ_3 respectively for γ_1, the product of the three is the circulant

$$\begin{vmatrix} a_1+a_4\epsilon_1+\ldots & (a_2+a_5\epsilon_1+\ldots)\epsilon_1^{\frac{1}{3}} & (a_3+a_6\epsilon_1+\ldots)\epsilon_1^{\frac{2}{3}} \\ (a_3+a_6\epsilon_1+\ldots)\epsilon_1^{\frac{2}{3}} & a_1+a_4\epsilon_1+\ldots & (a_2+a_5\epsilon_1+\ldots)\epsilon_1^{\frac{1}{3}} \\ (a_2+a_5\epsilon_1+\ldots)\epsilon_1^{\frac{1}{3}} & (a_3+a_6\epsilon_1+\ldots)\epsilon_1^{\frac{2}{3}} & a_1+a_4\epsilon_1+\ldots \end{vmatrix},$$

which, by multiplying the rows in order by $\epsilon_1^{\frac{1}{3}}$, $\epsilon_1^{\frac{2}{3}}$, $\epsilon_1^{\frac{3}{3}}$, and thereafter dividing the columns in order by the same, becomes

$$\begin{vmatrix} a_1+a_4\epsilon_1+\ldots & a_2+a_5\epsilon_1+\ldots & a_3+a_6\epsilon_1+\ldots \\ (a_3+a_6\epsilon_1+\ldots)\epsilon_1 & a_1+a_4\epsilon_1+\ldots & a_2+a_5\epsilon_1+\ldots \\ (a_2+a_5\epsilon_1+\ldots)\epsilon_1 & (a_3+a_6\epsilon_1+\ldots)\epsilon_1 & a_1+a_4\epsilon_1+\ldots \end{vmatrix}.$$

Performing the multiplications by ϵ_1, we see this to be equal to

$$\begin{vmatrix} a_1+a_4\epsilon_1+\ldots+a_{13}\epsilon_1^4 & a_2+a_5\epsilon_1+\ldots+a_{14}\epsilon_1^4 & a_3+a_6\epsilon_1+\ldots+a_{15}\epsilon_1^4 \\ a_{15}+a_3\epsilon_1+\ldots+a_{12}\epsilon_1^4 & a_1+a_4\epsilon_1+\ldots+a_{13}\epsilon_1^4 & a_2+a_5\epsilon_1+\ldots+a_{14}\epsilon_1^4 \\ a_{14}+a_2\epsilon_1+\ldots+a_{11}\epsilon_1^4 & a_{15}+a_3\epsilon_1+\ldots+a_{12}\epsilon_1^4 & a_1+a_4\epsilon_1+\ldots+a_{13}\epsilon_1^4 \end{vmatrix},$$

and therefore to be expressible as a sum of 5^3 determinants with monomial elements which can be grouped so as to take the form

$$P + Q\epsilon_1 + R\epsilon_1^2 + S\epsilon_1^3 + T\epsilon_1^4,$$

where each of the coefficients of the powers of ϵ_1 is a sum of 5^{3-1} such determinants. It is next observed that the fourth, fifth, sixth linear factors of the given circulant differ from the first, second, third merely in having ϵ_2 in place of ϵ_1: their product therefore is

$$P + Q\epsilon_2 + R\epsilon_2^2 + S\epsilon_2^3 + T\epsilon_2^4:$$

and hence finally the product of the five triads is equal to

$$C(P, Q, R, S, T),$$

as predicated. Here r is 3 and s is 5.

Had we written the first linear factor of the given circulant in the form

$$a_1 + a_2(\epsilon_1\gamma_1^{\frac{1}{5}}) + a_3(\epsilon_1\gamma_1^{\frac{1}{5}})^2 + \ldots + a_{15}(\epsilon_1\gamma_1^{\frac{1}{5}})^{14},$$

we should have changed it at the outset into

$$(a_1+a_6\gamma_1+a_{11}\gamma_1^2) + (a_2+a_7\gamma_1+a_{12}\gamma_1^2)\epsilon_1\gamma_1^{\frac{1}{5}} + \ldots + (a_5+a_{10}\gamma_1+a_{15}\gamma_1^2)\epsilon_1^4\gamma_1^{\frac{4}{5}},$$

and so have obtained for the product of the first five linear factors the circulant

$$C\{(a_1+a_6\gamma_1+a_{11}\gamma_1^2),\ (a_2+a_7\gamma_1+a_{12}\gamma_1^2)\gamma_1^{\frac{1}{5}}, \ldots, (a_5+a_{10}\gamma_1+a_{15}\gamma_1^2)\gamma_1^{\frac{4}{5}}\}.$$

This by multiplication of rows and division of columns we should have changed into

$$\begin{vmatrix} a_1+a_6\gamma_1+a_{11}\gamma_1^2 & a_2+a_7\gamma_1+a_{12}\gamma_1^2 & \ldots & a_5+a_{10}\gamma_1+a_{15}\gamma_1^2 \\ a_{15}+a_5\gamma_1+a_{10}\gamma_1^2 & a_1+a_6\gamma_1+a_{11}\gamma_1^2 & \ldots & a_4+a_9\gamma_1+a_{14}\gamma_1^2 \\ a_{14}+a_4\gamma_1+a_9\gamma_1^2 & a_{15}+a_5\gamma_1+a_{10}\gamma_1^2 & \ldots & a_3+a_8\gamma_1+a_{13}\gamma_1^2 \\ a_{13}+a_3\gamma_1+a_8\gamma_1^2 & a_{14}+a_4\gamma_1+a_9\gamma_1^2 & \ldots & a_2+a_7\gamma_1+a_{12}\gamma_1^2 \\ a_{12}+a_2\gamma_1+a_7\gamma_1^2 & a_{13}+a_3\gamma_1+a_8\gamma_1^2 & \ldots & a_1+a_6\gamma_1+a_{11}\gamma_1^2 \end{vmatrix},$$

from which we should have passed on to the form

$$X + Y\gamma_1 + Z\gamma_1^2.$$

Thereupon we should have seen the product of the next five linear factors to be

$$X + Y\gamma_2 + Z\gamma_2^2,$$

and the product of the last five to be

$$X + Y\gamma_3 + Z\gamma_3^2;$$

and therefore the product of the whole to be

$$C(X, Y, Z).$$

Here r is 5 and s is 3.

From this transformation of a circulant of composite order into one of lower order Torelli proceeds to deduce a circulant factor of the original. Thus, having reached C(P, Q, R, S, T) as an equivalent of the fifteen-line circulant, we know that

$$P + Q + R + S + T$$

must be a factor of it, and, looking back at the three-line determinant whence P, Q, . . . originated, we see that their sum equals

$$C(a_1+a_4+\ldots+a_{13},\ a_2+a_5+\ldots+a_{14},\ a_3+a_6+\ldots+a_{15}),$$

which is the factor desired. In like manner from the existence of the form C(X, Y, Z) there is obtained the factor

$$C(a_1+a_6+a_{11},\ a_2+a_7+a_{12},\ \ldots,\ a_5+a_{10}+a_{15}).$$

The same result is established by operating directly on the original circulant; and this procedure has the advantage of giving at the same time a convenient expression for the cofactor. Thus, by performing on $C(a_1, a_2, \ldots, a_{15})$ the operations

$$\begin{aligned} &\text{row}_1 + \text{row}_4 + \text{row}_7 + \text{row}_{10} + \text{row}_{13} \\ &\text{row}_2 + \text{row}_5 + \ldots \\ &\text{row}_3 + \text{row}_6 + \ldots, \end{aligned}$$

and writing

$$\xi,\ \eta,\ \zeta \quad \text{for} \quad a_1 + a_4 + \ldots,\ \ a_2 + a_5 + \ldots,\ \ a_3 + a_6 + \ldots,$$

we obtain

$$\begin{vmatrix} \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta \\ \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta \\ \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi & \eta & \zeta & \xi \\ a_{13} & a_{14} & a_{15} & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_1 \end{vmatrix}.$$

By column-subtraction twelve corresponding elements in the first three rows can then be made 0, and the determinant resolved into two; for example, if we choose the last twelve elements there is obtained

$$\mathrm{C}(\xi, \eta, \zeta) \cdot \begin{vmatrix} a_1 - a_{13} & a_2 - a_{14} & a_3 - a_{15} & a_4 - a_1 & \dots & a_{12} - a_9 \\ a_{15} - a_{12} & a_1 - a_{13} & a_2 - a_{14} & a_3 - a_{15} & \dots & a_{11} - a_8 \\ a_{14} - a_{11} & a_{15} - a_{12} & a_1 - a_{13} & a_2 - a_{14} & \dots & a_{10} - a_7 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_5 - a_2 & a_6 - a_3 & a_7 - a_4 & a_8 - a_5 & \dots & a_1 - a_{13} \end{vmatrix}.$$

Since 3 is prime to 5, the pair of circulant factors can themselves have no factor in common save $a_1+a_2+\dots+a_{15}$, and therefore

$$(a_1+a_2+\dots+a_{15})\,\mathrm{C}(a_1, a_2, \dots, a_{15})$$

must be divisible by the product of the said pair. As an expression for the resulting quotient, which must be of the eighth degree in the original elements, Torelli obtains a determinant of the eighth order. He also extends the reasoning to a circulant having its order-number resolvable into more than two mutually prime integers: and finally makes application (§§ 8, 9) of his results to generalize Stern's cofactor theorem of 1871 and a theorem of Kummer's of 1851 on complex numbers.

It will suffice to add that throughout his paper Torelli makes due reference to the corresponding results regarding *skew* circulants ("gobbi circolanti").

MUIR, T. (1882).

[A proposed general method for the solutions of equations. *Proceed. Philos. Soc. Glasgow*, xiii. p. 616.]

The property proposed to be utilized is the resolvability of $\mathrm{C}(x, a, b, c, \dots)$ into linear factors and the consequent solubility of the equation

$$\mathrm{C}(x, a, b, c, \dots) = 0.$$

The paper was withheld from the printer, as it was found that Cayley had in effect already applied the process to a specialized quintic in the *Quart. Journ. of Math.*, xviii. pp. 154–167. A. Legoux'

paper, ' Sur une application d'un déterminant,' which appeared later in 1882 (*Quart. Journ. of Math.*, xix. pp. 41–43) concerns the same proposal.*

BHUT, A. B. (1882).

[Question 7072. *Educ. Times*, xxxv. p. 155 : *Math. from Educ. Times*, xl. p. 48.]

The problem is to find x, y, z, u from the equations

$$\begin{vmatrix} x & y & z \\ u & x & y \\ z & u & x \end{vmatrix} = a^3, \quad \begin{vmatrix} y & z & u \\ x & y & z \\ u & x & y \end{vmatrix} = b^3, \quad \begin{vmatrix} z & u & x \\ y & z & u \\ x & y & z \end{vmatrix} = c^3, \quad \begin{vmatrix} u & x & y \\ z & u & x \\ y & z & u \end{vmatrix} = d^3,$$

and it is not observed that this is the same as to express x, y, z, u in terms of their complementary minors in $\mathrm{C}(x, y, z, u)$. Calling these when signed X, Y, Z, U, we have from the theorem regarding a minor of the adjugate

$$\begin{vmatrix} \mathrm{X} & \mathrm{Y} & \mathrm{Z} \\ \mathrm{U} & \mathrm{X} & \mathrm{Y} \\ \mathrm{Z} & \mathrm{U} & \mathrm{X} \end{vmatrix} = x \begin{vmatrix} x & y & z & u \\ u & x & y & z \\ z & u & x & y \\ y & z & u & x \end{vmatrix}^2,$$

and therefore

$$x = \begin{vmatrix} \mathrm{X} & \mathrm{Y} & \mathrm{Z} \\ \mathrm{U} & \mathrm{X} & \mathrm{Y} \\ \mathrm{Z} & \mathrm{U} & \mathrm{X} \end{vmatrix} \div \begin{vmatrix} \mathrm{X} & \mathrm{Y} & \mathrm{Z} & \mathrm{U} \\ \mathrm{U} & \mathrm{X} & \mathrm{Y} & \mathrm{Z} \\ \mathrm{Z} & \mathrm{U} & \mathrm{X} & \mathrm{Y} \\ \mathrm{Y} & \mathrm{Z} & \mathrm{U} & \mathrm{X} \end{vmatrix}^{\frac{3}{4}}.$$

The corresponding set of equations in the case of three unknowns is much older : one appearance was in a Cambridge ' tripos ' paper of 1860.

CESÀRO, E. (1883) : NEUBERG, J. (1883).

[Question 245. *Mathesis*, iii. pp. 118–119 ; vi. pp. 60–62.]

[Question 273. *Mathesis*, iii. p. 192 ; viii. p. 215 ; x. pp. 117–119.]

These concern special circulants which have already been fully dealt with.

* The still earlier paper noted above on p. 360 came to light at a much later date.

FORSYTH, A. R. (1884).

[On certain symmetric products involving prime roots of unity. *Messenger of Math.*, xiv. pp. 40–56.]

Forsyth begins in effect by equating two forms of the eliminant of

$$\left.\begin{aligned} x^n - p_1x^{n-1} + p_2x^{n-2} - \ldots &\equiv (x-a_1)(x-a_2)\ \ldots\ (x-a_n) = 0 \\ x^s - 1 &\equiv (x-\omega_1)(x-\omega_2)\ \ldots\ (x-\omega_s) = 0 \end{aligned}\right\},$$

obtaining naturally

$$\prod_{x=\omega_1}^{x=\omega_s} (x^n - p_1x^{n-1} + \ldots) = (-1)^{n(s+1)} \cdot (1-a_1{}^s)(1-a_2{}^s)\ \ldots\ (1-a_s{}^n),$$

and when $s = 3$

$$C(\sigma, \tau, \upsilon) = (1-a_1{}^3)(1-a_2{}^3)\ \ldots\ (1-a_n{}^3),$$

where

$$\begin{aligned} \sigma &= 1 - p_3 + p_6 - p_9 + \ldots \\ \tau &= -p_1 + p_4 - p_7 + p_{10} - \ldots \\ \upsilon &= p_2 - p_5 + p_8 - p_{11} + \ldots . \end{aligned}$$

Three paragraphs (pp. 43–46) are then devoted to the evaluation of circulants. The procedure adopted is essentially the same as Minozzi's of 1878. As illustrative examples of its effectiveness the five-line and seven-line circulants are taken, the result in the former case agreeing with Glaisher's, and in the latter being new.

MUIR, T. (1884).

[Note on the final expansion of circulants. *Messenger of Math.*, xiv. pp. 169–175.]

The mode of expansion referred to is that used by Minozzi and Forsyth, the main object being to correct, if necessary, the latter's expansion of $C(a_0, a_1, a_2, \ldots a_6)$; or, say, C(0, 1, 2, 3, 4, 5, 6). It is first pointed out that the procedure is considerably shortened by using an additional property, namely, that *when*

$$A a_0^{\alpha} a_1^{\beta} a_2^{\gamma} a_3^{\delta} a_4^{\epsilon} a_5^{\zeta} a_6^{\eta}$$

is a term of the expansion, then also is

$$A a_0^{\alpha} a_m^{\beta} a_n^{\gamma} a_p^{\delta} a_q^{\epsilon} a_r^{\zeta} a_s^{\eta}$$

a term, where m, n, p, q, r, s *are numbers less than* 7, *and such that*

$$m, 2m, 3m, 4m, 5m, 6m \equiv m, n, p, q, r, s \pmod 7.$$

Thus, having got the term $-27a_0^3a_1^2a_2a_3$ in the expansion of

$$C(0, 1, 2, 3, 4, 5, 6),$$

we operate on the suffixes 1, 2, 3 by multiplying in succession by the numbers 2, 3, 4, 5, 6 and casting out the sevens, the result being the new sets of suffixes 2, 4, 6 ; 3, 6, 2 ; 4, 1, 5 ; 5, 3, 1 ; 6, 5, 4 : and so finally the complete set of terms

$$-27a_0^3(a_1^2a_2a_3+a_2^2a_4a_6+a_3^2a_6a_2+a_4^2a_1a_5+a_5^2a_3a_1+a_6^2a_5a_4).$$

The " reason of the rule " is that

$$\begin{aligned} C(0, 1, 2, 3, 4, 5, 6) &= C(0, 2, 4, 6, 1, 3, 5), \\ &= C(0, 3, 6, 2, 5, 1, 4), \\ &= C(0, 4, 1, 5, 2, 6, 3), \\ &= C(0, 5, 3, 1, 6, 4, 2), \\ &= C(0, 6, 5, 4, 3, 2, 1) ; \end{aligned}$$

as we can show either by transposition of rows and columns, or by considering the circulant in its form as a product of seven linear factors and using the fact that, if ω be one of the imaginary seventh roots of unity, ω^2, ω^3, ω^4, ω^5, ω^6 are the others.

A caution is then given that *two terms having apparently the same form have not necessarily the same coefficient* ; and as an illustration it is shown that in Forsyth's expression half of the terms with the coefficient 35 should have the coefficient -14, and that the coefficient of his final term should be changed from -448 to -105, the whole expansion then being

$$\begin{aligned} &\mathring{\Sigma}a_0^7 \\ &-7\mathring{\Sigma}a_0^5(a_1a_6+a_2a_5+a_3a_4) \\ &+7\mathring{\Sigma}a_0^4(a_1^2a_5+a_2^2a_3+a_3^2a_1+a_4^2a_6+a_5^2a_4+a_6^2a_2) \\ &+14\mathring{\Sigma}a_0^4(a_1a_2a_4+a_3a_5a_6) \\ &-7\mathring{\Sigma}a_0(a_1^3a_6^3+a_2^3a_5^3+a_3^3a_4^3) \\ &-21\mathring{\Sigma}a_0^3(a_1^2a_2a_3+a_2^2a_4a_6+a_3^2a_6a_2+a_4^2a_1a_5+a_5^2a_3a_1+a_6^2a_5a_4) \\ &+14\mathring{\Sigma}a_0^3(a_1^2a_6^2+a_2^2a_5^2+a_3^2a_4^2) \\ &+7\mathring{\Sigma}a_0^3(a_1a_2a_5a_6+a_2a_4a_3a_5+a_3a_6a_1a_4) \\ &-7\mathring{\Sigma}a_0(a_1^2a_2^2a_4^2+a_3^2a_6^2a_5^2) \\ &+35\mathring{\Sigma}a_0^2(a_1^2a_3a_4a_5+a_2^2a_6a_1a_3+a_3^2a_2a_5a_1) \\ &-14\mathring{\Sigma}a_0^2(a_1^2a_2a_4a_6+a_2^2a_4a_1a_5+a_3^2a_6a_5a_4) \\ &-105a_0a_1a_2a_3a_4a_5a_6 ; \end{aligned}$$

or, with the help of the fresh property, in the still more condensed form

$$\sum_{\mu=0}^{\mu=6}\sum_{\nu=1}^{\nu=6}(a_\mu^7 - 7a_\mu^5 a_{\mu+\nu}a_{\mu+6\nu} + 7a_\mu^4 a_{\mu+\nu}^2 a_{\mu+5\nu} + \ldots),$$

where every suffix greater than 6 is to be made less than 7 by subtracting a multiple of 7 from it and duplicates are to be neglected.

An entirely different mode of saving labour in the computation rests on the fact that all the terms which contain an element in the first power can be got by evaluating a zero-axial determinant of the next lower order. For example, since every term of

$$C(a_0, a_1, a_2, a_3, a_4)$$

is of this kind except $\mathring{\Sigma}a_0^5$, we evaluate

$$5\begin{vmatrix} . & a_1 & a_2 & a_3 \\ a_4 & . & a_1 & a_2 \\ a_3 & a_4 & . & a_1 \\ a_2 & a_3 & a_4 & . \end{vmatrix},$$

and thus finding the cofactor of a_0 to be

$$-5(a_1^3a_2+a_2^3a_4+a_3^3a_1+a_4^3a_2) + 5(a_1^2a_4^2+a_2^2a_3^2) - 5a_1a_2a_3a_4,$$

we obtain the full expansion

$$\mathring{\Sigma}a_0^5 - 5\mathring{\Sigma}a_0^3(a_1a_4+a_2a_3) + 5\mathring{\Sigma}a_0^2(a_1^2a_3+a_2^2a_1) - 5a_0a_1a_2a_3a_4.$$

In the case of the seven-line continuant the saving is almost quite as great, since in addition to $\mathring{\Sigma}a_0^7$ the term $14\mathring{\Sigma}a_0^3a_1^2a_6^2$ is the only one that does not involve a first power of an element.

STUDNIČKA, F. J. (1884).

[O řešení rovnic třetího a čtortého stupně pomocí determinantů cyklických. *Casopis pro pěstování math. a fys.*, xiii. pp. 225–233.]

The subject is the same as that of Muir's paper of 1882, the cubic and quartic being taken as examples.

WARD, P. C. (1885): MUIR, T. (1886).

(See under this heading in Chap. XIII.)

MUIR, T. (1885).

[Detached theorems on circulants. *Trans. Roy. Soc. Edin.*, xxxii. pp. 639–643.]

The first of the theorems is that *if in a circulant the element in the place* (p, q) *be by the nature of the circulant the same as the element in the place* (r, s), *the complementary minor of the former element is to that of the latter as* $(-1)^{r+s}:(-1)^{p+q}$. But more than this is intended to be brought out, namely, that by cyclical transposition of rows and columns the element in the place (r, s) may be made to take the place (p, q) and the determinant be in outward form exactly the same as before. The second theorem is Stern's of 1871, although not so credited. The mode of establishing it is to alter the first column of $C(a, b, c, d, e)$ so as to make it possible to remove $a+b\omega+c\omega^2+d\omega^3+e\omega^4$ as a factor, and leave

$$1,\ \omega,\ \omega^2,\ \omega^3,\ \omega^4$$

for the elements of the column. The third theorem is that *if the linear factors of a circulant, say of the fifth order, be* $\alpha, \beta, \gamma, \ldots$, *and the complementary minors of the elements of the first row be* A, −B, C, − . . . , *then*

$$C(\alpha\beta\gamma\delta,\ \alpha\beta\gamma\epsilon,\ \alpha\beta\delta\epsilon,\ \alpha\gamma\delta\epsilon,\ \beta\gamma\delta\epsilon) = 5^5\text{ABCDE}.$$

This is proved by using the preceding theorem five times over in the form

$$\begin{aligned} A + \omega E + \omega^2 D + \omega^3 C + \omega^4 B &= \beta\gamma\delta\epsilon, \\ A + \omega^2 E + \omega^4 D + \omega C + \omega^3 B &= \gamma\delta\epsilon\alpha, \\ \cdots\cdots\cdots\cdots \end{aligned}$$

and thence obtaining the linear factors of

$$C(\alpha\beta\gamma\delta,\ \alpha\beta\gamma\epsilon,\ \alpha\beta\delta\epsilon,\ \alpha\gamma\delta\epsilon,\ \beta\gamma\delta\epsilon)$$

in the form

$$5A,\quad 5B,\quad 5C,\quad 5D,\quad 5E.$$

The fourth theorem is that *if the elements of the first row of a circulant of the* n^{th} *order be multiplied by* $\omega^n, \omega^{n-1}, \ldots, \omega$ *respectively, the elements of the second row by* $\omega^{n-1}, \omega^{n-2}, \ldots, \omega, \omega^n$ *respectively, the elements of the third row by* $\omega^{n-2}, \omega^{n-3}, \ldots, \omega, \omega^n, \omega^{n-1}$ *respectively, and so on, where* ω *is any* n^{th} *root of unity, the circulant is unaltered*

in value. Here for proof we have only to multiply the rows in order by ω^0, ω^1, ω^2, ω^3, ω^4, and then divide the columns by ω^5, ω^4, ω^3, ω^2, ω. The fifth theorem concerns skew circulants, being the analogue of the same author's theorem of 1881 regarding ordinary circulants.

In order to throw light on the law of the coefficients of the final expansion of $C(a, b, c, d, e)$, the determinant

$$\begin{vmatrix} a\alpha & b\beta & c\gamma & d\delta & e\epsilon \\ e\beta & a\gamma & b\delta & c\epsilon & d\alpha \\ d\gamma & e\delta & a\epsilon & b\alpha & c\beta \\ c\delta & d\epsilon & e\alpha & a\beta & b\gamma \\ b\epsilon & c\alpha & d\beta & e\gamma & a\delta \end{vmatrix}$$

is framed by attaching to each element of $C(a, b, c, d, e)$ the corresponding element of $(-1)^{3+2+1}C(\alpha, \beta, \gamma, \delta, \epsilon)$. This, which degenerates into the ordinary five-line circulant on putting

$$a = b = c = d = e = 1 \quad \text{or} \quad \alpha = \beta = \gamma = \delta = \epsilon = 1,$$

is found to be equal to

$$\begin{aligned} &\mathring{\Sigma}a^5 \cdot a\beta\gamma\delta\epsilon - \mathring{\Sigma}a^3be \cdot \mathring{\Sigma}a^3\gamma\delta + \mathring{\Sigma}a^2b^2d \cdot \mathring{\Sigma}a^2\beta\gamma^2 \\ &+\mathring{\Sigma}a^5 \cdot abcde - \mathring{\Sigma}a^3\beta\epsilon \cdot \mathring{\Sigma}a^3cd + \mathring{\Sigma}a^2\beta^2\delta \cdot \mathring{\Sigma}a^2bc^2 \end{aligned} - 10abcde \cdot a\beta\gamma\delta\epsilon,$$

where all the coefficients of the ordinary circulant except the last are in a manner "sifted" into their unit constituents.

SCHRADER, W. (1887): SPORER, B. (1887).

[Beitrage zur Theorie der Determinanten. vi+156 pp. Halle-a-S.]

[Einiges über gewisse Determinanten. *Math.-naturw. Mitt.* (Tübingen), ii. pp. 106–107.]

In the section on circulants (pp. 87–94) Schrader gives one fresh evaluation, namely,

$$C(1, 2a, 3a^2, \ldots, na^{n-1}) = \frac{\{1-(n+1)a^n\}^n - (-na^{n-1})^n}{(1-a^n)^2},$$

where we may note that the evident factor on the left,

$$1 + 2a + 3a^2 + \ldots + na^{n-1},$$

and the evident factor on the right,

$$\frac{na^{n+1} - (n+1)a^n + 1}{(a-1)^2},$$

are equivalents.

It is the well-known case of this where a is 1 that the other writer deals with.

ROBINSON, L. W. (1889): THYAGARAGAIYAR, V. R. (1899).

[Question 10254. *Educ. Times*, xlii. p. 332: *Math. from Educ. Times*, liv. p. 54.]

[Question 14240. *Educ. Times*, lii. p. 270: *Math. from Educ. Times*, lxxiii. p. 56.]

Already known results.

ROGERS, L. J. (1891).

[Question 10992. *Educ. Times*, xliv. p. 199.]

This concerns the first primary minor of $C(a, b, c, \ldots)$ when $a+b+c+\ldots = 0$, the problem being to resolve the said minor into linear factors. No solution ever appeared. Probably the proposer had in his mind special circulants of the form

$$C(a-b, b-c, c-d, d-a),$$

the first primary minor of which is

$$\begin{vmatrix} a-b & b-c & c-d \\ d-a & a-b & b-c \\ c-d & d-a & a-b \end{vmatrix} \quad \text{and } \therefore \quad = \begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ 1 & 1 & 1 & 1 \end{vmatrix},$$

so that its factors are factors of $C(a, b, c, d)$.

RAVUT, . (1894).

[Résolution des équations des deuxième, troisième, et quatrième degrés en prenant pour point de départ l'équation identique de Cayley sur les matrices. *Assoc. Franç.* . . . (Caen), xxiii. pp. 285–294.]

The method reached from the starting-point mentioned is essentially that referred to above under Muir (1882).

WOODALL, H. J. (1894).

[Question 12426. *Educ. Times*, xlvii. p. 305 : *Math. from Educ. Times*, lxiii. pp. 35–36.]

The real point of interest here concerns the product of two circulants. If the circulants be zero-axial, the ordinary multiplication-theorem, though of course producing a circulant, does not produce one that is zero-axial ; so that, strictly speaking, the product is in this case not of the same form as its factors. It is shown, however, that what is not true of the zero-axial circulant is true of the zero-axial circulant divided by the sum of its elements. Thus, by row-multiplication

$$\begin{vmatrix} . & a & b \\ b & . & a \\ a & b & . \end{vmatrix} \cdot \begin{vmatrix} . & x & y \\ y & . & x \\ x & y & . \end{vmatrix} = \begin{vmatrix} ax+by & bx & ay \\ ay & ax+by & bx \\ bx & ay & ax+by \end{vmatrix},$$

and consequently, on removing the factors $a+b$, $x+y$, $(a+b)(x+y)$ from the three determinants respectively, we have

$$\begin{vmatrix} 1 & 1 & 1 \\ b & . & a \\ a & b & . \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ y & . & x \\ x & y & . \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ ay & ax+by & bx \\ bx & ay & ax+by \end{vmatrix},$$

where, by diminishing in the determinant on the right the second and third rows by $ax+by$ times the first, there results

$$\begin{vmatrix} 1 & 1 & 1 \\ ay-ax-by & . & bx-ax-by \\ bx-ax-by & ay-ax-by & . \end{vmatrix}.$$

In other words,

$$(a^2-ab+b^2)(x^2-xy+y^2) = \mathrm{P}^2 - \mathrm{PQ} + \mathrm{Q}^2,$$

if P, Q be taken equal to $bx-ax-by$, $ay-ax-by$ respectively.

MUIR, T. (1896).

[On the resolution of circulants into rational factors. *Proc. Roy. Soc. Edin.*, xxi. pp. 369–382.]

Viewing the circulant $\mathrm{C}(a_1, a_2, \ldots, a_n)$ as the eliminant of the

equations

$$\left.\begin{aligned} a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n &= 0 \\ x^n - 1 &= 0 \end{aligned}\right\},$$

the author concludes that *corresponding to every rational factor of* x^n-1 *there must be a rational factor of the circulant*; and his object is to determine such factors and to present them, when found, in the most convenient forms. The means employed for the purpose is dialytic elimination.

Thus, to begin with, Catalan's factor corresponding to

$$(x^n-1)/(x-1)$$

is obtained by deducing from the equations

$$\left.\begin{aligned} ax^4+bx^3+cx^2+dx+e &= 0 \\ x^4+x^3+x^2+x+1 &= 0 \end{aligned}\right\}$$

the cubic

$$(b-a)\,x^3 + (c-a)\,x^2 + (d-a)\,x + (e-a) = 0,$$

thence by cyclical substitution three other equations, and finally

$$\begin{vmatrix} b-a & c-a & d-a & e-a \\ c-b & d-b & e-b & a-b \\ d-c & e-c & a-c & b-c \\ e-d & a-d & b-d & c-d \end{vmatrix},$$

which being inherently persymmetric* is more conveniently written

$$\mathrm{P}(b-c,\ c-d,\ d-e,\ e-a,\ a-b,\ b-c,\ c-d).$$

The process is then extended to find the factor corresponding to

$$(x^{2m}-1)/(x^2-1);$$

in other words, to find the cofactor of

$$(a_1+a_2+\ldots+a_{2m})(a_1-a_2+a_3-\ldots-a_{2m}) \text{ in } \mathrm{C}(a_1, a_2, \ldots, a_{2m}).$$

Thus, in the case where $2m = 8$, from the equation

$$ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h = 0$$

the seventh and sixth powers of x are removed with the help of the other equation

$$x^6 + x^4 + x^2 + 1 = 0,$$

---

* The operations required for the transformation are

$$\mathrm{col}_1 - \mathrm{col}_2, \quad \mathrm{col}_2 - \mathrm{col}_3, \quad \mathrm{col}_3 - \mathrm{col}_4;$$

and in the case of the determinant on next page

$$\mathrm{col}_1 - \mathrm{col}_3, \quad \mathrm{col}_2 - \mathrm{col}_4, \quad \mathrm{col}_3 - \mathrm{col}_5, \quad \mathrm{col}_4 - \mathrm{col}_6.$$

the result being

$$(c-a)x^5+(d-b)x^4+(e-a)x^3+(f-b)x^2+(g-a)x+(h-b) = 0,$$

whence by cyclical substitution and elimination we obtain

$$\begin{vmatrix} c-a & d-b & e-a & f-b & g-a & h-b \\ d-b & e-c & f-b & g-c & h-b & a-c \\ e-c & f-d & g-c & h-d & a-c & b-d \\ f-d & g-e & h-d & a-e & b-d & c-e \\ g-e & h-f & a-e & b-f & c-e & d-f \\ h-f & a-g & b-f & c-g & d-f & e-g \end{vmatrix},$$

which, again, can be expressed persymmetrically, namely,

$$\mathrm{P}(c-e,\ d-f,\ e-g,\ f-h,\ g-a,\ h-b,\ a-c,\ b-d,\ c-e,\ d-f,\ e-g).$$

In the next place, the factor of $\mathrm{C}(a_1, a_2, a_3, a_4, a_5, a_6)$ corresponding to the factor x^2+x+1 of x^6-1 is obtained. We simply, as before, remove in order from the equation

$$a_1x^5+a_2x^4+a_3x^3+a_4x^2+a_5x+a_6 = 0,$$

with the help of the equation

$$x^2+x+1 = 0,$$

the fifth, fourth, third, second powers of x, with the result

$$(a_5-a_4+a_2-a_1)x+(a_6-a_4+a_3-a_1) = 0,$$

whence the eliminant

$$\begin{vmatrix} a_5-a_4+a_2-a_1 & a_6-a_4+a_3-a_1 \\ a_6-a_5+a_3-a_2 & a_1-a_5+a_4-a_2 \end{vmatrix},$$

which, as before, can be expressed persymmetrically.

Proceeding in this way, the prime factors of the circulants up to and including that of the tenth order are calculated, and are tabulated for reference. By reason of the persymmetry and the notation for it, the table occupies only one page ; and it is pointed out how it might have been made still shorter by using the fact that when once the first element of any of the persymmetric determinants is obtained, the others follow at once in cyclical procession. For example, if we desire to have the factor of $\mathrm{C}(a_1, a_2, \ldots, a_{10})$ corresponding to the factor $x^4+x^3+x^2+x+1$ of $x^{10}-1$, and know from the table that the first element of it is $a_7-a_8+a_2-a_3$, we have the whole of it immediately, namely,

$$\begin{vmatrix} a_7-a_8+a_2-a_3 & a_8-a_9+a_3-a_4 & a_9-a_{10}+a_4-a_5 & a_{10}-a_1+a_5-a_6 \\ a_8-a_9+a_3-a_4 & a_9-a_{10}+a_4-a_5 & a_{10}-a_1+a_5-a_6 & a_1-a_2+a_6-a_7 \\ a_9-a_{10}+a_4-a_5 & a_{10}-a_1+a_5-a_6 & a_1-a_2+a_6-a_7 & a_2-a_3+a_7-a_8 \\ a_{10}-a_1+a_5-a_6 & a_1-a_2+a_6-a_7 & a_2-a_3+a_7-a_8 & a_3-a_4+a_8-a_9 \end{vmatrix}.$$

The process is seen to be general, and to consist in deducing from the equation of higher degree, by means of the other equation, an equation of lower degree than the latter, and in then using cyclical substitution and elimination.

The next part of the paper, however, makes it clear that the same results can be obtained, when known, by operating on the circulants themselves in accordance with the laws of determinants. Thus, from the eight-line circulant it is shown how to remove the second linear factor after the first, and how then to resolve the remaining six-line determinant into the two previously found per-symmetric determinants of the second and fourth orders respectively.

Finally, the results are applied to the special case of circulants whose elements, when the first is left aside, read forward the same as backward : and it is shown that, when from such circulants the rational linear factor or factors are removed, the remaining factor is a complete square. This is possible from the fact that the per-symmetric cofactors above obtained are then zero-axial and skew. Thus it is found that

$$C(a_1, a_2, a_3, a_4, a_4, a_3, a_2)$$

$$= (a_1+2a_2+2a_3+2a_4) \begin{vmatrix} a_3-a_4 & a_2-a_3 & a_1-a_2 & a_2-a_1 & a_3-a_2 \\ & a_3-a_4 & a_2-a_3 & a_1-a_2 & a_2-a_1 \\ & & a_3-a_4 & a_2-a_3 & a_1-a_2 \\ & & & a_3-a_4 & a_2-a_3 \\ & & & & a_3-a_4 \end{vmatrix}^2$$

and that

$$C(a_1, a_2, a_3, a_4, a_5, a_4, a_3, a_2)$$

$$= (a_1+2a_2+2a_3+2a_4+a_5)(a_1-2a_2+2a_3-2a_4+a_5)$$

$$\cdot \begin{vmatrix} a_3-a_1 & a_4-a_2 & a_5-a_3 & \cdot & a_3-a_5 \\ & a_3-a_1 & a_4-a_2 & a_5-a_3 & \cdot \\ & & a_3-a_1 & a_4-a_2 & a_5-a_3 \\ & & & a_3-a_1 & a_4-a_2 \\ & & & & a_3-a_1 \end{vmatrix}^2.$$

Further, it is shown that such circulants being centrosymmetric can be resolved into factors in quite a different way, the repeated factor now appearing not as a Pfaffian but as a determinant; so that by the equatement of the two resolutions we obtain equality between the two determinants

$$\begin{vmatrix} a_1-a_3 & a_2-a_4 & a_3-a_4 \\ a_2-a_4 & a_1-a_4 & a_2-a_3 \\ a_3-a_4 & a_2-a_3 & a_1-a_2 \end{vmatrix}, \qquad \begin{vmatrix} a_1-a_3 & a_2-a_4 & a_3-a_5 \\ a_2-a_4 & a_1-a_5 & a_2-a_4 \\ a_3-a_5 & a_2-a_4 & a_1-a_3 \end{vmatrix}$$

and the above-given Pfaffians respectively.

ANDRÉ, D. (1896).

[Théorème nouveau de reversibilité algébrique. *Bull. Soc. Math. de France*, xxiv. pp. 136–139: *Intermédiaire des Math.*, iii. pp. 214–217.]

The theorem here accurately stated and carefully established may very conveniently be put in much smaller compass, namely, *If any two circulants*, $C(x_1, x_2, \ldots, x_n)$, $C(y_1, y_2, \ldots, y_n)$ *be so related that*

$$\frac{X_1}{y_1} = \frac{X_2}{y_2} = \ldots = \frac{X_n}{y_n},$$

then also

$$\frac{x_1}{Y_1} = \frac{x_2}{Y_2} = \ldots = \frac{x_n}{Y_n}.$$

To prove it we have only to denote the common ratio of the first set by $1/r$, insert rX_1, rX_2, ... for y_1, y_2, ... in $C(y_1, y_2, \ldots, y_n)$, and use Jacobi's theorem regarding a primary minor of the adjugate.

BICKMORE, C. E. (1898).

[Question 13748. *Educ. Times*, li. p. 40: *Math. from Educ. Times*, lxix. p. 85.]

The result here recorded is that the skew circulant of the fourth order, $C'(x, y, z, w)$, whose final expansion is

$$x^4 + y^4 + z^4 + w^4 + 2(x^2z^2 + y^2w^2) + 4(x^2yw - y^2zx - z^2wy + w^2xz),$$

is expressible in each of the forms

$$a^2+b^2, \quad c^2+2d^2, \quad e^2-2f^2,$$

where a, b, c, d, e, f are definite quadratic functions of x, y, z, w, namely,

$$x^2+2yw-z^2, \quad y^2-2zx-w^2,$$
$$x^2-y^2+z^2-w^2, \quad xy-yz+zw+wx,$$
$$x^2+y^2+z^2+w^2, \quad xy+yz+zw-wx.$$

This is reached by taking the four linear factors

$$x+\xi y+\xi^2 z+\xi^3 w, \quad x+\xi^3 y-\xi^2 z+\xi w,$$
$$x-\xi y+\xi^2 z-\xi^3 w, \quad x-\xi^3 y-\xi^2 z-\xi w$$

of the circulant and multiplying them in the three different ways as a pair of pairs. For example, the product of the first and second factors being

$$x^2-y^2+z^2-w^2+(\xi+\xi^3)(xy-yz+zw+wx),$$

and the product of the third and fourth being

$$x^2-y^2+z^2-w^2-(\xi+\xi^3)(xy-yz+zw+wx),$$

the product of the four is got in the form

$$(x^2-y^2+z^2-w^2)^2+2(xy-yz+zw+wx)^2.$$

We note for ourselves that by the multiplication-theorem we obtain

$$\begin{vmatrix} x & y & z & w \\ -w & x & y & z \\ -z & -w & x & y \\ -y & -z & -w & x \end{vmatrix}^2 = \begin{vmatrix} e & f & . & -f \\ f & e & f & . \\ . & f & e & f \\ -f & . & f & e \end{vmatrix}$$

$$= \begin{vmatrix} e & f & . & . \\ 2f & e & . & . \\ . & f & e & 2f \\ -f & . & f & e \end{vmatrix} = (e^2-2f^2)^2;$$

also

$$\begin{vmatrix} x & y & z & w \\ -w & x & y & z \\ -z & -w & x & y \\ -y & -z & -w & x \end{vmatrix} \cdot \begin{vmatrix} x & -y & z & -w \\ -w & -x & y & -z \\ -z & w & x & -y \\ -y & z & -w & -x \end{vmatrix} = \begin{vmatrix} c & -d & . & -d \\ -d & -c & d & . \\ . & d & c & -d \\ -d & . & -d & -c \end{vmatrix},$$

i.e.

$$C'(x, y, z, w) \cdot (-1)^2 C'(x, y, z, w) = (c^2+2d^2)^2 ;$$

and

$$\begin{vmatrix} x & y & z & w \\ -w & x & y & z \\ -z & -w & x & y \\ -y & -z & -w & x \end{vmatrix} \cdot \begin{vmatrix} x & w & -z & y \\ -w & z & -y & x \\ -z & y & -x & -w \\ -y & x & w & -z \end{vmatrix} = \begin{vmatrix} a & . & b & . \\ . & -b & . & a \\ b & . & -a & . \\ . & a & . & b \end{vmatrix},$$

i.e.

$$C'(x, y, z, w) \cdot (-1)^2 C'(x, y, z, w) = (a^2+b^2)^2.$$

FONTENÉ, G. (1898).

[Sur un système remarquable de n relations entre deux systèmes de n quantités. *Nouv. Annales de Math.*, (3) xvii. pp. 317–328.]

The interest here lies in the fact that a set of homogeneous linear equations in $x_1, x_2, \ldots, x_n$ whose determinant is $C(a_1, a_2, \ldots, a_n)$ is also viewable as a set in $a_1, a_2, \ldots, a_n$ whose determinant is $C(x_1, x_2, \ldots, x_n)$. The close and full study made of the set does not, however, lead to the evolution of any fresh property of circulants.

STUDNIČKA, F. J. (1899): NEUBERG, J. (1899).

[Beitrag zur Theorie der cyclischen Determinanten. *Monatshefte für Math. u. Phys.*, x. pp. 193–197.]

[Question 1241. *Mathesis*, (2) ix. p. 240; x. pp. 100–101.]

The results of the first paper here are Roberts' of 1859 and Baehr's of 1860; that of the second is Painvin's of 1858.

Studnička also evaluates the skew circulant which is twin to Baehr's, and thus arrives at a case of Scott's theorem of 1880 regarding any odd-ordered skew circulant. His paper seems to be a second edition of the first part of one published in Czech in 1898 (*Věstník České Akad.* . . . , vii. pp. 477–493).* In a miscellaneous paper in the latter serial for 1899 he gives (pp. 199–201) the theorem that *the circulant whose elements are the coefficients of* $(a+b)^n$ *is* 2^n *or* 0 *according as* n *is odd or even.*

* The title is "Nové poučky o některých determinantech zvláštních."

BLOCK CIRCULANTS, FROM 1881 TO 1898.

MUIR, T. (1881).

[On new and recently discovered properties of certain symmetric determinants. *Quart. Journ. of Math.*, xviii. pp. 166–177.]

Fresh light is here (§ 6) thrown on Puchta's determinant of 1877 by pointing out that it is centrosymmetric and breaks up into two determinants that are centrosymmetric, that each of these breaks up in the same way, and so on: the linear factors of the determinant are thus got with the greatest readiness. Further, it is shown that the same determinant includes as a special case Brioschi's of 1853 (*Hist.*, ii. pp. 133–136).

To be noted also is the reference (§ 9) to the determinant

$$\begin{vmatrix} a & b & c & f & e & d \\ c & a & b & d & f & e \\ b & c & a & e & d & f \\ f & e & d & a & b & c \\ d & f & e & c & a & b \\ e & d & f & b & c & a \end{vmatrix},$$

which may be viewed as formed by cyclically permuting the arrays of the circulants $C(a, b, c)$, $C(f, e, d)$. On reversing the order of the last three rows and the last three columns, we find it to be centrosymmetric and resolvable into

$$C(a+f, b+e, c+d) \cdot C(a-f, b-e, c-d).$$

PUCHTA, A. (1881).

[Ein neuer Satz aus der Theorie der Determinanten. *Denkschr. . . . Akad. d. Wiss.* (Wien), xliv. pp. 277–282.]

Puchta here extends his own work of 1877 (*Hist.*, iii. pp. 388–389). He is uninfluenced by and apparently quite unaware of what Noether had done in 1879, proceeding leisurely as before from one case of his so-called new theorem to another, and doing this without any reference to elimination. The theorem itself he leaves unformulated, and indeed insufficiently described. In all probability, however,

what he has in his mind is identical with the generalization given in Noether's second paper.

His first example is a nine-line determinant stated to be in illustration of the case where the order-number is the square of a prime. The elements composing every row being

$$a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,$$

he begins by constructing the circulant arrays

$$\begin{matrix} a & b & c \\ c & a & b \\ b & c & a, \end{matrix} \qquad \begin{matrix} d & e & f \\ f & d & e \\ e & f & d, \end{matrix} \qquad \begin{matrix} g & h & i \\ i & g & h \\ h & i & g; \end{matrix}$$

he then cyclically permutes these, and finally brings the whole nine arrays together so as to form a single determinant, the result being

$$\begin{vmatrix} a & b & c & d & e & f & g & h & i \\ c & a & b & f & d & e & i & g & h \\ b & c & a & e & f & d & h & i & g \\ g & h & i & a & b & c & d & e & f \\ i & g & h & c & a & b & f & d & e \\ h & i & g & b & c & a & e & f & d \\ d & e & f & g & h & i & a & b & c \\ f & d & e & i & g & h & c & a & b \\ e & f & d & h & i & g & b & c & a \end{vmatrix}.$$

We may for convenience call it a *block circulant*, and denote it by

$$\mathrm{C}(abc,\ def,\ ghi).$$

The linear factors of it Puchta obtains by taking the coefficients of a, b, c in the linear factors of $\mathrm{C}(a, b, c)$, namely,

$$\begin{matrix} 1 & 1 & 1 \\ 1 & \gamma & \gamma^2 \\ 1 & \gamma^2 & \gamma, \end{matrix}$$

(where of course γ is a primitive third-root of unity), and multiplying each element of this array, L say, by the array itself, thus reaching

$$\begin{matrix} \mathrm{L} & \mathrm{L} & \mathrm{L} \\ \mathrm{L} & \gamma\mathrm{L} & \gamma^2\mathrm{L} \\ \mathrm{L} & \gamma^2\mathrm{L} & \gamma\mathrm{L}. \end{matrix}$$

This is understood to stand for the nine-by-nine array

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \gamma & \gamma^2 & 1 & \gamma & \gamma^2 & 1 & \gamma & \gamma^2 \\
1 & \gamma^2 & \gamma & 1 & \gamma^2 & \gamma & 1 & \gamma^2 & \gamma \\
1 & 1 & 1 & \gamma & \gamma & \gamma & \gamma^2 & \gamma^2 & \gamma^2 \\
1 & \gamma & \gamma^2 & \gamma & \gamma^2 & 1 & \gamma^2 & 1 & \gamma \\
1 & \gamma^2 & \gamma & \gamma & 1 & \gamma^2 & \gamma^2 & \gamma & 1 \\
1 & 1 & 1 & \gamma^2 & \gamma^2 & \gamma^2 & \gamma & \gamma & \gamma \\
1 & \gamma & \gamma^2 & \gamma^2 & 1 & \gamma & \gamma & \gamma^2 & 1 \\
1 & \gamma^2 & \gamma & \gamma^2 & \gamma & 1 & \gamma & 1 & \gamma^2
\end{array}$$

each row of which when multiplied by

$$a \quad b \quad c \quad d \quad e \quad f \quad g \quad h \quad i$$

gives one of the linear factors desired.

His next step is to the case where the number of triads of elements is *five*: in other words, to the block circulant

$$\mathrm{C}(abc,\ def,\ ghi,\ jkl,\ mno).$$

Here the only difference is that L is used to multiply each element of

$$\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
1 & \epsilon & \epsilon^2 & \epsilon^3 & \epsilon^4 \\
1 & \epsilon^2 & \epsilon^4 & \epsilon & \epsilon^3 \\
1 & \epsilon^3 & \epsilon & \epsilon^4 & \epsilon^2 \\
1 & \epsilon^4 & \epsilon^3 & \epsilon^2 & \epsilon,
\end{array}$$

—the array which corresponds to L in the case of the five-line circulant. In connection herewith there is noted as a result of transposition of columns the interesting property

$$\mathrm{C}(abc,\ def,\ ghi,\ jkl,\ mno)$$
$$= \mathrm{C}(adgjm,\ behkn,\ cfilo).$$

In the third place he states that the order-number of the determinant may be even, the example taken being

$$\mathrm{C}(abc,\ def),$$

where to obtain the factors, L is used to multiply each element of the array

$$\begin{array}{cc}
1 & 1 \\
1 & -1.
\end{array}$$

The mode of proof relied on throughout is verificatory, any particular linear function of $a, b, c, \ldots$ obtained in accordance with directions being shown to be a factor of the determinant by increasing the fresh column by suitably chosen multiples of the other columns.

The paper closes with the remark that it is easy to see how the property dealt with may be extended to determinants of order $m^\alpha n^\beta p^\gamma \ldots$ having $m^\alpha n^\beta p^\gamma \ldots$ linear factors.

For ourselves we note that Noether's is just such an extension as is here referred to, and that Puchta's three cases are in our notation (*Hist.*, iii. p. 392) the eliminants of

$$\left.\begin{aligned}((1, x, x^2)(1, y, y^2)\,|\,a, b, c, d, e, f, g, h, i) &= 0\\ x^3 = y^3 &= 1\end{aligned}\right\},$$

$$\left.\begin{aligned}((1, x, x^2)(1, y, y^2, y^3, y^4)\,|\,a, b, c, \ldots, m, n, o) &= 0\\ x^3 = y^5 &= 1\end{aligned}\right\},$$

$$\left.\begin{aligned}((1, x, x^2)(1, y)\,|\,a, b, c, d, e, f) &= 0\\ x^3 = y^2 &= 1\end{aligned}\right\}.$$

Further, it is necessary for us to give a warning that the arrays requiring to be permuted towards the formation of Puchta's determinants are not always to be permuted cyclically. For example, besides the 'block circulant' of

$$\begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ b & a & d & c & f & e & h & g \end{array}$$

there is included in the generalization Puchta's eight-line determinant of 1877, which is composed of the same arrays differently permuted. The former is the eliminant of

$$\left.\begin{aligned}((1, x)(1, y, y^2, y^3)\,|\,a, b, c, d, e, f, g, h) &= 0\\ x^2 = y^4 &= 1\end{aligned}\right\}$$

and has four of its linear factors involving $\sqrt{-1}$: whereas the latter is the eliminant of

$$\left.\begin{aligned}((1, x)\,(1, y)\,(1, z)\,|\,a, b, c, d, e, f, g, h) &= 0\\ x^2 = y^2 = z^2 &= 1\end{aligned}\right\}$$

and has all its factors rational.

HAMMOND, J. (1882).

[Solution of question 5244. *Math. from Educ. Times*, xxxvii. p. 75.]

The point worth noting here is that if A, B, C, D be the elements of the adjugate of

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}, \quad \text{or } \Delta \text{ say,}$$

then

$$\begin{aligned} \mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D} &= (a+b-c-d)(a-b+c-d)(a-b-c+d) \\ &= \Delta \div (a+b+c+d), \\ \mathrm{A}+\mathrm{B}-\mathrm{C}-\mathrm{D} &= \Delta \div (a+b-c-d), \\ \mathrm{A}-\mathrm{B}+\mathrm{C}-\mathrm{D} &= \Delta \div (a-b+c-d), \\ \mathrm{A}-\mathrm{B}-\mathrm{C}+\mathrm{D} &= \Delta \div (a-b-c+d). \end{aligned}$$

HAMMOND, J. (1882).

[Question 7114. *Educ. Times*, xxxv. p. 202: *Math. from Educ. Times*, xxxviii. pp. 98–99.]

This concerns the determinant

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$$

originally spoken of as 'biaxisymmetric,' but better viewed as *the circulant of two two-line circulant arrays.* The full result obtained is that *if* p, q, r, s *be the linear factors of the determinant, and* P, Q, R, S *the corresponding factors of its adjugate, then*

$$\mathrm{P} = \mathrm{qrs}, \quad \mathrm{Q} = \mathrm{prs}, \quad \mathrm{R} = \mathrm{pqs}, \quad \mathrm{S} = \mathrm{pqr},$$

and $$\mathrm{p}^3 = \frac{\mathrm{QRS}}{\mathrm{P}^2}, \quad \mathrm{q}^3 = \ldots\ldots$$

These are made to rest ultimately on the fact that

$$\left.\begin{aligned} a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+d\mathrm{D} &= pqrs \\ b\mathrm{A}+a\mathrm{B}+d\mathrm{C}+c\mathrm{D} &= 0 \\ c\mathrm{A}+d\mathrm{B}+a\mathrm{C}+b\mathrm{D} &= 0 \\ d\mathrm{A}+c\mathrm{B}+b\mathrm{C}+a\mathrm{D} &= 0 \end{aligned}\right\};$$

for it is thereby shown that on the multiplication of any one of p, q, r, s by the corresponding one of P, Q, R, S, both being in their quadrinomial form, the product reduces to $a\mathrm{A}+b\mathrm{B}+c\mathrm{C}+d\mathrm{D}$.

PAIGE, C. LE (1884).

[Sur l'équation du quatrième degré. *Časopis pro pěstování math. a fys.*, xiv. pp. 26–28.]

The solubility of

$$\begin{vmatrix} x & a & b & c \\ a & x & c & b \\ b & c & x & a \\ c & b & a & x \end{vmatrix} = 0$$

is here used in the same way as the solubility of $\mathrm{C}(x, a, b, c) = 0$ by others: and the next determinant of the same special type,—Puchta's of 1881,—is noted as if for the first time, and its nine linear factors given.

DEDEKIND, R. (1886).

[*Sitzungsb. . . . Ges. d. Wiss.* (Berlin), 1897, p. 1007.]

Frobenius, quoting a private communication of Dedekind's, gives here by implication the factorization of a circulant of two three-line circulant arrays similar to that dealt with by Muir in 1881. Using ρ for a primitive third root of 1, so that

$$\mathrm{C}(x_1, x_2, x_3) = (x_1+x_2+x_3)(x_1+\rho x_2+\rho^2 x_3)(x_1+\rho^2 x_2+\rho x_3) = u_1u_2u_3, \text{ say,}$$

and

$$\mathrm{C}(x_4, x_5, x_6) = (x_4+x_5+x_6)(x_4+\rho x_5+\rho^2 x_6)(x_4+\rho^2 x_5+\rho x_6) = v_1v_2v_3, \text{ say,}$$

and denoting the matrices

$$\begin{vmatrix} x_1 & x_3 & x_2 & x_4 & x_5 & x_6 \\ x_2 & x_1 & x_3 & x_5 & x_6 & x_4 \\ x_3 & x_2 & x_1 & x_6 & x_4 & x_5 \\ x_4 & x_5 & x_6 & x_1 & x_3 & x_2 \\ x_5 & x_6 & x_4 & x_2 & x_1 & x_3 \\ x_6 & x_4 & x_5 & x_3 & x_2 & x_1 \end{vmatrix}, \quad \begin{vmatrix} 1 & -1 & 1 & . & . & 1 \\ 1 & -1 & \rho^2 & . & . & \rho \\ 1 & -1 & \rho & . & . & \rho^2 \\ 1 & 1 & . & 1 & 1 & . \\ 1 & 1 & . & \rho & \rho^2 & . \\ 1 & 1 & . & \rho^2 & \rho & . \end{vmatrix}, \quad \begin{vmatrix} u_1+v_1 & . & . & . & . & . \\ . & u_1-v_1 & . & . & . & . \\ . & . & u_2 & v_2 & . & . \\ . & . & v_3 & u_3 & . & . \\ . & . & . & . & u_2 & v_2 \\ . & . & . & . & u_3 & v_3 \end{vmatrix}$$

by X , L , U

he points out that

$$XL = LU,$$

from which of course it follows that the determinant of X,—that is, the block circulant,—is equal to

$$(u_1+v_1)(u_1-v_1)(u_2u_3-v_2v_3)^2.$$

To this we add that by interchanging the 4th and 5th rows and the 2nd and 3rd columns the determinant is made centrosymmetric and resolves at once into

$$\begin{vmatrix} x_1+x_6 & x_2+x_5 & x_3+x_4 \\ x_2+x_4 & x_3+x_6 & x_1+x_5 \\ x_3+x_5 & x_1+x_4 & x_2+x_6 \end{vmatrix} \cdot \begin{vmatrix} x_1-x_6 & x_2-x_5 & x_3-x_4 \\ x_2-x_4 & x_3-x_6 & x_1-x_5 \\ x_3-x_5 & x_1-x_4 & x_2-x_6 \end{vmatrix}$$

as in the other case referred to.

GEGENBAUR, L. (1888).

[Ueber Determinanten. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), xcvii. II A, pp. 154–163.]

The fundamental theorem here is—though not so stated—that *the circulant of two general* n*-line arrays is resolvable into the product of two* n*-line determinants*; for example, when n is 3 and the two given arrays are those of $|\, a_1b_2c_3\,|$, $|\, \alpha_1\beta_2\gamma_3\,|$, we have

$$\begin{vmatrix} a_1 & a_2 & a_3 & \alpha_1 & \alpha_2 & \alpha_3 \\ b_1 & b_2 & b_3 & \beta_1 & \beta_2 & \beta_3 \\ c_1 & c_2 & c_3 & \gamma_1 & \gamma_2 & \gamma_3 \\ \alpha_1 & \alpha_2 & \alpha_3 & a_1 & a_2 & a_3 \\ \beta_1 & \beta_2 & \beta_3 & b_1 & b_2 & b_3 \\ \gamma_1 & \gamma_2 & \gamma_3 & c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha_1 & a_2+\alpha_2 & a_3+\alpha_3 \\ b_1+\beta_1 & b_2+\beta_2 & b_3+\beta_3 \\ c_1+\gamma_1 & c_2+\gamma_2 & c_3+\gamma_3 \end{vmatrix} \cdot \begin{vmatrix} a_1-\alpha_1 & a_2-\alpha_2 & a_3-\alpha_3 \\ b_1-\beta_1 & b_2-\beta_2 & b_3-\beta_3 \\ c_1-\gamma_1 & c_2-\gamma_2 & c_3-\gamma_3 \end{vmatrix}$$

By way of proof it suffices to perform the operations

$$\text{row}_1+\text{row}_4, \quad \text{row}_2+\text{row}_5, \quad \text{row}_3+\text{row}_6$$

followed by

$$\text{col}_4-\text{col}_1, \quad \text{col}_5-\text{col}_2, \quad \text{col}_6-\text{col}_3,$$

and this is in effect what the author does. Note, however, should have been taken that the determinant is inherently centrosym-

metric, this being made evident by reversing the order of the last three rows and thereafter the order of the last three columns.

If a companion determinant of the same construction be taken and the circulant of their two arrays formed, the theorem may be repeatedly applied, and the said circulant thereby resolved into the product of 2^2 determinants of the n^{th} order. From the general result obtained by continuing in this way the similar theorems of Puchta, Voigt, and Gegenbaur himself are shown to be deducible.

JANISCH, E. (1890).

[Bemerkungen zum Rationalmachen der Nenner. *Archiv d. Math. u. Phys.*, (2) x. pp. 420–440.]

The first fraction considered here is of the type

$$\frac{a+\beta\sqrt{x}+\gamma\sqrt{y}+\delta\sqrt{xy}}{a+b\sqrt{x}+c\sqrt{y}+d\sqrt{xy}}.$$

Rationalization is taken to mean that the fraction can be transformed into

$$\mathrm{P}+\mathrm{Q}\sqrt{x}+\mathrm{R}\sqrt{y}+\mathrm{S}\sqrt{xy},$$

where P, Q, R, S are rational: and as such a transformation implies that

$$\left.\begin{aligned} a\mathrm{P}+bx\mathrm{Q}+cy\mathrm{R}+dxy\mathrm{S} &= \alpha \\ b\mathrm{P}+a\mathrm{Q}+dy\mathrm{R}+cy\mathrm{S} &= \beta \\ c\mathrm{P}+dx\mathrm{Q}+a\mathrm{R}+bx\mathrm{S} &= \gamma \\ d\mathrm{P}+c\mathrm{Q}+b\mathrm{R}+a\mathrm{S} &= \delta \end{aligned}\right\},$$

it is clear that P, Q, R, S are determinable as required. In the more complicated case which the author deals with he actually solves the set of eight equations, taking note of the axisymmetry of the determinant of the set and establishing the relations existing between its primary minors. Curiously enough he does not mention that the determinant is the denominator resulting from rationalization.

A fraction involving cube roots is treated in similar fashion, the roots being

$$\sqrt[3]{x},\ \sqrt[3]{y},\ \sqrt[3]{x^2},\ \sqrt[3]{xy},\ \sqrt[3]{y^2},\ \sqrt[3]{x^2y},\ \sqrt[3]{xy^2},\ \sqrt[3]{x^2y^2},$$

and the outcome quite analogous.

BURNSIDE, W. (1893).

[On a property of certain determinants. *Messenger of Math.*, (2) xxiii. pp. 112–114.]

Though the property here noted is not essentially new, the fresh point of view makes the treatment of it interesting. In Burnside's own words it is: *If in a determinant of* n *rows the successive rows proceed from the first by permutations which form an Abelian group of order* n (*including identity*), *the determinant is expressible as the product of* n *linear factors.* A clearly worded proof is given with the requisite elementary explanation regarding cyclical and Abelian groups. The illustrative example chosen is the 9-line circulant of three circulant arrays, that is to say, the first of those brought forward by Puchta in 1881. Denoting

$$\begin{array}{lll} a+b+c & a+b\gamma+c\gamma^2 & a+b\gamma^2+c\gamma \\ d+e+f & d+e\gamma+f\gamma^2 & d+e\gamma^2+f\gamma \\ g+h+i & g+h\gamma+i\gamma^2 & g+h\gamma^2+i\gamma \end{array} \quad \text{by} \quad \begin{array}{lll} \mathrm{P} & \mathrm{P}' & \mathrm{P}'' \\ \mathrm{Q} & \mathrm{Q}' & \mathrm{Q}'' \\ \mathrm{R} & \mathrm{R}' & \mathrm{R}'' \end{array}$$

the result given is

$$\begin{aligned} \mathrm{C}(abc, def, ghi) = {} & (\mathrm{P}+\mathrm{Q}+\mathrm{R})(\mathrm{P}+\mathrm{Q}\gamma+\mathrm{R}\gamma^2)(\mathrm{P}+\mathrm{Q}\gamma^2+\mathrm{R}\gamma) \\ & \cdot(\mathrm{P}'+\mathrm{Q}'+\mathrm{R}')(\mathrm{P}'+\mathrm{Q}'\gamma+\mathrm{R}'\gamma^2)(\mathrm{P}'+\mathrm{Q}'\gamma^2+\mathrm{R}'\gamma) \\ & \cdot(\mathrm{P}''+\mathrm{Q}''+\mathrm{R}'')(\mathrm{P}''+\mathrm{Q}''\gamma+\mathrm{R}''\gamma^2)(\mathrm{P}''+\mathrm{Q}''\gamma^2+\mathrm{R}''\gamma), \end{aligned}$$

in substantial agreement of course with Puchta's.

PASCAL, E. (1897).

[I Determinanti: . . . . viii+330 pp. Milano.]

In dealing with Puchta's determinant of order 2^m it is noted that each minor of order 2^{m-1} differs at most only in sign from its complementary.

SCHULZE, E. (1897, 1899).

[Eine Determinantenformel. *Zeitschr. f. Math. u. Phys.*, xlii. pp. 313–322.]

[Eine Determinantenformel. *Zeitschr. f. Math. u. Phys.*, xliv. pp. 167–175.]

The problem here solved is the generalizing of Spottiswoode's property of the simple circulant. The solution originates in so far

specializing the units $e_1, e_2, \ldots, e_n$ of a system of complex numbers as to make the product of any two a linear function of them all, namely,

$$\left.\begin{aligned} e_1 \cdot e_1 &= C_{111}e_1 + C_{211}e_2 + \ldots + C_{n11}e_n \\ e_1 \cdot e_2 &= C_{112}e_1 + C_{212}e_2 + \ldots + C_{n12}e_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ e_h \cdot e_k &= C_{1hk}e_1 + C_{2hk}e_2 + \ldots + C_{nhk}e_n \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right\};$$

and the result itself is that *The* n-*line determinant,* whose* $(r, s)^{th}$ *element is*

$$C_{r1s}a_1 + C_{r2s}a_2 + \ldots + C_{rns}a_n,$$

is equal to

$$(e_1^{(1)}a_1 + e_2^{(1)}a_2 + \ldots + e_n^{(1)}a_n)(e_1^{(2)}a_1 + e_2^{(2)}a_2 + \ldots + e_n^{(2)}a_n) \ldots (e_1^{(n)}a_1 + \ldots + e_n^{(n)}a_n)$$

where $e_k^{(1)}, e_k^{(2)}, \ldots, e_k^{(n)}$ *are the* n *values of* e_k *to be obtained from the equation*

$$\begin{vmatrix} C_{1k1} - e_k & C_{1k2} & \ldots & C_{1kn} \\ C_{2k1} & C_{2k2} - e_k & \ldots & C_{2kn} \\ \ldots & \ldots & \ldots & \ldots \\ C_{nk1} & C_{nk2} & \ldots & C_{nkn} - e_k \end{vmatrix} = 0.$$

The author is himself conscious that the determinant is of " ziemlich komplizierten Bau," and by painstaking selection of simple and fitting values for the C's provides four or five examples with rational elements, one of the determinants being equal to

$$(a_1 + a_2 + \ldots + a_{2m})^m (a_1 - a_2 + a_3 - a_4 + \ldots - a_{2m})^m,$$

and another being Puchta's of 1877.

The second paper is on the same subject as the first, but adds little of any consequence, even the examples being mainly confined to special cases of the simple circulant already dealt with by Scott, Schrader, and others.

MUIR, T. (1898).

[The relations between the coaxial minors of a determinant of the fourth order. *Transac. R. Soc. Edinburgh*, xxxix. pp. 323–339.]

In the third part of this paper (§§ 12–16) the first subject is the rationalization of the equation

$$x + h\sqrt{bc} + k\sqrt{ca} + l\sqrt{ab} = 0,$$

* Mentioned by Weierstrass in the *Göttinger Nachrichten* for 1884, p. 397.

the result obtained being

$$\begin{vmatrix} x & h\sqrt{bc} & k\sqrt{ca} & l\sqrt{ab} \\ h\sqrt{bc} & x & l\sqrt{ab} & k\sqrt{ca} \\ k\sqrt{ca} & l\sqrt{ab} & x & h\sqrt{bc} \\ l\sqrt{ab} & k\sqrt{ca} & h\sqrt{bc} & x \end{vmatrix} \text{ or } \begin{vmatrix} x & h & k & l \\ hbc & x & lb & kc \\ kca & la & x & hc \\ lab & ka & hb & x \end{vmatrix} = 0.$$

Viewed as a case of this and derived either from

$$\cos(\alpha+\beta+\gamma) - \cos\alpha\cos\beta\cos\gamma + \Sigma\cos\alpha\sin\beta\sin\gamma = 0$$

or from

$$\cos(\alpha+\beta+\gamma) + 2\cos\alpha\cos\beta\cos\gamma + \Sigma\cos\alpha\cos(\beta+\gamma) = 0$$

is the relation between $\cos(\alpha+\beta+\gamma)$, $\cos\alpha$, $\cos\beta$, $\cos\gamma$, namely,

$$\begin{vmatrix} \cos(\alpha+\beta+\gamma)+2\cos\alpha\cos\beta\cos\gamma & \cos\alpha & \cos\beta & \cos\gamma \\ \cos\alpha+2\cos\beta\cos\gamma\cos(\alpha+\beta+\gamma) & \cos(\alpha+\beta+\gamma) & \cos\gamma & \cos\beta \\ \cos\beta+2\cos\gamma\cos\alpha\cos(\alpha+\beta+\gamma) & \cos\gamma & \cos(\alpha+\beta+\gamma) & \cos\alpha \\ \cos\gamma+2\cos\alpha\cos\beta\cos(\alpha+\beta+\gamma) & \cos\beta & \cos\alpha & \cos(\alpha+\beta+\gamma) \end{vmatrix} = 0.$$

This leads to consideration of the relations between

$$\begin{aligned} &\sin(\alpha+\beta+\gamma),\ \sin\alpha,\ \sin\beta,\ \sin\gamma, \\ &\cos(\alpha+\beta),\ \cos\alpha,\ \cos\beta, \\ &\sin(\alpha+\beta),\ \sin\alpha,\ \sin\beta, \end{aligned}$$

and the rationalization of

$$a + b\sqrt{xy} + c\sqrt{xz} + d\sqrt{xw} + e\sqrt{yz} + f\sqrt{yw} + g\sqrt{zw} + h\sqrt{xyzw} = 0$$

in three forms. It also suggests the expression of the relation

$$\cos^{-1}x + \cos^{-1}y + \cos^{-1}z + \cos^{-1}w = 0$$

in purely algebraical form, namely,

$$\begin{vmatrix} x & y & z & w+2xyz \\ y & x & w & z+2yxw \\ z & w & x & y+2zwx \\ w & z & y & x+2wzy \end{vmatrix} = 0$$

or

$$\Sigma x^4 - 2\Sigma x^2y^2 + 8xyzw + 4(\Sigma x^2y^2z^2 - \Sigma x^3yzw) = 0.$$

CHAPTER XVII.

CONTINUANTS, FROM 1881 TO 1899.

THE progress made in the study of continuants is unabated, but at the same time shows no acceleration, the number of writings, though large, being not in excess of the number for the preceding period.

MUIR, T. (1881).

[On some transformations connecting general determinants with continuants. *Transac. R. Soc. Edinburgh*, xxx. pp. 5–12.]

Although the transformation here brought to notice is of some considerable importance, it is ultimately dependent merely on the appropriate use of a very simple proposition, namely, that *any element of a determinant may be made* 0 *by multiplying its column* (and thereby of course the determinant) *by another element belonging to the same row, and then subtracting from the altered column the other column multiplied by the given element*: for example,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2b_3 & a_3 \\ b_1 & b_2b_3 & b_3 \\ c_1 & c_2b_3 & c_3 \end{vmatrix} \div b_3 = \begin{vmatrix} a_1 & |a_2b_3| & a_3 \\ b_1 & . & b_3 \\ c_1 & |c_2b_3| & c_3 \end{vmatrix} \div b_3.$$

With this in mind it is readily seen that

$$|a_1b_2c_3d_4| = \begin{vmatrix} |a_1d_2| & |a_2d_3| & a_3 & a_4 \\ |b_1d_2| & |b_2d_3| & b_3 & b_4 \\ |c_1d_2| & |c_2d_3| & c_3 & c_4 \\ . & . & d_3 & d_4 \end{vmatrix} \div d_2d_3 = \begin{vmatrix} |a_1c_2d_3| & |a_2d_3| & a_3 & a_4 \\ |b_1c_2d_3| & |b_2d_3| & b_3 & b_4 \\ . & |c_2d_3| & c_3 & c_4 \\ . & . & d_3 & d_4 \end{vmatrix} \div d_3|c_2$$

the three-line determinants making their appearance in virtue of the identity

$$\begin{vmatrix} |x_\alpha z_\beta| & |x_\beta z_\gamma| \\ |y_\alpha z_\beta| & |y_\beta z_\gamma| \end{vmatrix} = z_\beta |x_\alpha y_\beta z_\gamma|.$$

By proceeding in the same way on the other side of the diagonal there is obtained the desired result, namely,

$$|a_1b_2c_3d_4| = \begin{vmatrix} |a_1c_2d_3| & |a_2d_3| & . & . \\ |b_1c_2d_3| & |b_2d_3| & |a_2b_3| & . \\ . & |c_2d_3| & |a_2c_3| & |a_2b_3c_4| \\ . & . & |a_2d_3| & |a_2b_3d_4| \end{vmatrix} \div |a_2b_3|\ |a_2d_3|\ |c_2d_3|.$$

The corresponding result is then given for the case of the fifth order, namely,

$$(1') \quad |a_1b_2c_3d_4e_5| = \frac{\begin{vmatrix} |a_1c_2d_3e_4| & |a_2d_3e_4| & . & . & . \\ |b_1c_2d_3e_4| & |b_2d_3e_4| & |a_2b_3e_4| & . & . \\ . & |c_2d_3e_4| & |a_2c_3e_4| & |a_2b_3c_4| & . \\ . & . & |a_2d_3e_4| & |a_2b_3d_4| & |a_2b_3c_4d_5| \\ . & . & . & |a_2b_3e_4| & |a_2b_3c_4e_5| \end{vmatrix}}{|a_2b_3c_4|\ |a_2b_3e_4|\ |a_2d_3e_4|\ |c_2d_3e_4|}$$

and a rule for writing it out in any case. A slightly more general result is also noted, and is exemplified in

$$(2) \quad |a_1b_2c_3d_4| = \frac{\begin{vmatrix} |a_1c_2d_3| & |m_1a_2d_3| & . & . \\ |b_1c_2d_3| & |m_1b_2d_3| & |m_1a_2b_3| & . \\ . & |m_1c_2d_3| & |m_1a_2c_3| & |m_1a_2b_3c_4| \\ . & . & |m_1a_2d_3| & |m_1a_2b_3d_4| \end{vmatrix}}{|m_1a_2b_3|\ |m_1a_2d_3|\ |m_1c_2d_3|}.$$

In the next place, by using on these identities the law of complementaries there are obtained the correlative identities

$$|a_1b_4|\ |b_1c_4|\ |c_1d_4| = \begin{vmatrix} b_4 & |b_1c_4| & . & . \\ a_4 & |a_1c_4| & |c_1d_4| & . \\ . & |a_1b_4| & |b_1d_4| & d_1 \\ . & . & |b_1c_4| & c_1 \end{vmatrix},$$

$$|a_1b_5|\ |b_1c_5|\ |c_1d_5|\ |d_1e_5| = \begin{vmatrix} b_5 & |b_1c_5| & . & . & . \\ a_5 & |a_1c_5| & |c_1d_5| & . & . \\ . & |a_1b_5| & |b_1d_5| & |d_1e_5| & . \\ . & . & |b_1c_5| & |c_1e_5| & e_1 \\ . & . & . & |c_1d_5| & d_1 \end{vmatrix},$$

$$m_4|a_1b_4|\ |b_1c_4|\ |c_1d_4| = \begin{vmatrix} |m_1b_4| & |b_1c_4| & . & . \\ |m_1a_4| & |a_1c_4| & |c_1d_4| & . \\ . & |a_1b_4| & |b_1d_4| & d_4 \\ . & . & |b_1c_4| & c_4 \end{vmatrix};$$

and by using on the first of them (1) the law of extensible minors, an alternative expression for $|\,a_1b_2c_3d_4e_5\,|$ is got, namely,

$$\frac{\begin{vmatrix} |\,a_1c_2d_3e_5\,| & |\,a_2d_3e_5\,| & . & . \\ |\,b_1c_2d_3e_5\,| & |\,b_2d_3e_5\,| & |\,a_2b_3e_5\,| & . \\ . & |\,c_2d_3e_5\,| & |\,a_2c_3e_5\,| & |\,a_2b_3c_4e_5\,| \\ . & . & |\,a_2d_3e_5\,| & |\,a_2b_3d_4e_5\,| \end{vmatrix}}{|\,a_2b_3e_5\,|\;|\,a_2d_3e_5\,|\;|\,c_2d_3e_5\,|}.$$

To bring transformations of this kind into connection with continued fractions it is necessary that one of the new continuants be a principal minor of another. By altering the lettering in (2) with an eye on (1′), such a pair is obtained, and division then gives the result

$$\frac{|\,a_1b_2c_3d_4e_5\,|\;|\,c_2d_3e_4\,|}{|\,b_2c_3d_4e_5\,|} = \frac{\begin{vmatrix} |\,a_1c_2d_3e_4\,| & |\,a_2d_3e_4\,| & . & . & . \\ |\,b_1c_2d_3e_4\,| & |\,b_2d_3e_4\,| & |\,a_2b_3e_4\,| & . & . \\ . & |\,c_2d_3e_4\,| & |\,a_2c_3e_4\,| & |\,a_2b_3c_4\,| & . \\ . & . & |\,a_2d_3e_4\,| & |\,a_2b_3d_4\,| & |\,a_2b_3c_4d_5 \\ . & . & . & |\,a_2b_3e_4\,| & |\,a_2b_3c_4e_5 \end{vmatrix}}{\begin{vmatrix} |\,b_2d_3e_4\,| & |\,a_2b_3e_4\,| & . & . \\ |\,c_2d_3e_4\,| & |\,a_2c_3e_4\,| & |\,a_2b_3c_4\,| & . \\ . & |\,a_2d_3e_4\,| & |\,a_2b_3d_4\,| & |\,a_2b_3c_4d_5\,| \\ . & . & |\,a_2b_3e_4\,| & |\,a_2b_3c_4e_5\,| \end{vmatrix}}$$

whence, by Sylvester's fundamental theorem regarding the application of continuants, there comes

$$\frac{|\,a_1b_2c_3d_4e_5\,|\;|\,c_2d_3e_4\,|}{|\,b_2c_3d_4e_5\,|}$$

$$= |\,a_1c_2d_3e_4\,| - \cfrac{|\,a_2d_3e_4\,|\;|\,b_1c_2d_3e_4\,|}{|\,b_2d_3e_4\,| - \cfrac{|\,a_2b_3e_4\,|\;|\,c_2d_3e_4\,|}{|\,a_2c_3e_4\,| - \cfrac{|\,a_2b_3e_4\,|\;|\,a_2d_3e_4\,|}{|\,a_2b_3d_4\,| - \cfrac{|\,a_2b_3c_4d_5\,|\;|\,a_2b_3e_4\,|}{|\,a_2b_3c_4e_5\,|}}}}$$

the corresponding identity for the next lower order being

$$\frac{|\,a_1b_2c_3d_4\,|\;|\,c_2d_3\,|}{|\,b_2c_3d_4\,|} = |\,a_1c_2d_3\,| - \cfrac{|\,a_2d_3\,|\;|\,b_1c_2d_3\,|}{|\,b_2d_3\,| - \cfrac{|\,a_2b_3\,|\;|\,c_2d_3\,|}{|\,a_2c_3\,| - \cfrac{|\,a_2b_3c_4\,|\;|\,a_2d_3}{|\,a_2b_3d_4\,|}}}$$

and the complementary

$$\frac{|\,a_1b_5\,|}{a_1} = |\,a_1c_5\,| - \cfrac{|\,c_1d_5\,|\,|\,a_1b_5\,|}{|\,b_1d_5\,| - \cfrac{|\,d_1e_5\,|\,|\,b_1c_5\,|}{|\,c_1e_5\,| - \cfrac{e_1\,|\,c_1d_5\,|}{d_1}}}.$$

The rest of the paper gives in condensed form a second series of results analogous to the foregoing. Of these we have room for only one, namely, that corresponding to (1′) above,

$$|\,a_1b_2c_3d_4f_5\,| = \begin{vmatrix} y_1 & x_1 & . & . & . \\ z_1 & y_2 & x_2 & . & . \\ . & z_2 & y_3 & x_3 & . \\ . & . & z_3 & y_4 & x_4 \\ . & . & . & z_4 & y_5 \end{vmatrix} \div a_2\,|\,a_2b_3\,|\,|\,a_2b_3c_4\,| \cdot b_1\,|\,b_1c_2\,|\,|\,b_1c_2d_3\,|$$

where

$$\begin{aligned} x_1 &= a_2, & z_1 &= b_1, \\ x_2 &= |\,a_2b_3\,|, & z_2 &= |\,b_1c_2\,|, \\ x_3 &= b_1\,|\,a_2b_3c_4\,|, & z_3 &= a_2\,|\,b_1c_2d_3\,|, \\ x_4 &= |\,b_1c_2\,|\,|\,a_2b_3c_4d_5\,|, & z_4 &= |\,a_2b_3\,|\,|\,b_1c_2d_3f_4\,|, \end{aligned}$$

and

$$\begin{aligned} y_1 &= a_1, \\ y_2 &= b_2, \\ y_3 &= a_2\,|\,b_1c_3\,| - a_3\,|\,b_1c_2\,|, \\ y_4 &= |\,a_2b_3\,|\,|\,b_1c_2d_4\,| - |\,a_2b_4\,|\,|\,b_1c_2d_3\,|, \\ y_5 &= |\,a_2b_3c_4\,|\,|\,b_1c_2d_3f_5\,| - |\,a_2b_3c_5\,|\,|\,b_1c_2d_3f_4\,|. \end{aligned}$$

CHRYSTAL, G. (1881).

[Note on Mr. Muir's transformation of a determinant into a continuant. *Transac. R. Soc. Edinburgh*, xxx. pp. 13–14.]

Chrystal's interest in Muir's paper led him to reinvestigate the subject, his method being to start with a general set of linear homogeneous equations and obtain the requisite alternative form of eliminant to give rise to the fundamental identity. Succeeding in this, he indicates how a generalization may be effected.

KLUG, L. (1881).

[Die Entwickelung des Euler'schen Algorithmus. *Archiv d. Math. u. Phys.*, lxvii. pp. 337–342.]

A simple inquiry into the types of terms in the final development: determinants not used.

JADANZA, N. (1882, 1884).

[Sopra un determinante gobbo che si presenta nello studio dei cannochiali. *Atti . . . Accad. . . . di Torino*, xvii. pp. 714–723.]

[Teorica dei cannochiali, ... 182 pp. Torino.]

The determinant in question is the continuant, and makes its appearance here in the same connection as it did with Casorati in 1872.

MUIR, T. (1883).

[Question 7344. *Educ. Times*, xxxvi. pp. 131, 200: *Math. from Educ. Times*, xl. p. 26.]

The identity here established is that exemplified by

$$\mathrm{K}(a-1,\ a,\ a,\ a+1) \ = \ a\mathrm{K}(a,\ a,\ a).$$

MUIR, T. (1883).

[On the general equation of differences of the second order. *Philos. Magazine*, (5) xvii. pp. 115–118.]

The equation in question being

$$u_x \ = \ a_{x-1}u_{x-1} + b_{x-2}u_{x-2},$$

Cayley had recently * expressed u_n in the form

$$\mathrm{A}u_1 + \mathrm{B}u_0,$$

and stated more or less arbitrarily a rule for writing out the values

* Cayley, A. On the general equation of differences of the second order. *Quart. Journ. of Math.*, xiv. pp. 23–25; or *Collected Math. Papers*, x. pp. 47–49.

of A and B. This rule Muir here justifies, showing that A and B are continuants, for example,

$$u_6 = \mathrm{K}\begin{pmatrix} b_4 & b_3 & b_2 & b_1 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{pmatrix} u_1 + b_0 \mathrm{K}\begin{pmatrix} b_4 & b_3 & b_2 \\ a_5 & a_4 & a_3 & a_2 \end{pmatrix} u_0.$$

He then takes the more general equation

$$u_x = a_{x-1}u_{x-1} + b_{x-2}u_{x-2} + c_{x-2},$$

and deduces

$$\begin{aligned} u_6 = \mathrm{K}\begin{pmatrix} b_4 & b_3 & b_2 & b_1 \\ a_5 & a_4 & a_3 & a_2 & a_1 \end{pmatrix} u_1 + b_0 \mathrm{K}\begin{pmatrix} b_4 & b_3 & b_2 \\ a_5 & a_4 & a_3 & a_2 \end{pmatrix} u_0 \\ + c_0 \mathrm{K}\begin{pmatrix} b_4 & b_3 & b_2 \\ a_5 & a_4 & a_3 & a_2 \end{pmatrix} + c_1 \mathrm{K}\begin{pmatrix} b_4 & b_3 \\ a_5 & a_4 & a_3 \end{pmatrix} \\ + \ldots + c_4, \end{aligned}$$

illustrating the result by appropriately applying it to an equation connected with another special form of determinant.

WOLSTENHOLME, J. (1884): MUIR, T. (1884).

[Questions 7574, 7607. *Educ. Times*, xxxvii. pp. 28, 59, 188; l. p. 194: *Math. from Educ. Times*, xli. pp. 112–113; xlii. pp. 95–96.]

The net result here is

$$\begin{vmatrix} a & b & . & \ldots & . & . \\ c & a & b & \ldots & . & . \\ . & c & a & \ldots & . & . \\ . & . & . & . & . & . \\ . & . & . & \ldots & a & b \\ . & . & . & \ldots & c & a \end{vmatrix}_n = \left\{a - 2(bc)^{\frac{1}{2}} \cos\frac{\pi}{n+1}\right\}\left\{a - 2(bc)^{\frac{1}{2}} \cos\frac{2\pi}{n+1}\right\} \ldots \left\{a - 2(bc)^{\frac{1}{2}} \cos\frac{n\pi}{n+1}\right\}$$

the determinant being shown from its recurrence-formula to be the coefficient of y^n in the expansion of $(1 - ay + bcy^2)^{-1}$. It is alternative to a result obtained in 1878 (*Hist.*, iii. pp. 420–421). Both forms of course make evident that the determinant is really a function of only two variables.

The condensation of the same continuant we shall refer to presently.

MUIR, T. (1884).

[Note on the condensation of a special continuant. *Proceed. Edinburgh Math. Soc.*, ii. pp. 16–18.]

The note opens with the statement and proof of two lemmas. The one is the identity exemplified by

$$\mathrm{K}\begin{pmatrix} & -b^2 & -b^2 & -b^2 & -b^2 & \\ a+b & a & a & a & a & a+b \end{pmatrix} = (a+2b)\,\mathrm{K}\begin{pmatrix} & -b^2 & -b^2 & -b^2 & \\ a & a & a & a & a \end{pmatrix}.$$

The other is to the effect that *the determinant whose elements are all zeros except those in the main diagonal and in the two diagonals drawn through the places* (1, 3), (3, 1) *parallel to the main diagonal is expressible as the product of two continuants* ; and that *the quotient of two such determinants is, like the quotient of two continuants, expressible as a continued fraction.* For example,

$$\begin{vmatrix} a_1 & . & c_1 & . & . & . & . \\ . & a_2 & . & c_2 & . & . & . \\ b_1 & . & a_3 & . & c_3 & . & . \\ . & b_2 & . & a_4 & . & c_4 & . \\ . & . & b_3 & . & a_5 & . & c_5 \\ . & . & . & b_4 & . & a_6 & . \\ . & . & . & . & b_5 & . & a_7 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & . & . \\ b_1 & a_3 & c_3 & . \\ . & b_3 & a_5 & c_5 \\ . & . & b_5 & a_7 \end{vmatrix} \cdot \begin{vmatrix} a_2 & c_2 & . \\ b_2 & a_4 & c_4 \\ . & b_4 & a_6 \end{vmatrix},$$

and consequently also

$$\begin{vmatrix} a_1 & . & c_1 & . & . & . \\ . & a_2 & . & c_2 & . & . \\ b_1 & . & a_3 & . & c_3 & . \\ . & b_2 & . & a_4 & . & c_4 \\ . & . & b_3 & . & a_5 & . \\ . & . & . & b_4 & . & a_6 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & . \\ b_1 & a_3 & c_3 \\ . & b_3 & a_5 \end{vmatrix} \cdot \begin{vmatrix} a_2 & c_2 & . \\ b_2 & a_4 & c_4 \\ . & b_4 & a_6 \end{vmatrix} :$$

so that by division we find the quotient of the seven-line by the six-line determinant equal to

$$\begin{vmatrix} a_1 & c_1 & . & . \\ b_1 & a_3 & c_3 & . \\ . & b_3 & a_5 & c_5 \\ . & , & b_5 & a_7 \end{vmatrix} \div \begin{vmatrix} a_1 & c_1 & . \\ b_1 & a_3 & c_3 \\ . & b_3 & a_5 \end{vmatrix},$$

and therefore equal to

$$a_7 - \frac{b_5c_5}{a_5 - \dfrac{b_3c_3}{a_3 - \dfrac{b_1c_1}{a_1}}}.$$

Then denoting by $\mathrm{F}(b, x, c\,;\ 2n+1)$ the continuant of the $(2n+1)^{\text{th}}$ order having x's for the elements of the main diagonal, b's for those of the upper minor diagonal, and c's for those of the lower, we multiply it by its equivalent $\mathrm{F}(-c,\ x,\ -b\,;\ 2n+1)$, and thus obtain for its square a determinant of the type dealt with in the second lemma, the result being that after application of that lemma and its fellow the square root can be extracted, and we have

$$\mathrm{F}(b, x, c\,;\ 2n+1) \ = \ \mathrm{F}(b^2,\ x^2-2bc,\ c^2\,;\ n).$$

By a similar process it is found that

$$\begin{aligned}\mathrm{F}(b, x, c\,;\ 2n) \ &= \ \mathrm{F}(b^2,\ x^2-2bc,\ c^2\,;\ n) \\ &\quad + bc\mathrm{F}(b^2,\ x^2-2bc,\ c^2\,;\ n-1).\end{aligned}$$

The cases of these equalities in which $b = c = 1$ are due to Wolstenholme (*Educ. Times*, xxxvii. p. 28).*

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* The opportunity may here be suitably taken to point out that the similar equality given by Günther (*Hist.*, iii. p. 407) is readily established as follows :

$$\begin{vmatrix} x & b_1 & . & . & . & . \\ -1 & 1 & b_2 & . & . & . \\ . & -1 & x & b_3 & . & . \\ . & . & -1 & 1 & b_4 & . \\ . & . & . & -1 & x & b_5 \\ . & . & . & . & -1 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & . & . & . & . \\ . & -b_2 & 1 & 1 & . & . \\ . & . & 1 & 1 & . & . \\ . & . & . & -b_4 & 1 & 1 \\ . & . & . & . & 1 & 1 \\ . & . & . & . & . & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x+b_1 & -b_1b_2 & . & . & . & . \\ . & . & b_2 & . & . & . \\ -1 & b_2+x+b_3 & x+b_3 & -b_3b_4 & . & . \\ . & . & . & . & b_4 & . \\ . & -1 & -1 & b_4+x+b_5 & x+b_5 & b_5 \\ . & . & . & . & . & 1 \end{vmatrix}$$

$$= (-b_2)(-b_4) \begin{vmatrix} x+b_1 & -b_1b_2 & . \\ -1 & b_2+x+b_3 & -b_3b_4 \\ . & -1 & b_4+x+b_5 \end{vmatrix},$$

all that remains being the division of both sides by b_2b_4. The case where the order is the $(2n)^{\text{th}}$ need not be separately considered, either case being deducible from the other on putting the last b of the latter equal to 0.

MUIR, T. (1885).

[On bipartite functions. *Transac. R. Soc. Edinburgh*, xxxii. pp. 461–482.]

These functions we have already spoken of in reporting their fundamental connection with the multiplication-theorem of determinants. We have now to note that they include continuants as a special case, and that, as they consist of terms that are all essentially positive, it would seem more natural to classify continuants under them than under determinants : for example,

$$\begin{array}{cc|cc} p & 1 & & \\ q & b & r & 1 \\ \hline 1 & . & c & . \\ & & s & d \end{array} = pqrs + pqd + pcs + brs + bd,$$

$$= \begin{vmatrix} p & b & . & . \\ -1 & q & c & . \\ . & -1 & r & d \\ . & . & -1 & s \end{vmatrix}.$$

We may add that it is on this account that we have suggested the resuscitation of Sylvester's word ' cumulant ' to take the place of ' bipartite ' as used in the memoir under review.

STUDNIČKA, F. J. (1886).

[Eine neue Anwendung der Kettenbruchdeterminanten. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), 1886, pp. 1–6.]

The result here is

$$\cos n\theta = \begin{vmatrix} \cos\theta & 1 & . & . & \dots \\ 1 & 2\cos\theta & 1 & . & \dots \\ . & 1 & 2\cos\theta & 1 & \dots \\ . & . & . & . & \dots \end{vmatrix}_n,$$

obtained, curiously enough, from applying in connection with the equation $x^2+ax+b = 0$ Brioschi's recurrent expression of 1854 for the sum of the n^{th} powers of the roots of an equation. It is of course much simpler to use

$$\cos(n+1)\theta - 2\cos\theta \cdot \cos n\theta - \cos(n-1)\theta = 0$$

as a recurrence-formula, and simpler still to go back to Wolstenholme's result of 1874.

STUDNIČKA, F. J. (1886).

[O nekih vlastitostih determinanta verižnika. *Rad jugoslavenske akad. . . . u Zagrebu*, lxxviii. pp. 105–115.]

In essence this consists of notes originating in the observation that the coefficients in the expansion of $(a_0+a_1x+\ldots+a_nx^n)^{-1}$ are continuants when n is 2, and a similar observation in regard to another recurrent (*Hist.*, ii. pp. 211–212).

GEGENBAUR, L. (1888).

[Ueber die Functionen $C_n^{\nu}(x)$. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), xcvii. Abth. II a, pp. 259–270.]

The function referred to being the well-known coefficient of α^n in the expansion of $(1-2\alpha x+\alpha^2)^{-\nu}$, it suffices to recall, as the author does, the related results given above under Wolstenholme (1874, 1884), and to record the new equality *

$$C_n^{\nu}(\cos x) = 2^n \frac{(n+\nu-1)!}{n!(\nu-1)!} F,$$

where F is the continuant whose diagonal elements are all $\cos x$, whose bordering elements on the one side are all 1, and on the other

$$\frac{1}{2(\nu+1)}, \quad \frac{2\nu+1}{2(\nu+1)(\nu+2)}, \quad \ldots, \quad \frac{(n-1)(2\nu+n-2)}{4(\nu+n-2)(\nu+n-1)}.$$

MUIR, T. (1889).

[On a rapidly converging series for the extraction of the square root. *Proceed. R. Soc. Edinburgh*, xvii. pp. 14–18.]

The theorem here established by using properties of simple continuants is that if

$$\sqrt{N} = A + \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_z} + \underset{*}{\frac{1}{2A}} + \ldots,$$

then

$$\sqrt{N} = A + \frac{k(a_2, \ldots, a_z)}{k_0} + \frac{(-1)^z}{2k_0k_1} - \frac{1}{2^2k_0k_1k_2} - \frac{1}{2^3k_0k_1k_2k_3} - \ldots,$$

* This, however, should have been noted under the year 1877, when the author published a paper under the same title as the above (*Sitzungsb.*, lxxv. pp. 891–905).

where

$$k_0 = k(a_1, \ldots, a_z),$$
$$k_1 = k(a_1, \ldots, a_z, \mathrm{A}),$$
$$k_2 = k(a_1, \ldots, a_z, 2\mathrm{A}, a_1, \ldots, a_z, \mathrm{A}),$$
$$k_3 = k(a_1, \ldots, 2\mathrm{A}, \ldots, 2\mathrm{A}, \ldots, 2\mathrm{A}, \ldots, a_z, \mathrm{A}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

For example, since

$$\sqrt{300} = 17 + \frac{1}{\underset{*}{3}} + \frac{1}{8} + \frac{1}{3} + \frac{1}{\underset{*}{34}} + \ldots,$$

we have

$$10\sqrt{3} = 17 + \frac{25}{78} - \frac{1}{78\times2702} - \frac{1}{78\times2702\times7300802} - \ldots$$
$$= 17\cdot320508\ 075688\ 772935\ 2744634\ \ldots,$$

giving $\sqrt{3}$ correct to the twenty-sixth fractional place.

GAMBIOLI, D. (1889); . . . (1890).

[Sulle frazioni continue. *Rendic. . . . Accad. delle Sci. di Bologna*, anno 1889, pp. 33–55.]

[Question 230. *American Math. Monthly*, xii. (1905) p. 134.]

Although continuants are used throughout Gambioli's paper, nothing fresh is noted regarding them.

The result given under the other head is merely a case of the persymmetric continuant whose repeated elements are x, x^2+1, x, and whose value for the n^{th} order is

$$(x^{2n+2}-1) \div (x^2-1).$$

NOVARESE, H. (1890): CESÀRO, E. (1892).

[Question 680. *Mathesis*, x. p. 72; (2) i. pp. 24–25.]

[Remarques sur un continuant. *Mathesis*, (2) ii. pp. 5–12.]

The originating proposition here is that *if* $\mathrm{a}_0, \mathrm{a}_1, \ldots, \mathrm{a}_n$ *be all positive, and* $\lambda_1^2, \lambda_2^2, \ldots, \lambda_{n-1}^2$ *be none of them greater than* 1, *the continuant*

$$\begin{vmatrix} a_0+a_1 & -\lambda_1 a_1 & . & \dots & . \\ -\lambda_1 a_1 & a_1+a_2 & -\lambda_2 a_2 & \dots & . \\ . & -\lambda_2 a_2 & a_2+a_3 & \dots & . \\ . & . & . & \dots & a_{n-1}+a_n \end{vmatrix}, \quad \text{or } D_n \text{ say,}$$

is necessarily positive. It is readily seen to hold for the first two cases, and indeed that $D_2 > a_2 D_1$. It is then noted that

$$\begin{aligned} D_r &= (a_{r-1}+a_r)D_{r-1} - \lambda_{r-1}^2 a_{r-1}^2 D_{r-2} \\ &= a_r D_{r-1} + a_{r-1}(D_{r-1} - \lambda_{r-1}^2 a_{r-1} D_{r-2}), \end{aligned}$$

from which it follows that if $D_{r-1} > a_{r-1} D_{r-2}$, we must also have

$$D_r > a_r D_{r-1}.$$

The full requirements of a gradational proof are thus to hand.

Cesàro's treatment is not so purely determinantal. Observing that D_n is axisymmetric, he views it as the discriminant of a quadric, namely, the quadric

$$a_0 x_1^2 + a_1(x_1^2+x_2^2-2\lambda_1 x_1 x_2) + a_2(x_2^2+x_3^2-2\lambda_2 x_2 x_3) + \dots + a_n x_n^2,$$

in which the cofactor of every a being not less than the square of the difference of two x's is positive. The quadric as a whole is thus essentially positive, and therefore also is its discriminant. The rest of the paper is in keeping with the title, the points taken up being such as would seem natural in connection with any continuant,—its 'value,' transformations of it, its adjugate, interesting particular cases, generalizations, the related continued fraction, the convergence of the latter, and so on. Of the results obtained we may note (1) that the value of the continuant *

$$\begin{vmatrix} a_0+a_1 & -a_1 & . & \dots \\ -a_1 & a_1+a_2 & -a_2 & \dots \\ . & -a_2 & a_2+a_3 & \dots \\ . & . & . & \dots \end{vmatrix}_n$$

* This suggests continuant forms for Ferrers' determinants of 1855 (*Hist.*, ii. pp. 141–142), namely,

$$\begin{vmatrix} a_1 & -a_1 & . & \dots \\ -a_1 & a_1+a_2 & -a_2 & \dots \\ . & -a_2 & a_2+a_3 & \dots \\ . & . & . & \dots \end{vmatrix}_n, \quad \begin{vmatrix} 1+a_1 & -a_1 & . & \dots \\ -a_1 & a_1+a_2 & -a_2 & \dots \\ . & -a_2 & a_2+a_3 & \dots \\ . & . & . & \dots \end{vmatrix}_n.$$

is

$$a_0a_1a_2 \ldots a_n\left(\frac{1}{a_0}+\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right);$$

(2) that the sum of its signed primary minors is

$$a_0a_1a_2 \ldots a_n(\sigma_1+4\sigma_2+9\sigma_3+\ldots+n^2\sigma_n),$$

where

$$\sigma_h = \frac{1}{a_0a_h}+\frac{1}{a_1a_{h+1}}+\frac{1}{a_2a_{h+2}}+\ldots+\frac{1}{a_{n-h}a_n},$$

and (3) that therefore

$$\begin{vmatrix} x+a_0+a_1 & x-a_1 & . & \ldots \\ x-a_1 & x+a_1+a_2 & x-a_2 & \ldots \\ . & x-a_2 & x+a_2+a_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}_n$$

$$= a_0a_1a_2 \ldots a_n\left\{\frac{1}{a_0}+\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}+x(\sigma_1+4\sigma_2+\ldots+n^2\sigma_n)\right.$$

Also (4) that if the non-zero elements in the arrays

$$\begin{array}{cccc} . & a_{12} & a_{13} & \ldots \\ a_{21} & . & a_{23} & \ldots \\ a_{31} & a_{32} & . & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array} \qquad \begin{array}{cccc} . & b_{12} & b_{13} & \ldots \\ b_{21} & . & b_{23} & \ldots \\ b_{31} & b_{32} & . & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{array}$$

be all positive, a_h standing for the sum of the h^{th} row of the former and β_h for the sum of the h^{th} column of the latter, and if a similar array of c's be such as to satisfy the condition

$$c_{rs}^2 < a_{rs}\,b_{rs},$$

then the determinant

$$\begin{vmatrix} a_1+\beta_1 & c_{12}+c_{21} & c_{13}+c_{31} & \ldots \\ c_{12}+c_{21} & a_2+\beta_2 & c_{23}+c_{32} & \ldots \\ c_{13}+c_{31} & c_{23}+c_{32} & a_3+\beta_3 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

is essentially positive: to which there is added the enlightening remark that to construct such a determinant we have only to take a sum of positive multiples of essentially positive quadrics and form the discriminant.

MUIR, T. (1892).

[Note on a theorem regarding a series of convergents to the roots of a number. *Proceed. R. Soc. Edinburgh*, xix. pp. 15–19.]

This is an investigation of Sang's theorem* that the ratio of the rational portion of the expansion of $(\sqrt{n}+1)^r$ to the coefficient of $\sqrt{n}$ in the remaining portion is an approximation to $\sqrt{n}$. In the course of it proof is given that the numerators and denominators of the ratios are continuants of the forms

$$\begin{vmatrix} 1 & n-1 & . & . & \dots \\ -1 & 2 & n-1 & . & \dots \\ . & -1 & 2 & n-1 & \dots \\ . & . & -1 & 2 & \dots \\ . & . & . & . & \dots \end{vmatrix}, \quad \begin{vmatrix} 2 & n-1 & . & \dots \\ -1 & 2 & n-1 & \dots \\ . & -1 & 2 & \dots \\ . & . & . & \dots \end{vmatrix},$$

respectively, these when of the r^{th} order being equal to

$$\frac{(\sqrt{n}+1)^r + (-1)^r(\sqrt{n}-1)^r}{2}, \qquad \frac{(\sqrt{n}+1)^{r+1} + (-1)^r(\sqrt{n}-1)^{r+1}}{2\sqrt{n}},$$

and the convergents obtained being those of

$$1 + \frac{n-1}{2 +} \frac{n-1}{2 +} \frac{n-1}{2 +} \dots,$$

an already well-known continued fraction for $\sqrt{n}$.

MOLLAME, V. (1892).

[Sviluppo di una determinante e relazioni notevole che ne derivano. *Rivista di Mat.*, iii. pp. 47–53.]

The evaluation of $\mathrm{K}(a, {}^{-1}\ a, {}^{-1} \dots {}^{-1}, a)$ here given is not essentially different from what had already been repeatedly reached. By specializing in Scott's result of 1879, for example, the value obtained is

$$\frac{\{a+\sqrt{a^2-4}\}^{n+1} - \{a-\sqrt{a^2-4}\}^{n+1}}{2^{n+1}\sqrt{a^2-4}},$$

* *Sang, E.* On the extension of Brounker's method to the comparison of several magnitudes. *Proceed. R. Soc. Edinburgh*, xviii.

whereas Mollame gives

$$\text{cofactor of } \sqrt{-1} \text{ in } \frac{(a+\sqrt{-1}\sqrt{4-a^2})^{n+1}}{2^n\sqrt{4-a^2}}.$$

They both lead to

$$\frac{1}{2^n}\left\{(n+1)a^n + (n+1)_3a^{n-2}(a^2-4) + (n+1)_5a^{n-4}(a^2-4)^2 + \ldots\right\}.$$

SEGAR, H. W. (1893).

[On the roots of certain continuants. *Messenger of Math.*, (2) xxii. pp. 171–181.]

By multiplying the r^{th} row and the r^{th} column by a_r, Segar's first deduction from the familiar proposition is that if the b's and a's be real and the determinant be axisymmetric, the equation

$$\begin{vmatrix} b_{11}+a_1^2\lambda & b_{12} & b_{13} & \ldots. \\ b_{21} & b_{22}+a_2^2\lambda & b_{23} & \ldots. \\ b_{31} & b_{32} & b_{33}+a_3^2\lambda & \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0$$

has all its roots real, and therefore also the equation

$$\begin{vmatrix} b_{11}\lambda+a_1^2 & b_{12}\lambda & b_{13}\lambda & \ldots. \\ b_{21}\lambda & b_{22}\lambda+a_2^2 & b_{23}\lambda & \ldots. \\ b_{31}\lambda & b_{32}\lambda & b_{33}\lambda+a_3^2 & \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0.$$

His next step is that in the case of a continuant we need not insist on $b_{r,r+1}$ and $b_{r+1,r}$ being equal but merely of the same sign; thus, this condition being fulfilled, the equation

$$\begin{vmatrix} a_1^2 & b_{12}\lambda^2 & . & \ldots. \\ b_{21} & a_2^2 & b_{23}\lambda^2 & \ldots. \\ . & b_{32} & a_3^2 & \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0, \text{ or } \begin{vmatrix} a_1^2 & (b_{12}b_{21})^{\frac{1}{2}}\lambda & . & \ldots. \\ (b_{12}b_{21})^{\frac{1}{2}}\lambda & a_2^2 & (b_{23}b_{32})^{\frac{1}{2}}\lambda & \ldots. \\ . & (b_{23}b_{32})^{\frac{1}{2}}\lambda & a_3^2 & \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

has all its roots real; and the equation

$$\begin{vmatrix} a_1^2 & b_{12}\lambda & . & \ldots. \\ b_{21} & a_2^2 & b_{23}\lambda & \ldots. \\ . & b_{32} & a_3^2 & \ldots. \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0$$

has all its roots real and positive. On the other hand, if $b_{r,r+1}$ differ in sign from $b_{r+1,r}$ the roots of the last equation are all real and negative.

It is then pointed out that if any one of the determinants under consideration become of infinite order but remain convergent for all finite values of λ, it must have an infinite number of roots that are all real; and this result being reached, the next six pages are devoted to the discussion of the character of the roots of a variety of important functions.

At the close the original subject is reverted to, and we learn that in exactly the same way we can show that if the products $b_{13}b_{31}$, $b_{24}b_{42}$, ... be real and of like sign, the equation

$$\begin{vmatrix} a_1^2 & . & b_{13} & . & . & . & \dots \\ . & a_2^2 & . & b_{24} & . & . & \dots \\ b_{31} & . & a_3^2 & . & b_{35} & . & \dots \\ . & b_{42} & . & a_4^2 & . & b_{46} & \dots \\ . & . & b_{53} & . & a_5^2 & . & \dots \\ . & . & . & b_{64} & . & a_6^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0$$

has all its latent roots real, and of the same sign as the said products. Not only so, but the determinant which has r cross-rows of zeros between the a's and the b's has the same properties.

PEIRCE, B. O. (1893).

[Question 11930. *Educ. Times*, xlvi. pp. 232–233 : or *Math. from Educ. Times*, lx. pp. 49–50.]

The algebraical equality which is at the basis of the result here is

$$\Gamma_{n+1} = x\mathrm{K}_n,$$

where K_n is the continuant

$$(x+2k, -^{k^2}x+2k, -^{2k^2}x+2k, \dots .)$$

and Γ_n is got from K_n by changing the first and last $x+2k$ into $x+k$. This, however, is the same as Muir's first lemma of 1884, as is seen on putting in the latter

$$b = -k \quad \text{and} \quad u = x + 2k.$$

PALMSTRØM, A. (1897).

[Sur une généralisation de l'équation de Lamé. *Bergens Museums Aarbog for* 1897, no. xi. 9 pp.]

As might be expected from the title, the matter of interest to us here is a determinant resolvable into linear factors. In form it is a recurrent with one diagonal more than a continuant, the principal diagonal being

$$\mathrm{B},\ \mathrm{B},\ \mathrm{B},\ \ldots\ldots\ (n+1 \text{ in number}),$$

the adjacent diagonal on the one side

$$(2n+t_1-2)\beta,\quad 2(2n+t_1-3)\beta,\quad 3(2n+t_1-4)\beta,\quad \ldots\ldots$$

and the two diagonals on the other side

$$n(3n+t_1-3)\beta,\quad (n-1)(3n+t_1-6)\beta,\quad (n-2)(3n+t_1-9)\beta,\quad \ldots\ldots$$

$$2n(n-1)\beta,\quad 2(n-1)(n-2)\beta,\quad 2(n-2)(n-3)\beta,\quad \ldots\ldots$$

The method employed to obtain the factors of the result is addition of multiples of columns, the first operation being

$$\mathrm{col}_1 + (n)_1\,\mathrm{col}_2 + (n)_2\,\mathrm{col}_3 + \ldots.$$

and the removed factor

$$\mathrm{B} + \{2n^2 + (t_1-2)n\}\beta.$$

We note for ourselves that by certain slight modifications of notation much apparent complication can be done away with: for example, the writing of x for B/β and reversal of the order of the rows followed by reversal of the order of the columns give

$$\begin{vmatrix} x & 1\cdot a & 2(1\cdot 2) \\ 2(a+1) & x & 2(a+3) \\ . & 1(a+2) & x \end{vmatrix} = (x+2a+4)(x-2)(x-2a-2),$$

$$\begin{vmatrix} x & 1\cdot a & 2(1\cdot 2) & . \\ 3(a+2) & x & 2(a+3) & 2(2\cdot 3) \\ . & 2(a+3) & x & 3(a+6) \\ . & & 1(a+4) & x \end{vmatrix} = (x+3a+12)(x+a)(x-a-6)(x-3a$$

CALDARERA, F. (1897).

[Sull' equazioni lineari ricorrenti trinomie. *Giornale di Mat.*, xxxv. pp. 333–348.]

The continuant $\mathrm{K}(b, {}^{-a}b, {}^{-a}b, \ldots)$ is here considered, but nothing fresh is brought to light.

We may also note that the already familiar case where $a = -1$ and $b = 2 \cos x$ is dealt with in the *Periodico di Matematica* for the same year (xii. p. 196).

CANDIOTI, M. R. (1898): BICKMORE, C. E. (1898): CHRISTIE, R. W. D. (1899).

[Fracciones continuas. *Anales . . . Soc. Cient. Argentina,* xlvi. pp. 149–158.]

[Question 14040. *Educ. Times,* li. p. 472.]

[Question 14131. *Educ. Times,* lii. p. 94 : *Math. from Educ. Times,* lxxiii. p. 71.]

The first 'question' here gives properties of the continuant $K(a, a, a, \ldots)$, two of them being included in Smith's of 1854 (*Hist.*, ii. p. 425) : the second in reality, if not in appearance, concerns the same simple function.

Candioti's paper is merely an introductory exposition in which continuants are used from the outset.

BORINI, B. (1899).

[I Continuanti. 117 pp. 4°. Forli. 1900.]

This monograph, large in page, type and margin, gives a clearly written account of the theory and application of continuants. It forms indeed, in the author's own words, 'una specie di trattato,' the material being freely taken over from the original sources.

It consists of an historical introduction and seven chapters. The subjects of the latter are determinants of finite and of infinite order (pp. 7–18, 19–29) ; continuants (pp. 31–44) ; connection between continuants and continued fractions (pp. 45–66) ; periodic continued fractions (pp. 67–73) ; connection of series and continued fractions (pp. 75–95) ; generalization of the continued fraction algorithm (pp. 97–117).

So far as continuants are concerned there is nothing fresh to report.

CHAPTER XVIII.

MULTILINEANTS, FROM 1883 TO 1895.

ALTHOUGH we have found it expedient to make the period here four years shorter than in the other cases, the number of writings reported on is nevertheless three times the corresponding number in the preceding volume. The remarkable increase of interest thus evidenced will be seen to be almost entirely due to the new departure made in Hill's paper of 1877.

PINCHERLE, S. (1883).

[Sui sistemi di funzioni analitiche e le serie formate coi medesimi. *Annali di Mat.*, (2) xii. pp. 11–41, 107–137.]

The theory of so-called 'associated groups' dealt with in §§ 19–21 of the first paper here is closely connected with our subject. It really concerns a special set of linear equations infinite in number and connecting an infinite number of unknowns, the word 'gruppo' being used for an oblong array, and the use of 'associato' being in effect similar to that of Bellavitis (*Hist.*, i. p. 95).

APPELL, P. (1884).

[Sur une méthode élémentaire pour obtenir les développements en série trigonométrique des fonctions elliptiques. *Bull. Soc. Math. de France*, xiii. pp. 13–18.]

The method in question is that which is primarily applicable when we have available for the determination of the infinity of coefficients a like infinity of equations connecting them linearly; and it consists in taking only the first m equations, deleting therefrom all the terms

whose coefficients are higher than the m^{th}, solving the set as thus curtailed, and finally putting m equal to ∞ in the results. In the case dealt with by Appell the curtailed set of equations is of the type

$$\left.\begin{array}{rcl} A_1 + A_2 + \ldots + A_{m+1} & = & 1 \\ A_1\cos a_1 + A_2\cos a_2 + \ldots + A_{m+1}\cos a_{m+1} & = & \cos\lambda \\ A_1\cos 2a_1 + A_2\cos 2a_2 + \ldots + A_{m+1}\cos 2a_{m+1} & = & \cos 2\lambda \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ A_1\cos ma_1 + A_2\cos ma_2 + \ldots + A_{m+1}\cos ma_{m+1} & = & \cos m\lambda \end{array}\right\},$$

of which the solution is

$$\begin{aligned} A_1 &= \frac{(\cos\lambda-\cos a_2)(\cos\lambda-\cos a_3)\ldots(\cos\lambda-\cos a_{m+1})}{(\cos a_1-\cos a_2)(\cos a_1-\cos a_3)\ldots(\cos\lambda-\cos a_{m+1})} \\ A_2 &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

and the final outcome of the process is identical with the expansion as previously known : to the extent, therefore, of this one case the process is justified.

POINCARÉ, H. (1884).

[Remarques sur l'emploi de la méthode précédente. *Bull. Soc. Math. de France*, xiii. pp. 19–27.]

Poincaré raises the question of the legitimacy of the general process exemplified in Appell's paper. He first sets for consideration the infinite set of equations

$$\left.\begin{array}{l} a_1X_1 + a_2X_2 + \ldots + a_nX_n + \ldots. = 0 \\ a_1^2X_1 + a_2^2X_2 + \ldots + a_n^2X_n + \ldots. = 0 \\ a_1^3X_1 + a_2^3X_2 + \ldots + a_n^3X_n + \ldots. = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\},$$

where the a's are in ascending order of magnitude with the limit ∞. By finding the residues of an appropriately constructed function he obtains the solution

$$X_i = \frac{-a_i}{\left(1-\frac{a_i}{a_1}\right)\left(1-\frac{a_i}{a_2}\right)\ldots\left(1-\frac{a_i}{a_n}\right)\ldots.}$$

already long familiar in the case where n is finite. It is at once belittled, however, as not being unique, another solution being clearly got by multiplying by a_i the value of X_i just found, another by using the multiplier a_i^2, and so on. Next he takes up the infinite set whose array of coefficients is

$$\begin{array}{ccccccc} a_{01} & a_{02} & \cdots & a_{0n} & \cdots \\ a_{11} & a_{12} & \cdots & a_{1n} & \cdots \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

and arrives at the like final result. He then takes a set which, if we bear in mind the cosine's exponential value, is seen to have a closer resemblance to Appell's, namely, the set

$$\cdots + a^p_{-2}X_{-2} + a^p_{-1}X_{-1} + a^p_0X_0 + a^p_1X_1 + a^p_2X_2 + \ldots = 0 \Big\}_{p=0}^{p=\infty}$$

and deals with it in similar fashion. Finally, he applies what he has obtained to Appell's actual problem, and while obtaining the same solution as Appell, directs attention to an infinity of other solutions and to the fact that, unlike Appell's, the expansions corresponding to the latter are not convergent.

POINCARÉ, H. (1886).

[Sur les déterminants d'ordre infini. *Bull. Soc. Math. de France*, xiv. pp. 77–90.]

Having read Hill's paper of 1877 and been recalled by it to his own 'remarks' of 1884, Poincaré resumes the subject. First of all he devotes considerable space (pp. 77–82) to the elucidation of a point formerly left obscure. He then attacks directly the question of the convergence of a determinant of infinite order, opening with two lemmas regarding ordinary determinants, namely :

Any n*-line determinant is less in absolute value than the product of the sums got by adding the absolute values of the elements of each row* : in symbols, say,

$$\overline{\mathrm{D}}_n < \Pi_n.$$

If a certain number of elements of the determinant be replaced by zeros, $\overline{\mathrm{D}}_n, \Pi_n$ *thereby becoming* $\overline{\mathrm{D}}_n', \Pi_n'$, *then* $\Pi_n' < \Pi_n$ *and the diminu-*

tion $\Pi_n - \Pi_n'$ *is in absolute value greater than the difference of* $\bar{D}_n$ *and* $\bar{D}_n'$.

The former lemma is made to depend on Grassmann and Cauchy's actual product expression for a determinant (*Hist.*, ii. pp. 43–45, 77–78). The theorems established on convergence are :

(*a*) *Any unit-axial multilineant will converge if the sum of its non-diagonal elements converges absolutely.*

This is proved for multilineants of the form $|\, a_{11} a_{22} \ldots a_{nn} \ldots . |$ and separately for those of the form

$$|\, \ldots a_{-n,-n} \ldots a_{-1,-1}\, a_{0,0}\, a_{1,1} \ldots a_{nn} \ldots . |.$$

(*b*) *If a convergent multilineant have the elements of a line replaced by quantities all less in absolute value than a certain positive number, it will still be convergent.*

As an illustrative example he naturally takes Hill's

$$\begin{vmatrix}
\ldots & 1 & \frac{\theta_1}{\theta_0 - 4^2} & \frac{\theta_2}{\theta_0 - 4^2} & \frac{\theta_3}{\theta_0 - 4^2} & \frac{\theta_4}{\theta_0 - 4^2} & \ldots \\
\ldots & \frac{\theta_1}{\theta_0 - 2^2} & 1 & \frac{\theta_1}{\theta_0 - 2^2} & \frac{\theta_2}{\theta_0 - 2^2} & \frac{\theta_3}{\theta_0 - 2^2} & \ldots \\
\ldots & \frac{\theta_2}{\theta_0 - 0^2} & \frac{\theta_1}{\theta_0 - 0^2} & 1 & \frac{\theta_1}{\theta_0 - 0^2} & \frac{\theta_2}{\theta_0 - 0^2} & \ldots \\
\ldots & \frac{\theta_3}{\theta_0 - 2^2} & \frac{\theta_2}{\theta_0 - 2^2} & \frac{\theta_1}{\theta_0 - 2^2} & 1 & \frac{\theta_1}{\theta_0 - 2^2} & \ldots \\
\ldots & \frac{\theta_4}{\theta_0 - 4^2} & \frac{\theta_3}{\theta_0 - 4^2} & \frac{\theta_2}{\theta_0 - 4^2} & \frac{\theta_1}{\theta_0 - 4^2} & 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{vmatrix},$$

pointing out in effect that the sum of the non-diagonal elements is resolvable into the product of two infinite series both of which are convergent.

KOCH, H. v. (1890).

[Om upplösningen af ett system lineära likheter mellan ett oändligt antal obekanta. *Öfversigt . . . Akad. Förhandl.* (Stockholm), xlvii. pp. 109–129.]

Koch's position on taking up the subject was different from that of preceding writers, as he had first made himself familiar with

previous work. In an introductory page he describes briefly what had already been done by Hill, Appell, and Poincaré. Strangely enough, however, his mode of attack is less reminiscent of these writers than of Kötteritzsch (*Hist.*, iii. pp. 426–427), whose name he does not mention.

His first section, corresponding in a way to Kötteritzsch's third, is summed up in the proposition that *the infinite set of equations*

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + \ldots. &= u_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + \ldots. &= u_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + \ldots. &= u_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \end{aligned}\right\},$$

where $|\,a_{1n}\,|$ *vanishes for no value of* n *from* 1 *to* ∞, *is equivalent to the set*

$$\left.\begin{aligned} B_{11}x_1 + B_{12}x_2 + \ldots + B_{1n}x_n + \ldots. &= V_1 \\ B_{22}x_2 + \ldots + B_{2n}x_n + \ldots. &= V_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ B_{nn}x_n + \ldots. &= V_n \\ \ldots\ldots\ldots & \end{aligned}\right\},$$

where

$$B_{1\mu} = a_{1\mu}, \quad B_{2\mu} = |\,a_{11}a_{2\mu}\,|, \quad B_{\nu\mu} = |\,a_{11}a_{22}\ldots a_{\nu\mu}\,|,$$
$$V_1 = u_1, \quad V_2 = |\,a_{11}u_2\,|, \quad V_\nu = |\,a_{11}\ldots a_{\nu-1,\nu-1}\,u_\nu\,|.$$

There is also appended the more general result in which the first p equations of the original set are retained in the derived set because of the fact that it is only for $n > p$ that $|\,a_{1n}\,|$ is known not to vanish.

In § 2 (pp. 114–117) he first seeks to find an expression for the difference between the value of any one of the x's in the set

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n = u_r \Big\}_{r=1}^{r=n}$$

and the value of the same x obtained from the set

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n + a_{r,n+1}x_{n+1} = u_r \Big\}_{r=1}^{r=n+1}.$$

Solving for x_p in the n-line set and calling its value $(x_p)_n$, he has of course

$$(x_p)_n = \frac{|\, a_{11} \ldots a_{p-1,\,p-1} u_p a_{p+1,\,p+1} \ldots a_{nn} \,|}{|\, a_{1n} \,|},$$

and substituting in the numerator here the equivalents of the u's given by the other set, he sees that these lengthy elements of the p^{th} column are reducible to binomials, namely, to

$$\begin{aligned} &a_{1p}(x_p)_{n+1} + a_{1,\,n+1}(x_{n+1})_{n+1} \\ &a_{2p}(x_p)_{n+1} + a_{2,\,n+1}(x_{n+1})_{n+1} \\ &\quad \ldots\ldots\ldots\ldots, \end{aligned}$$

and that therefore the numerator becomes

$$|\, a_{1n} \,| \cdot (x_p)_{n+1} + (-1)^{n-p} \cdot |\, a_{1,\,n+1} \,|_{\,n+1,\,p)} \cdot (x_{n+1})_{n+1},$$

where $|\, a_{1,\,n+1} \,|_{(n+1,\,p)}$ stands for the minor of $|\, a_{1,\,n+1} \,|$ got by deleting the $(n+1)^{\text{th}}$ row and the p^{th} column. The result desired thus is

$$(x_p)_n - (x_p)_{n+1} = (-1)^{n-p} \frac{|\, a_{1,\,n+1} \,|_{(n+1,\,p)}}{|\, a_{1n} \,|} (x_{n+1})_{n+1}.$$

By repeated use of it, followed by addition, there is readily deduced the more important result

$$(x_p)_n = (x_p)_p - \frac{|\, a_{1,\,p+1} \,|_{\,p+1,\,p}}{|\, a_{1p} \,|} (x_{p+1})_{p+1} + \frac{|\, a_{1,\,p+2} \,|_{\,p+2,\,p}}{|\, a_{1,\,p+1} \,|} (x_{p+2})_{p+2} - \ldots\ldots,$$

which for $p = 1$ becomes

$$\frac{|\, u_1 a_{22} \ldots\ldots a_{nn} \,|}{|\, a_{11} a_{22} \ldots\ldots a_{nn} \,|} = \frac{u_1}{a_{11}} - \frac{a_{12} \cdot \begin{vmatrix} a_{11} & u_1 \\ a_{21} & u_2 \end{vmatrix}}{a_{11} \cdot |\, a_{12} \,|} + \frac{|\, a_{12} a_{23} \,| \cdot |\, a_{11} a_{22} u_3 \,|}{|\, a_{12} \,| \cdot |\, a_{13} \,|} - \ldots\ldots$$

This last recalls previous equalities of a similar character (*Hist.*, ii. p. 40; iii. pp. 51, 115), and we thus get to see that the result in all its generality was originally formulated by Schweins in 1825 (*Hist.*, i. pp. 173–174). Examination of the two other results of the section has a like outcome, one of them indeed being but a disguised case of the foregoing, and the other the kind of 'extensional' which began to be formulated by Desnanot in 1819.

In § 3 (pp. 118–121) consideration of the infinite set of equations is resumed, the basis of treatment being found in the peculiar form of solution * just obtained for an n-line set, namely,

$$\left.\begin{aligned}(x_1)_n &= \psi_{11} - \psi_{12} + \psi_{13} - \ldots + (-1)^{n-1}\psi_{1n}\\ (x_2)_n &= \psi_{22} - \psi_{23} + \ldots + (-1)^{n-2}\psi_{2n}\\ (x_3)_n &= \psi_{33} - \ldots + (-1)^{n-3}\psi_{3n}\\ &\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\},$$

where $$\psi_{p,r} = \frac{|\,a_{1r}\,|_{(r,p)} \cdot |\,a_{11}\ldots a_{r-1,\,r-1}u_r\,|}{|\,a_{1,\,r-1}\,| \cdot |\,a_{1r}\,|}.$$

The result reached and formulated is essentially as follows: *If the series*

$$\psi_{p,\,p} - \psi_{p,\,p+1} + \psi_{p,\,p+2} - \ldots\ldots$$

be convergent for all positive integral values of p, *the limits being*

$$\xi_1,\ \xi_2,\ \xi_3,\ \ldots\ldots,$$

---

* To avoid any chance of vagueness as to the character of this solution the case where n is 4 may be stated. The equations being

$$\left.\begin{aligned}a_1x + a_2y + a_3z + a_4w &= a_0\\ &\ldots\ldots\ldots\\ d_1x + d_2y + d_3z + d_4w &= d_0\end{aligned}\right\}$$

the solution is

$$\begin{aligned}x &= \frac{a_0}{a_1} - \frac{a_2 \cdot |\,a_1b_0\,|}{a_1 \cdot |\,a_1b_2\,|} + \frac{|\,a_2b_3\,| \cdot |\,a_1b_2c_0\,|}{|\,a_1b_2\,| \cdot |\,a_1b_2c_3\,|} - \frac{|\,a_2b_3c_4\,| \cdot |\,a_1b_2c_3d_0\,|}{|\,a_1b_2c_3\,| \cdot |\,a_1b_2c_3d_4\,|},\\ y &= \frac{|\,a_1b_0\,|}{|\,a_1b_2\,|} - \frac{|\,a_1b_3\,| \cdot |\,a_1b_2c_0\,|}{|\,a_1b_2\,| \cdot |\,a_1b_2c_3\,|} + \frac{|\,a_1b_3c_4\,| \cdot |\,a_1b_2c_3d_0\,|}{|\,a_1b_2c_3\,| \cdot |\,a_1b_2c_3d_4\,|},\\ z &= \frac{|\,a_1b_3c_0\,|}{|\,a_1b_2c_3\,|} - \frac{|\,a_1b_2c_4\,| \cdot |\,a_1b_2c_3d_0\,|}{|\,a_1b_2c_3\,| \cdot |\,a_1b_2c_3d_4\,|},\\ w &= \frac{|\,a_1b_2c_3d_0\,|}{|\,a_1b_2c_3d_4\,|}.\end{aligned}$$

While noting the difference in form of the values of the unknowns, it must not be overlooked that any one of them can be put into the same form as each of the three others: for example, the value of x can by successive condensations be changed into

$$\begin{aligned}&\frac{|\,a_0b_2\,|}{|\,a_1b_2\,|} - \frac{|\,a_2b_3\,| \cdot |\,a_2b_1c_0\,|}{|\,a_1b_2\,| \cdot |\,a_1b_2c_3\,|} + \frac{|\,a_2b_3c_4\,| \cdot |\,a_2b_1c_3d_0\,|}{|\,a_1b_2c_3\,| \cdot |\,a_1b_2c_3d_4\,|},\\ &\qquad \frac{|\,a_2b_3c_0\,|}{|\,a_1b_2c_3\,|} - \frac{|\,a_2b_3c_4\,| \cdot |\,a_1b_2c_3d_0\,|}{|\,a_1b_2c_3\,| \cdot |\,a_1b_2c_3d_4\,|},\\ &\qquad\qquad \frac{|\,a_0b_2c_3d_4\,|}{|\,a_1b_2c_3d_4\,|},\end{aligned}$$

and the value of w by successive dilatations into the form of the value of x.

then

$$x_1, x_2, x_3, \ldots = \xi_1, \xi_2, \xi_3, \ldots .$$

will be a solution of the infinite set of equations

$$a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{r,n}x_n + \ldots = u_r \Big\}_{r=1}^{r=\infty}$$

provided that, any positive quantity δ having been fixed on, it is possible to find a set of positive integers

$$m^\nu : n_1^\nu, n_2^\nu, \ldots, n_m^\nu$$

such that for every index ν the inequality

$$\begin{vmatrix} u_\nu & a_{\nu 1} & \ldots & a_{\nu m} & 0 & \ldots & 0 \\ u_1 & a_{11} & \ldots & a_{1m} & a_{1,m+1} & \ldots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_n & a_{n1} & \ldots & a_{nm} & a_{n,m+1} & \ldots & a_{nn} \end{vmatrix} \div |a_{1n}| < \delta$$

holds good when

$$m > m^\nu \quad \textit{and} \quad n > n^\nu .$$

In regard to this it is only worth noting that the quotient in it which is to be $< \delta$ is readily seen to be merely another form of

$$u - \{a_{\nu 1}(x_1)_n + a_{\nu 2}(x_2)_n + \ldots + a_{\nu m}(x_m)_n\}.$$

The question of the uniqueness of the solution is next considered (pp. 121–124), and then the author following Poincaré strikes off in search of another with the help of the Theory of Functions.*

KOCH, H. v. (1890, 1892).

[Bidrag till teorin för oändliga determinanter. *Öfversigt . . . Vet.-Akad. Förhandl.*, xlvii. pp. 411–431.]

[Sur les déterminants infinis, et . . . . *Acta Math.*, xvi. pp. 217–249, ...]

In these papers, the second of which may be viewed as an enlarged edition of the first, Koch takes Poincaré as his guide, and is consequently on surer ground than before.

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* In the same year (1890) Koch has a paper in the *Öfversigt* (xlvii. pp. 225–236, 499–525) and another in the *Acta Math.* (xv. pp. 53–63), but their subject is the *application* of multilineants to the theory of linear differential equations. The second of the two papers is quite a close translation of the first part of the first.

The determinant considered is in Hill's form

$$| a_{-m,-m} \ldots a_{-1,-1} a_{00} a_{11} \ldots a_{mm} |,$$

that is to say, a determinant extending in four directions with the growth of m, having a central element a_{00} which it is convenient to call the 'origin,' and being of the $(2m+1)^{\text{th}}$ order although appropriately enough denoted by D_m. In regard to convergency the definition used is that D_∞ *is convergent if for every positive quantity* δ *there corresponds a positive integer* n′ *such that the difference between* D_{n+p} *and* D_n *is less than* δ *for every value of* n *greater than* n′ *and for any values of the positive integer* p : *and if such a number* n′ *does not exist* D_∞ *is not convergent.* The initial and fundamental theorem is a widening of Poincaré's first theorem, namely, *The multilineant* $| a_{-\infty,-\infty} \ldots a_{00} \ldots a_{\infty,\infty} |$ *is convergent if both the product of the diagonal elements and the sum of the non-diagonal elements be absolutely convergent.* It is then shown that for convergent multilineants of this type *it is immaterial which diagonal element is taken as the origin,* that *Poincaré's second theorem holds* and *will still hold when the change made in one line is made in any finite number of lines.* Also, that certain of the elementary theorems regarding determinants of *finite* order hold good : namely, those concerning the effect of interchanging two rows or two columns, of transposing rows and thereafter transposing columns so as to have the diagonal elements still in the diagonal, and of increasing a line by multiples of parallel lines : the theorem regarding partition when the elements of a line are polynomials : the multiplication-theorem : Jacobi's theorem regarding a minor of the adjugate : and a case of the extension of Laplace's expansion-theorem as arrived at by Muir in 1873 (*Hist.*, iii. pp. 79–80) and by others. Along with these are discussed (pp. 224–228) permissible developments of the multilineant, the fundamental one being

$$a_{00} + \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11}-1 \end{vmatrix} + \begin{vmatrix} a_{-1,-1}-1 & a_{-10} & a_{-11} \\ a_{0,-1} & a_{00} & a_{01} \\ a_{1,-1} & a_{10} & a_{11} \end{vmatrix} + \begin{vmatrix} a_{-1,-1} & a_{-1,0} & a_{-1,1} & a_{-1,2} \\ a_{0,-1} & a_{00} & a_{01} & a_{02} \\ a_{1,-1} & a_{1,0} & a_{1,1} & a_{12} \\ a_{2,-1} & a_{2,0} & a_{2,1} & a_{2,2}-1 \end{vmatrix} + \ldots$$

or, say,

$$\nabla_1 + \nabla_2 + \nabla_3 + \nabla_4 + \ldots,$$

where manifestly the first two terms have the sum $|a_{00}a_{11}|$, the first three terms the sum $|a_{-1,-1}a_{0,0}a_{1,1}|$, the first four terms the sum $|a_{-1,-1}a_{0,0}a_{1,1}a_{2,2}|$, and the first $2m+1$ terms the sum

$$|a_{-m,-m}\ldots a_{0,0}\ldots a_{m,m}|, \quad i.e. \quad \mathrm{D}_m.$$

The second is the double series

$$\sum_m \sum_k \nabla_{m,k},$$

the third the development originally meant to be indicated by the old notation

$$\sum(\pm\ldots a_{-m,-m}\ldots a_{00}\ldots a_{m,m}\ldots),$$

the fourth, fifth and sixth the various developments got from Laplace's expansion-theorem, the seventh the development according to elements of a chosen row and a chosen column, and the eighth

$$\sum a_{pp} + \sum |a_{pp}a_{qq}| + \sum |a_{pp}a_{qq}a_{rr}| + \ldots,$$

where $p > q > r > \ldots$.

BROWN, E. W. (1892, 1894): POINCARÉ, H. (1893): TISSERAND, F. (1894).

[The elliptic inequalities in the lunar theory. *American Journ. of Math.*, xv. pp. 244–263.]

[Investigations in the lunar theory. *American Journ of Math.*, xvii. pp. 318–358.]

[Les méthodes nouvelles de la Mécanique Céleste. ii. §§ 185–187, pp. 260–275.]

[Traité de Mécanique Céleste. iii. pp. 18–26.]

Section III. (pp. 254–258) of the first paper here and Section V. (pp. 331–334) of the second are closely connected with Hill's work of 1877, the two companion determinants arrived at in Hill's manner being interesting for comparison with the original.

The other writings also deal with Hill's work, but are of course mainly expositional.

ROBERTS, E. H. (1894).

[Note on infinite determinants. *Annals of Math.*, x. pp. 35–49.]

The author here, though familiar with Koch's paper of 1892, proceeds on his own lines. With him a multilineant is viewed as a series of an infinite number of terms, each of which is a product of an infinite number of factors. As a consequence the question of the order of the terms and the question of the order of the factors in a term cannot be left out of consideration. The convention is adopted that the order of the elements in a term shall be that of the rows to which they belong. The difficulty of choosing a companion convention in regard to the order of the terms leads to attention being concentrated on multilineants in which the order of the terms is immaterial. Such multilineants are said to be 'absolutely convergent,' and it is immediately concluded that their values are independent of the choice of a diagonal and an origin. It is then shown that to an absolutely convergent multilineant as thus defined, Poincaré's second theorem applies; also certain of the familiar theorems regarding n-line determinants, namely, the theorem regarding the interchange of two parallel lines, the theorem regarding the multiplication of a line, and the theorem regarding partition when the elements of a line are polynomials. The corresponding theorems in the case of 'semi-convergent' multilineants are given along with these as corollaries. All this leads up to a section (pp. 46–49) on actual 'tests for convergency' with which the paper closes. The summary of it is that three conditions are necessary, namely, (α) that each term be convergent, (β) that each term contain only a finite number of negative elements whose absolute value $\geqq 1$, (γ) that after a certain term is reached every subsequent term must either have an evanescent factor or an infinite number of factors each less than 1. These conditions, however, as is properly pointed out, are not sufficient, so that the reader has a natural feeling of not having been carried far.

KOCH, H. v. (1895).

[Sur la convergence des déterminants d'ordre infini et des fractions continues. *Comptes Rendus . . . Acad. des Sci.* (Paris), cxx. pp. 144–147.]

[Quelques théorèmes concernant la théorie générale des fractions continues. *Öfversigt . . . Akad. Förhandl.* (Stockholm), lii. pp. 101–112.]

The second paper here is merely an extension of the first. In the present connection its only result of interest is the first of three lemmas regarding the multilineant

$$\begin{vmatrix} 1 & \alpha_1 & . & . & \dots \\ \beta_1 & 1 & \alpha_2 & . & \dots \\ . & \beta_2 & 1 & \alpha_3 & \dots \\ . & . & . & . & . \end{vmatrix}.$$

namely, that *for convergence it is necessary and sufficient that the sum of the absolute values of the products* $\alpha_1\beta_1$, $\alpha_2\beta_2$, ... *be convergent.*

CHAPTER XIX.

N-DIMENSIONAL DETERMINANTS, FROM 1880 TO 1900.

In accordance with the action taken and explained in the preceding volume, the present chapter is reduced to a purely bibliographical list. The number of titles is two-thirds more than for the preceding period.

1880. Escherich, G. v. Die Determinanten höheren Ranges und ihre Verwendung zur Bildung von Invarianten. *Denkschr. . . . Akad. d. Wiss.* (Wien) : math.-nat. Cl., xliii. (2) pp. 1–12.

1880. Zajaczkowski, W. Theory of determinants with p suffixes (in Polish). *Pamietnik . . . Akad. . . . Krakowie*, vi. pp. 1–31.

1880. Gegenbaur, L. Ueber Determinanten höheren Ranges. *Denkschr. Akad. d. Wiss* (Wien) : math.-nat. Cl., xliii. (2) pp. 17–32.

1881. Scott, R. F. On some forms of cubic determinants. *Proceed. London Math. Soc.*, xiii. pp. 33–42.

1882. Gegenbaur, L. Zur Theorie der Determinanten höheren Ranges. *Denkschr. . . . Akad. d. Wiss.* (Wien) : math.-nat. Cl., xlvi. (2) pp. 291–298.

1884. Gegenbaur, L. Ueber Determinanten höheren Ranges. *Denkschr. . . . Akad. d. Wiss.* (Wien) : math.-nat. Cl., xlix. (2) pp. 225–230.

1885. Gegenbaur, L. Zur Theorie der Determinanten höheren Ranges. *Denkschr. . . . Akad. d. Wiss.* (Wien) : math.-nat. Cl., l. pp. 145–152.

1885. Schendel, L. Die r-stufige Determinante n^{ten} Grades. *Zeitschrift f. Math. u. Phys.*, xxxii. pp. 185–187.

1888. Gegenbaur, L. Ueber windschiefe Determinanten höheren Ranges. *Denkschr. . . . Akad. d. Wiss.* (Wien) : math.-nat. Cl., lv. pp. 39–48.

1890. Szüts, N. v. On cubic determinants (in Magyar). *Math. és term. tud. értésitö* (Budapest), viii. pp. 220–237 ; or *Math. u. naturw. Berichte aus Ungarn*, viii. pp. 199–217.

1890. Gegenbaur, L. Einige Sätze über Determinanten höheren Ranges. *Denkschr. . . . Akad. d. Wiss.* (Wien) : math.-nat. Cl., lvii. pp. 735–752.

1892. Gegenbaur, L. Ueber einige arithmetische Determinanten höheren Ranges. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), ci. (II a) pp. 425–484.

1892. Gegenbaur, L. Ueber den grössten gemeinsamen Theiler. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), ci. (II a) pp. 1143–1221.

1892. Campbell, J. E. Notes on determinants. *Proceed. London Math. Soc.*, xxiv. pp. 67–79.

1893. Gegenbaur, L. Einige mathematische Theoreme. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), cii. (II a) pp. 549–564.

1895. Szüts, N. v. Zur Theorie der Determinanten höheren Ranges. *Zeitschrift f. Math. u. Phys.*, xl. pp. 113–117.

1898. Cazzaniga, T. Précis d'une théorie élémentaire des déterminants cubiques d'ordre infini. *Math. Annalen*, liii. pp. 272–288.

1898. Waelsch, E. Ueber eine geometrische Behandlungsweise der Elemente der Determinantentheorie. *Monatshefte f. Math. u. Phys.*, ix. pp. 207–214.

1900. Hedrick, E. R. On three-dimensional determinants. *Annals of Math.*, (2) i. pp. 49–67.

Of the numerous text-books published during the period, Scott's (1880) and Pascal's (1897) are the only ones that follow the example of Günther in the preceding period (see above, pp. 78–79, 94).

CHAPTER XX.

BORDERED DETERMINANTS, UP TO 1900.

In the introduction to the former chapter on this branch of our subject reference should have been made to the notice of Binet's bordered determinant of 1811 (*Hist.*, i. pp. 70-71), and to the fuller account of Cayley's memoir of 1843 (*Hist.*, ii. pp. 18–20). The first of the two sections of the latter memoir (pp 64–75) may be said to be wholly concerned with a bordered determinant, namely, the special bilinear function

$$\begin{vmatrix} . & x & y & z & \dots \\ \xi & a_1 & a_2 & a_3 & \dots \\ \eta & b_1 & b_2 & b_3 & \dots \\ \zeta & c_1 & c_2 & c_3 & \dots \\ . & . & . & . & \dots \end{vmatrix}_{n+1}$$

viewed in connection with its linear transformation. His opening and main result is that the discriminant, which prior to transformation is seen to be the adjugate of $|\, a_1 b_2 c_3 \dots |$, A say, is after transformation

$$|\, h_1 k_2 l_3 \dots |\, ' \cdot \mathrm{A} \cdot |\, p_1 q_2 r_3 \dots |\,,$$

the linear substitutions used being

$$\begin{aligned} x &= h_1 x' + h_2 y' + h_3 z' + \dots & \xi &= p_1 \xi' + p_2 \eta' + p_3 \zeta' + \dots \\ y &= k_1 x' + k_2 y' + k_3 z' + \dots & \eta &= q_1 \xi' + q_2 \eta' + q_3 \zeta' + \dots \\ z &= l_1 x' + l_2 y' + l_3 z' + \dots & \zeta &= r_1 \xi' + r_2 \eta' + r_3 \zeta' + \dots \\ \dots & \dots & \dots & \dots \end{aligned}$$

We may add that, from this mode of statement, the corresponding theorem for the general bilinear is easily anticipated.

LINDELÖF, L. (1880).

(See under this heading in Chap. I.)

BURNSIDE, W. S. AND PANTON, A. W. (1881).

[The Theory of Equations, ... xvi + 387 pp. Dublin.]

It is noted (pp. 258, 259) that the product of an even-ordered binary quantic by the corresponding persymmetric determinant (*i.e.* the 'catalecticant') can be expressed by bordering the latter; for example,

$$(ax^4+4bx^3y+\ldots+ey^4)\cdot\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = -\begin{vmatrix} a & b & c & \xi \\ b & c & d & \eta \\ c & d & e & \zeta \\ \xi & \eta & \zeta & . \end{vmatrix},$$

where

$$\xi = ax^2 + 2bxy + cy^2, \quad \eta = bx^2 + 2cxy + dy^2, \quad \ldots\ldots$$

If, however, we write the quantic in the form

x^2	$2xy$	y^2	
a	b	c	x^2
b	c	d	$2xy$
c	d	e	y^2,

the equality is seen to be included in one of Cayley's of 1848 (*Hist.*, ii. pp. 116–117, 341).

KRETKOWSKI, W. (1881).

[O przekształceniach pewnych wielomianów jednorodnych drugiego stopnia. *Pamietnik Akad. . . . w Krakowie*, vii. pp. 69–73.]

What is perhaps worth noting here is the natural extension made from a quadric to a bilinear in the case of the two following theorems (*Hist.*, iii. pp. 95–96): (1) *The determinant got by bordering the discriminant of a quadric by the first differential-coefficients of the quadric is equal to* 4 *times the negative product of the quadric and its discriminant*, (2) *the determinant got by bordering the adjugate of the discriminant of a quadric by the independent variables is equal to the negative product of the quadric by the* $(n-2)^{th}$ *power of the discriminant*, n *being the number of variables*. For example, taking the bilinear

x	y	z	
a_1	a_2	a_3	ξ, or u say,
b_1	b_2	b_3	η
c_1	c_2	c_3	ζ

we now have

$$|a_1b_2c_3| \cdot u = -\begin{vmatrix} a_1 & a_2 & a_3 & \partial u/\partial\xi \\ b_1 & b_2 & b_3 & \partial u/\partial\eta \\ c_1 & c_2 & c_3 & \partial u/\partial\zeta \\ \partial u/\partial x & \partial u/\partial y & \partial u/\partial z & . \end{vmatrix},$$

and

$$|a_1b_2c_3|^{3-2} \cdot u = -\begin{vmatrix} A_1 & A_2 & A_3 & \xi \\ B_1 & B_2 & B_3 & \eta \\ C_1 & C_2 & C_3 & \zeta \\ x & y & z & . \end{vmatrix}.$$

In regard to proofs nothing need be added to what we have already said.

MUIR, T. (1882).

[On a determinant formed by bordering the product of two determinants. *Messenger of Math.*, xi. pp. 161–165.]

The theorem here established is, when stated for the third order, that if *the row-by-row product of* $|a_1b_2c_3|$ *and* $|\alpha_1\beta_2\gamma_3|$ *be bordered* by* 0, a_0, b_0, c_0 *and* 0, α_0, β_0, γ_0 *as row and column respectively, the resulting determinant with* $-$ *prefixed is equal to*

$$|a_0b_1c_2| \cdot |\alpha_0\beta_1\gamma_2| + |a_0b_1c_3| \cdot |\alpha_0\beta_1\gamma_3| + |a_0b_2c_3| \cdot |\alpha_0\beta_2\gamma_3|.$$

As for the proof it may suffice to say that it starts with replacing the 0 of the new row and column by -1 and making at the same time a suitable subtraction. There is a certain advantage in writing the result in the form

$$\left\|\begin{matrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \end{matrix}\right\| \cdot \left\|\begin{matrix} a_0 & a_1 & a_2 & a_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \end{matrix}\right\| - |a_1b_2c_3| \cdot |\alpha_1\beta_2\gamma_3|.$$

* It should have been noted that the same procedure is applicable with a similar effect in the case where it is the product of two *oblong* arrays that is bordered. Thus, starting with the determinant which is got by multiplying

$$\left\|\begin{matrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{matrix}\right\| \quad \text{and} \quad \left\|\begin{matrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{matrix}\right\|$$

we should have reached a sum of *six* products instead of three, the additional three being

$$|a_0b_1c_4| \cdot |\alpha_0\beta_1\gamma_4| + |a_0b_2c_4| \cdot |\alpha_0\beta_2\gamma_4| + |a_0b_3c_4| \cdot |\alpha_0\beta_3\gamma_4|.$$

MUIR, T. (1885).

[On bipartite functions. *Transac. R. Soc. Edinburgh*, xxxii. pp. 461–482.]

What has to be noted here is the theorem (p. 475) that *the product of a so-called bipartite of degree-order* (3, n) *by the* $(n-2)^{th}$ *power of the determinant of its square array is expressible as a bordered determinant of the* $(n+1)^{th}$ *order*: for example, when n is 4, we have

$$\begin{array}{cccc|c} a_1 & a_2 & a_3 & a_4 & \\ \hline b_1 & c_1 & d_1 & e_1 & f_1 \\ b_2 & c_2 & d_2 & e_2 & g_1 \\ b_3 & c_3 & d_3 & e_3 & h_1 \\ b_4 & c_4 & d_4 & e_4 & k_1 \end{array} \cdot |\, b_1c_2d_3e_4 \,|^2 = - \begin{vmatrix} . & a_1 & a_2 & a_3 & a_4 \\ f_1 & B_1 & C_1 & D_1 & E_1 \\ g_1 & B_2 & C_2 & D_2 & E_2 \\ h_1 & B_3 & C_3 & D_3 & E_3 \\ k_1 & B_4 & C_4 & D_4 & E_4 \end{vmatrix}.$$

The theorem is said to hold when the degree-order is (m, n) and the determinant taken is *any one* of the square arrays of the bipartite.

SCHRADER, W. (1887).

[Beiträge zur Theorie der Determinanten. vi+156 pp. Halle.]

In his section on null determinants (pp. 75–80) Schrader gives a theorem regarding $|\, a_1b_2c_3 \ldots . \,|$ when the cofactor of its last element vanishes; for example, when n is 4 and $|\, a_1b_2c_3 \,| = 0$, he has

$$\frac{|\, b_1c_2d_3 \,|}{|\, b_1c_2 \,|} = \frac{|\, a_1c_2d_3 \,|}{|\, a_1c_2 \,|} = \frac{|\, a_1b_2d_3 \,|}{|\, a_1b_2 \,|}.$$

It would be better, however, to view this apart from $|\, a_1b_2c_3d_4 \,|$, portion of which it does not concern, and to class it as a property of an $(n+1)$-by-n array having one of its n-line determinants equal to 0; or, better still, to say simply that *if* $|\, a_1b_2c_3 \,| = 0$, *then*

$$\frac{|\, b_1c_2x_3 \,|}{|\, b_1c_2 \,|} = \frac{|\, a_1c_2x_3 \,|}{|\, a_1c_2 \,|} = \frac{|\, a_1b_2x_3 \,|}{|\, a_1b_2 \,|}$$

whatever x_1, x_2, x_3 *may be*, the reason of course being the proportionality of the rows of $|\, A_1B_2C_3 \,|$.

KÜHNE, H. (1892).

[Beiträge zur Lehre von der n-fachen Mannigfaltigkeit. *Archiv d. Math. u. Phys.*, (2) xi. pp. 353–407.]

Extensive use is here made of determinants, those in the first

three sections (pp. 354–376) being unit-bordered and originating as the cofactor of b in

$$\begin{vmatrix} a_{11}-b & a_{12}-b & a_{13}-b & \dots \\ a_{21}-b & a_{22}-b & a_{23}-b & \dots \\ a_{31}-b & a_{32}-b & a_{33}-b & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

No fresh property is referred to.

ARNALDI, M. (1896).

[Sui determinanti orlati, e sullo sviluppo di un determinanti per determinanti orlati. *Giornale di Mat.*, xxxiv. pp. 209–214.]

The first theorem here concerns the determinant

$$\begin{vmatrix} \cdot & \dots & \cdot & \xi_{11} & \xi_{12} & \dots & \xi_{1n} \\ \cdot & \dots & \cdot & \xi_{r1} & \xi_{r2} & \dots & \xi_{rn} \\ \eta_{11} & \dots & \eta_{1r} & a_{11} & a_{12} & \dots & a_{1n} \\ \eta_{21} & \dots & \eta_{2r} & a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \dots & \cdot & \cdot & \cdot & \dots & \cdot \\ \eta_{n1} & \dots & \eta_{nr} & a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix},$$ or B say.

It may be conveniently enunciated as follows: *If* M *be any* r*-line minor of the* ξ *array,* N *any* r*-line minor of the* η *array, and* P *the minor of* $|a_{1n}|$ *got by deleting the columns and rows of* $|a_{1n}|$ *that are continuations of the columns of* M *and the rows of* N, *then*

$$B = \sum(-1)^{r+\sigma}MNP,$$

where σ *is the sum of the numbers of the columns taken from the* ξ *array and the numbers of the rows taken from the* η *array to form* M *and* N. The proof is simply a double application of Laplace's expansion-theorem.

The second theorem is one less likely to be expected, and is more interesting. It concerns the determinant, D say, got from B by substituting the array

$$\begin{matrix} \delta_{11} & \dots & \delta_{1r} \\ \cdot & \cdot & \cdot \\ \delta_{r1} & \dots & \delta_{rr} \end{matrix}$$

for the array of zeros, and consequently it concerns any determinant whatever. It may be shortly described as giving an expression for D in terms of the minors of $|\delta_{1r}|$ and minors of B. As formulated by the author in symbols it is a little unwieldy, and the gain in clearness is not quite in proportion to the increase in bulk. It simply affirms that D *is equal to an aggregate of products, the first factor of which is any minor of* $|\delta_{1r}|$, *from the order* r *to the order* 0, *and the second factor that minor of* B *which is got by deleting the rows and columns of* B *that are occupied in* D *by the first factor, the first term of the aggregate being thus* $|\delta_{1r}| \cdot |a_{1n}|$ *and the last* $1 \cdot$ B. It is reached somewhat artificially but effectively by changing each element δ_{hk} into $\delta_{hk}+0$ and each element η_{hk} into $0+\eta_{hk}$, and then expanding D as a sum of 2^r determinants with monomial elements. For example, when n is 3 and r is 2, the five-line D

$$= \begin{vmatrix} \delta_{11}+0 & \delta_{12}+0 & \xi_{11} & \xi_{12} & \xi_{13} \\ \delta_{21}+0 & \delta_{22}+0 & \xi_{21} & \xi_{22} & \xi_{23} \\ 0+\eta_{11} & 0+\eta_{12} & a_{11} & a_{12} & a_{13} \\ 0+\eta_{21} & 0+\eta_{22} & a_{21} & a_{22} & a_{23} \\ 0+\eta_{31} & 0+\eta_{32} & a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} \delta_{11} & \delta_{12} & \xi_{11} & \cdots \\ \delta_{21} & \delta_{22} & \xi_{21} & \cdots \\ . & . & a_{11} & \cdots \\ . & . & a_{21} & \cdots \\ . & . & a_{31} & \cdots \end{vmatrix} + \begin{vmatrix} \delta_{11} & . & \cdots \\ \delta_{21} & . & \cdots \\ . & \eta_{12} & \cdots \\ . & \eta_{22} & \cdots \\ . & \eta_{32} & \cdots \end{vmatrix} + \begin{vmatrix} . & \delta_{12} & \cdots \\ . & \delta_{22} & \cdots \\ \eta_{11} & . & \cdots \\ \eta_{21} & . & \cdots \\ \eta_{31} & . & \cdots \end{vmatrix} + \begin{vmatrix} . & . & \cdots \\ . & . & \cdots \\ \eta_{11} & \eta_{12} & \cdots \\ \eta_{21} & \eta_{22} & \cdots \\ \eta_{31} & \eta_{32} & \cdots \end{vmatrix}$$

$$= |\delta_{11}\ \delta_{22}|\ |a_{11}\ a_{22}\ a_{33}|$$

$$+\delta_{11}\begin{vmatrix} . & \xi_{21} & \xi_{22} & \xi_{23} \\ \eta_{12} & a_{11} & a_{12} & a_{13} \\ \eta_{22} & a_{21} & a_{22} & a_{23} \\ \eta_{32} & a_{31} & a_{32} & a_{33} \end{vmatrix} - \delta_{21}\begin{vmatrix} . & \xi_{11} & \xi_{12} & \xi_{13} \\ \eta_{12} & a_{11} & a_{12} & a_{13} \\ \eta_{22} & a_{21} & a_{22} & a_{23} \\ \eta_{32} & a_{31} & a_{32} & a_{33} \end{vmatrix} - \delta_{12}\begin{vmatrix} . & \xi_{21} & \xi_{22} & \xi_{23} \\ \eta_{11} & a_{11} & a_{12} & a_{13} \\ \eta_{21} & a_{21} & a_{22} & a_{23} \\ \eta_{31} & a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$+\delta_{22}\begin{vmatrix} . & \xi_{11} & \xi_{12} & \xi_{13} \\ \eta_{11} & a_{11} & a_{12} & a_{13} \\ \eta_{21} & a_{21} & a_{22} & a_{23} \\ \eta_{31} & a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} . & . & \xi_{11} & \xi_{12} & \xi_{13} \\ . & . & \xi_{21} & \xi_{22} & \xi_{23} \\ \eta_{11} & \eta_{12} & a_{11} & a_{12} & a_{13} \\ \eta_{21} & \eta_{22} & a_{21} & a_{22} & a_{23} \\ \eta_{31} & \eta_{32} & a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

BONOLIS, A. (1896).

(See under this heading in Chap. I.)

NANSON, E. J. (1898).

[On partial compounds. *Messenger of Math.*, xxviii. pp. 17–19.]

Here we have a wide generalization of Muir's theorem (1882) on the bordering of a product-determinant, the border, for one thing, being no longer restricted to a single line. The more important part of the result is in effect as follows: *If the row-by-row product of the first* k *columns of two* m*-by-*n *arrays* A, B *be bordered by the remaining columns, the determinant of the result is equal to the sum of the terms in the expansion of* AB *which involve all the bordering elements.* The mode of proof and the theorem itself will be readily understood from considering the case where $m, n, k = 3, 6, 4$. The arrays being

$$\begin{matrix} a_1 & a_2 & \dots & a_6 \\ b_1 & b_2 & \dots & b_6 \\ c_1 & c_2 & \dots & c_6 \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha_1 & \alpha_2 & \dots & \alpha_6 \\ \beta_1 & \beta_2 & \dots & \beta_6 \\ \gamma_1 & \gamma_2 & \dots & \gamma_6 \end{matrix},$$

we have

$$\begin{vmatrix} A_1 & A_2 & A_3 & a_5 & a_6 \\ B_1 & B_2 & B_3 & b_5 & b_6 \\ C_1 & C_2 & C_3 & c_5 & c_6 \\ a_5 & \beta_5 & \gamma_5 & . & . \\ a_6 & \beta_6 & \gamma_6 & . & . \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & . & . \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & . & . \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & . & . \\ . & . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . & 1 \end{vmatrix} \cdot \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & . & . & a_5 & a_6 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & . & . & \beta_5 & \beta_6 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & . & . & \gamma_5 & \gamma_6 \\ . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . & . \end{vmatrix}$$

$$= |a_1 b_5 c_6|\,|\alpha_1 \beta_5 \gamma_6| + |a_2 b_5 c_6|\,|\alpha_2 \beta_5 \gamma_6| + |a_3 b_5 c_6|\,|\alpha_3 \beta_5 \gamma_6| + |a_4 b_5 c_6|\,|\alpha_4 \beta_5 \gamma_6|.$$

CHAPTER XXI.

DETERMINANTS HAVING INVARIANT FACTORS, FROM 1851 TO 1900.

THE first recognition of the importance of the factors here referred to dates from 1851, and was due to Sylvester: but the methodical study of them under a special name came only after a considerable interval. The different varieties of determinant that fall to be dealt with will appear as we proceed; but it may be said generally that a determinant having invariant-factors is one whose elements are such that every set of its minors may legitimately be expected to have a highest-common-factor, and that the invariant-factors are factors selected from the determinant's ordinary factors in accordance with specified conditions in order to be of use for a definite purpose. Thus, while the determinant

$$\begin{vmatrix} \lambda-\mu & 3\lambda & 4\lambda \\ 2\lambda & 2\lambda-\mu & 4\lambda \\ 2\lambda & 3\lambda & 3\lambda-\mu \end{vmatrix},$$

which is equal to

$$(\lambda+\mu)^2(8\lambda-\mu),$$

is viewed in elementary algebra as having the six factors

$$1, \quad \lambda+\mu, \quad (\lambda+\mu)^2, \quad 8\lambda-\mu, \quad (\lambda+\mu)(8\lambda-\mu), \quad (\lambda+\mu)^2(8\lambda-\mu),$$

and is also there spoken of as having the *linear* factors

$$\lambda+\mu, \quad \lambda+\mu, \quad 8\lambda-\mu,$$

we are now face to face with the group

$$(\lambda+\mu)(8\lambda-\mu), \quad \lambda+\mu,$$

the members of which are called in English—in accordance with one usage at least, and the other usage is analogous—the *invariant-factors* of the determinant. The name is not altogether commendable, as there are other factors to which it might appropriately be given: it is preferable, however, to a translation of the German word 'Elementartheiler.' The fact of the group being singled out for use in the application of certain criteria suggests that the name 'critical factors' would be fairly suitable, and that at any rate in the final adjustment of nomenclature the word 'critical' might be used where a second adjective is wanted to promote clearness.

SYLVESTER, J. J. (1851).

[An enumeration of the contacts of lines and surfaces of the second order. *Philos. Magazine*, (4) i. pp. 119–140; or *Collected Math. Papers*, i. pp. 219–240.] *

Sylvester's mode of attacking his geometrical problem involves the consideration of $\square(U+\lambda V)$, the so-called determinant of $U+\lambda V$, where

$$U \equiv ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy,$$
$$V \equiv ax^2 + \beta y^2 + \gamma z^2 + 2a'yz + 2\beta' zx + 2\gamma' xy;$$

—that is to say, the discriminant

$$\begin{vmatrix} a+a\lambda & c'+\gamma'\lambda & b'+\beta'\lambda \\ c'+\gamma'\lambda & b+\beta\lambda & a'+a'\lambda \\ b'+\beta'\lambda & a'+a'\lambda & c+\gamma\lambda \end{vmatrix}.$$

His words are that "$U+\lambda V$ will be the function, the properties of whose complete determinant and of the minor systems of determinants belonging to it will serve to specify the nature of the contact." In the three pages of introduction, from which these words

* This is the last of a series of four papers touching more or less on the same subject. The three others are:

On the intersections, contacts, ... *C. and D. Math. Journ.*, v. (1850) pp. 262–282;

On a new class of theorems .... *Philos. Magazine*, xxxvii. (1850) pp. 213–218.

Additions to the Articles .... *Philos. Magazine*, xxxvii. (1850) pp. 363–370.

The four are numbered 22, 24, 25, 36 in the *Collected Math. Papers*, i.

are taken, he gives very little account of the theory on which his procedure depends: this has to be gathered in considerable part from the sixteen pages devoted to the critical examination of the various cases which he skilfully succeeds in bringing to light. There are indeed only two results actually formulated. The first is the property of invariance, namely, that "if all the minors of any order of $\square(U+\lambda V)$ have a factor $\lambda+\epsilon$ in common, this factor will continue common to the same system of minors when U and V are simultaneously transformed" by a linear substitution performed on their variables. The other is that "if any factor K^e enter into all the r^{th} minors of W,* and if K^i be the highest power of K common to all the $(r+1)^{th}$ minors, then K^{2e-i} will be a common factor to all the $(r-1)^{th}$ minors." By the r^{th} minors Sylvester means the minors having $n-r$ rows: his corollary, when we follow the other usage, is *If the* k*-line minors have a common factor* p^a, *the* (k+h)*-line minors contain at lowest the factor* $p^{(h+1)a}$. In a footnote he points to the natural extension of the theory from quadric to bilinear functions (the latter called by him 'binary'), giving the instance where the twin sets of variables are x, y, z and x', y', z', and the determinant is $|\, ab'c'' \,|$.

Through using the requisite factors for purposes of classification in ordinary geometry, he is led to think of the corresponding problem of enumeration for higher dimensions, and to give the number of cases when the determinant is of the 5^{th}, 6^{th}, 7^{th}, 8^{th} orders.

CAYLEY, A. (1854).

[Recherches sur les matrices dont les termes sont des fonctions linéaires d'une seule indeterminée. *Crelle's Journ.*, l. pp. 313–317; or *Collected Math. Papers*, ii. pp. 216–220.]

Cayley avowedly continues a particular part of the work of Sylvester. His n-line determinant (or square matrix) is any one whose elements are all linear functions of one and the same variable,† and thus differs from Sylvester's only in not being restricted to

* More properly $\square(W)$, W being any quadratic function.

† Or, any one whose array is a sum of multiples of two arrays,—a *bimatricant*, afterwards occurring in a quite different connection (see below p. 492).

axisymmetry. In the first place he seeks to find out, in the case of such a determinant having $(\lambda-a)^{\alpha}$ for a factor, the highest power of $\lambda-a$ that may be a factor of all the $(n-1)$-line minors, the highest power that may be a factor of all the $(n-2)$-line minors, and so on. The result, in effect already reached by Sylvester, is that every three consecutive exponents α, β, γ of the series of such powers is subject to the condition $\alpha-2\beta+\gamma \geqq 0$; or, as Cayley puts it, that the series of exponents has not only its first differences non-negative but its second differences also. From knowing this and that $\alpha \leqq n$, it is not difficult to find by trial, as doubtless Sylvester did, the various different ways in which the factors of a determinant may be distributed. Cayley, however, methodizes the work, gives the generating function from which can be found the number of different possible cases for any particular value of n, and devises a symbolical label for representing briefly the facts in each case. Thus, n being 4, the number of possible cases is found to be 12, in agreement with Sylvester, and one of the 12 labels is

$$\begin{array}{|ccc|}\hline 3 & 2 & 1 \\ 1 & & \\ \hline\end{array},$$

the first row of which indicates that the determinant contains $(\lambda-a)^3$, that its three-line minors all contain $(\lambda-a)^2$, and its two-line minors all contain $(\lambda-a)^1$: and the second row to indicate that the determinant contains the factor $(\lambda-b)^1$ while none of its minors does. Sylvester's numbers for the cases where n is greater than 4 receive slight corrections.

SYLVESTER, J. J. (1854).

[Note on the "Enumeration of the contacts of lines and surfaces of the second order." *Philos. Magazine*, (4) vii. pp. 331–334; or *Collected Math. Papers*, ii. pp. 30–33.]

A further advance, suggested by Cayley's paper, is here made in the denumeration problem, the first eleven numbers of cases being given, namely,

1, 3, 6, 14, 27, 58, 111, 223, 424, 817, 1527.

SMITH, H. J. S. (1861).*

[On systems of linear indeterminate equations and congruences. *Philos. Transac. R. Soc. London*, cli. pp. 293–326 ; or *Coll. Math. Papers*, i. pp. 367–406.]

The equations here dealt with have integers for their coefficients, and the solutions are wanted in integers ; naturally, therefore, the special form of determinant or oblong array which occurs is that having only integers for elements. Although almost the whole of the memoir is of more or less interest to us (*Hist.*, iii. pp. 5, 83–84), the sections which concern our present special heading are §§ 12–16. A beginning is made with the theorem that *If a rectangular array be pre-multiplied by a unit determinant, the highest-common-factor of any column of minors is the same in the product as in the given array.* A kind of companion theorem to this is next given, namely, that *If a determinant be post-multiplied by a rectangular array whose primary minors are mutually prime, the highest-common-factor of any row of minors is the same in the product as in the given determinant.* Then by a combination of the pair there is deduced the important result that *In any* n*-line determinant of integers the numbers* $\nabla_n, \nabla_{n-1}, \ldots, \nabla_1$, *which represent the highest-common-factors of all the minors of order* n, n−1, . . . , 1 *respectively, will remain unchanged when the determinant is pre-multiplied by any unit-determinant and post-multiplied by any* n*-row rectangular array whose primary minors are mutually prime.* The theorems of §§ 13, 14 form a similar set of three, their subject being possible transformations of a determinant of integers through the multiplication of it by unit-determinants. The first of them is that *Any determinant of*

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* In view of what has already been said (*Hist.*, iii. pp. 5, 11) on this paper and another of Smith's, it is just worth noting by way of correction in regard to notation that in it (p. 297) he writes a set of equations in the form

$$\left.\begin{matrix} a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = 0, \\ i = 1, 2, 3, \ldots, n, \end{matrix}\right\}$$

or even (p. 298) in the form $\sum_{k=1}^{k=n} a_{ik}x_k = 0$;

and a determinant (p. 302) in the form

$$|\, a_{rs} \,|, \quad \begin{matrix} r = 1, 2, 3, \ldots, n, \\ s = 1, 2, 3, \ldots, n. \end{matrix}$$

integers can through post-multiplication by a unit-determinant be transformed into

$$\begin{vmatrix} \mu_1 & r_{12} & r_{13} & \cdots & r_{1n} \\ . & \mu_2 & r_{23} & \cdots & r_{2n} \\ . & . & \mu_3 & \cdots & r_{3n} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & \mu_n \end{vmatrix},$$

where the μ*'s are positive, the* r*'s not negative, and each* μ *is greater than any* r *in the same row with it.* The companion theorem is obtained by substituting in this 'pre-' for 'post-' and 'column' for 'row': and the important deduced result is that *If* $|a_{rs}|_n$ *be a determinant of integers, and* $\nabla_n, \nabla_{n-1}, \ldots, \nabla_1$ *the highest-common-factors of its successive sets of minors, then two unit-determinants* U′, U″ *can be found such that*

$$|a_{rs}|_n \equiv U' \times \begin{vmatrix} \nabla_n/\nabla_{n-1} & . & . & \cdots & . \\ . & \nabla_{n-1}/\nabla_{n-2} & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & \nabla_1/1 \end{vmatrix} \times U''.$$

Involved in the proof of this is a property of the ∇'s, namely, that

$$\nabla_s/\nabla_{s-1} \textit{ is divisible by } \nabla_{s-1}/\nabla_{s-2}:$$

and, in a short paragraph immediately following, it is shown that the same holds good for the ∇'s of an oblong array. Of like incidental character is the theorem that *If in any rectangular array each* s*-line minor be divided by the highest-common-factor of its own* (s−1)*-line minors, the highest-common-factor of all the quotients thus obtained is* ∇_s/∇_{s-1}—a theorem which, he says, will bear the extension corresponding to the substitution herein of $s-k$ for $s-1$. The unextended theorem practically defines ∇_s/∇_{s-1} as a highest-common-factor; and, viewing it in this light, he draws the conclusion that *If a matrix be contained in another, not only is* ∇_s *in the case of the former divisible by the corresponding highest-common-factor in the case of the latter. but also* ∇_s/∇_{s-1} *is divisible by the corresponding quotient.* In conclusion, using p^{i_s} to stand for 'the highest power of a prime p contained in all the s-line minors of an array,' he gives as an alternative to one of the later results: *If all the primary minors of a particular* s*-line minor be divisible by* $p^{i_{s-1}+m}$, *the latter minor is divisible by* p^{i_s+m}.

WEIERSTRASS, K. (1868).

[Zur Theorie der bilinearen und quadratischen Formen. *Monatsb. . . . Akad. d. Wiss.* (Berlin), Jahrg. 1868, pp. 310–338; or *Math. Werke*, ii. pp. 19–44.]

With Weierstrass we find ourselves brought back to the viewpoint of Sylvester (1851), the subject-matter now being a family of loci, $pP+qQ$, of which the parent forms P, Q are both bilinears in $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. At the outset a page (p. 111) is devoted to explaining his use of the term *elementary-divisor* ('Elementartheiler'). In condensed form what he says is that *If* $ap+bq$ *be any linear factor of* [P, Q] *the determinant of* $pP+qQ$, *and* $(ap+bq)^{l_k}$ *be the highest power of* $ap+bq$ *contained in all the* $(n-k)$*-line minors of* [P, Q], *then*

$$(ap+bq)^{l_0-l_1}, \quad (ap+bq)^{l_1-l_2}, \quad \ldots$$

are called 'elementary-divisors' of [P, Q], it having been previously proved that

$$l_{k-1} > l_k \quad \text{if} \quad l_k > 0$$

and

$$l_{k-1} = 0 \quad \text{if} \quad l_k = 0.$$

The property of invariance, imperfectly stated by Sylvester, is then carefully formulated and proved (pp. 312–313), the enunciation being in effect that *If bilinear forms* P, Q *be changed by a pair of linear substitutions of non-zero determinant into* P′, Q′, *then the determinants of* $pP+qQ$, $pP'+qQ'$ *agree in their elementary divisors.* The converse theorem is next taken up, namely, that *If the determinant of a family of bilinears* $pP+qQ$ *agree in their elementary-divisors with the determinant of the family* $pP'+qQ'$, *then a pair of linear substitutions can be found which will simultaneously transform* P *into* P′ *and* Q *into* Q′. The proof of it is very much more troublesome than the other, more than half of the whole paper being occupied directly or indirectly with the discussion of it. In the first place an auxiliary transformation of P and Q has to be brought about, the most prominent peculiarity of which is that the new companion-like expressions obtained for P and Q involve entities connected with the determinant of $pP+qQ$,—indeed, the expressions are fully determinable as soon as the 'elementary divisors'

of the said determinant are known. To the establishment and elucidation of this transformation all the second section (pp. 314–324) is devoted,—a fact that alone tends to suggest something of special difficulty or complexity in the process. When it has been accomplished, however, the application of the result to achieve the desired proof is comparatively easy, and along with related matters, such as the change of P, Q into the forms

$$a_1X_1Y_1 + \ldots + a_nX_nY_n, \quad b_1X_1Y_1 + \ldots + b_nX_nY_n,$$

is fully dealt with in §§ 3, 4 (pp. 325–331). The remaining section (pp. 332–338) is given up to the consideration of the analogous theorems which hold when P and Q, instead of being bilinears, are quadrics.

SMITH, H. J. S. (1873).

[On the arithmetical invariants of a rectangular matrix of which the constituents are integral numbers. *Proceed. London Math. Soc.*, iv. pp. 236–240 ; or *Coll. Math. Papers*, ii. pp. 67–71.]

This is a short but not unimportant continuation of the writer's paper of 1861, the invariants referred to being the ∇'s of that paper,—that is to say, ∇_s is the highest-common-factor of the s-line minors of the given rectangular array of integers. Restating one of the last theorems of his former paper, namely, that defining ∇_s/∇_{s-1} as a highest-common-factor, he thence makes the deduction that *Any* s-*line minor, which is not divisible by a higher power of a prime* p *than* ∇_s *is, contains at least one primary minor which is not divisible by a higher power of* p *than* ∇_{s-1} *is*. This he also extends a little, while shortening the enunciation into *Any minor which is not divisible in excess by* p^σ *contains at least one primary minor which is not divisible in excess by* p^σ. He then supplies to each of these three theorems a companion, or, as he terms it, a 'reciprocal' theorem, introducing for the purpose the use of *major* as the counter term to minor,—that is to say, so that B being a minor of A, A is a major of B. Of the new three the basic theorem is that *If each* $(s-1)$-*line minor be divided by the highest-common-factor of its own first majors, the highest-common-factor of all the fractions thus obtained is* ∇_{s-1}/∇_s: *in other words,* ∇_s/∇_{s-1} *is the lowest-common-multiple*

of the denominators of the fractions. The second of the two others is that *Any minor which is not divisible in excess by* p^σ *is contained in at least one first major which is not divisible in excess by* p^σ. When the given array is square, the adjugate array can be taken into consideration, and there thus originates a self-reciprocal theorem, which in its final extended form is that *In any determinant of integers the highest-common-factor of the* (s−k)*-line minors pertaining to a given* s*-line minor is identical with the highest-common-factor of the* (s+k)*-line majors pertaining to the same minor, so far as factors prime to the determinant are concerned.* The same final extension, namely, from 1 to k, is given also to the basic theorems of the two sets of three above.

SMITH, H. J. S. (1873).

[On systems of linear congruences. *Proceed. London Math. Soc.*, iv. pp. 241–249 ; or *Collected Math. Papers*, ii. pp. 71–80.]

Here (pp. 244–245) is dealt with a theorem which belongs in strictness to the subject of the foregoing paper, namely, *If in an* m*-by-*n *array of integers* $(m \leqq n)$ Q_m, Q_{m-1}, ... *be the highest-common-factors of the various sets of minors, and* q_s, q_{s-1}, ... *those of an* s*-by-*n *array forming part of the former, then* Q_n/Q_{n-t} *is divisible by* q_s/q_{s-t}. It is also worth noting that at the close (§ 11) he gives a proof that his important transformation-theorem of 1861 has, as he says, "an infinite number of solutions,"—in other words, that U′, U″ in it admit of endless variety.

STICKELBERGER, L. (1878).

[Ueber Schaaren von bilinearen und quadratischen Formen. *Crelle's Journ.*, lxxxvi. pp. 20–43.]

Seeing that Stickelberger in the study of his subject had read Darboux as well as Weierstrass and Kronecker, we are not unprepared to find him resort to the bordering of the discriminant in dealing with the different sets of its minors (*Hist.*, iii. p. 440 ; p. 301 note *). Unfortunately, this does not lead to conciseness in the statement of results. Of the few theorems that directly concern us, the fundamental one (p. 33) is in effect as follows : *The dis-*

criminant of the family of bilinears $\lambda A - B$ *being non-zero and denoted by* $|c_{1n}|$, *or* W *say, and*

$$\begin{vmatrix} c_{11} & \dots & c_{1n} & v_{11} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n1} & \dots & c_{nn} & v_{n1} \\ u_{11} & \dots & u_{1n} & . \end{vmatrix}, \quad \begin{vmatrix} c_{11} & \dots & c_{1n} & v_{11} & v_{12} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{n1} & \dots & c_{nn} & v_{n1} & v_{n2} \\ u_{11} & \dots & u_{1n} & . & . \\ u_{21} & \dots & u_{2n} & . & . \end{vmatrix}, \quad \dots\dots$$

be denoted by

$$W_{+1} \quad , \quad W_{+2} \quad , \quad \dots\dots$$

if each of the W*'s up to and including* $W_{+(\nu-1)}$ *be not divisible by a higher power of* $\lambda - c$ *than are the coefficients of the bilinear form got from the said* W *by substituting the variables* $\xi_1, \dots, \xi_n$ *for the last row of* u*'s and* $\eta_1, \dots, \eta_n$ *for the last column of* v*'s, then* $W_{+\nu}$ *is not divisible by a higher power of* $\lambda - c$ *than are all the* $(n-\nu)$*-line minors of* W. The proof, it has to be noted, is dependent on the change of $\lambda A - B$ into Weierstrass' 'reduced' form. What we are finally led up to is the following (pp. 38–39): *The discriminant of the family of bilinears* $\lambda A - B$ *being non-zero and denoted by* W, *and*

$$W, w_1, w_2, \dots, w_{\nu-1}$$

being such that each w *is a primary minor of the one immediately preceding it, and contains no higher power of* $\lambda - c$ *than any other of the set of primary minors to which it belongs, then* w_ν *contains* $\lambda - c$ *in no higher power than any other* $(n-\nu)$*-line minor of* W.

FROBENIUS, G. (1878).

[Theorie der linearen Formen mit ganzen Coefficienten. *Crelle's Journ.*, lxxxvi. pp. 146–208.]

The title of this extensive paper at once recalls Smith's of 1861; but, as the writer had been led to the investigation by the study of a problem of bilinear forms, it is not unnatural to find that a goodly portion of his space is occupied with the latter. This being the case, we note with interest that he is acquainted with practically all the papers which we have dealt with up to this point.

So far as our special subject is concerned, he opens (p. 148) quite in the style of Smith, namely, with the highest-common-factors of the various sets of minors of a determinant of integers $|a_{\alpha\beta}|_n$.

Smith's quotients of invariants, ∇_s/∇_{s-1}, he deals with under the symbol E_s; but, using Weierstrass' word, he calls E_s the s^{th} 'elementary-divisor.' It must be noted, too, that the determinant is not limited to being non-zero, but is taken of non-zero rank r: we are also given to understand in passing that the elements, instead of being taken as integers, might have been taken as 'integral functions of a parameter.' But the greatest difference of all is that $|a_{\alpha\beta}|_n$ is not studied directly by itself: its every element $a_{\alpha\beta}$ is at once attached to a product of variables $x_\alpha y_\beta$, and thereafter the determinant appears merely as the discriminant of a bilinear form. The transformation of bilinears is then dealt with, particularly the reduction to the form

$$h_1x_1y_1 + h_2x_2y_2 + \ldots . + h_rx_ry_r :$$

and it is only when p. 159 is reached that we arrive at elementary theorems like the divisibility of ∇_s by ∇_{s-1} and of E_s by E_{s-1}. Following on this comes a section (§ 6) headed 'Simple * and Compound Elementary-Divisors,' opening with a theorem stated in Cayley's manner, namely, that *The exponents which any particular prime bears in* ∇_r, ∇_{r-1}, ... *form a series whose first and second differences are non-negative*; but containing on next page an equally familiar theorem stated quite differently, namely, *If* $\Sigma a_{\alpha\beta}x_\alpha y_\beta$ *be in any way by unimodular substitutions transformed into* $\Sigma h_\lambda x_\lambda y_\lambda$, *the number of pairs of variables in the latter is* r, *and the* h's *form a system of compound elementary-divisors of* $\Sigma a_{\alpha\beta}x_\alpha y_\beta$. Here it will even be observed that it is the form, not the determinant, that is credited with the elementary-divisors.

Most of the remaining sections (pp. 165–208) are concerned with linear equations and congruences.

FROBENIUS, G. (1879).

[Theorie der linearen Formen mit ganzen Coefficienten. *Crelle's Journ.*, lxxxviii. pp. 96–116.]

This is a continuation of the paper just reported on. The section which most directly concerns us is the last (pp. 113–116). It opens

* The power to which any single prime is raised in any elementary-divisor is called a '*simple* elementary-divisor.'

with what is called an extension of a theorem of Smith's, namely, *In order that a bilinear form* B *may be linearly transformable from another* A *it is necessary and sufficient that* B *does not exceed* A *in non-zero rank and that any invariant-factor of* B *be divisible by the corresponding invariant-factor of* A. A more important theorem, however, is his own, being only foreshadowed by Smith, namely, *The highest-common-factor of the* s*-line minors of* ABC . . . *is divisible by the product of the highest-common-factors of the* s*-line minors of* A, B, C, . . . ; *and the* s^{th} *invariant-factor of* ABC . . . *is the lowest-common-multiple of the* s^{th} *invariant-factors of* A, B, C, ... This immediately leads up to Smith's theorem about ∇_s/∇_{s-1} being itself a highest-common-factor, and thence to its author's deduction from it which Frobenius here puts in the form—*If an* s*-line minor of* A *contains a prime in no higher power than do the other* s*-line minors, then among the* τ*-line minors of the said minor there is one that contains the said prime in no higher power than do all the* τ*-line minors of* A—pointing out also that from it there readily follows Stickelberger's theorem of the year before.

VOSS, A. (1889).

[Ueber einen Satz aus der Theorie der Determinanten. *Sitzungsb. . . . Akad. d. Wiss.* (München), xix. pp. 329–339.]

The theorem in question is Stickelberger's first, paraphrased by us above, the writer's merit being that besides improving it he establishes it, as seems natural, on a purely determinantal basis. Although he shortens the writing of the bordered determinants by using Salmon's notation (*Hist.*, iii. pp. 434–435, 437), the proof occupies much space.

SAUVAGE, L. (1891, 1893).

[Théorie des diviseurs élémentaires et applications. *Ann. de l'Éc. norm. sup.*, (3) viii. pp. 285–340.]

[Compléments à la théorie des diviseurs élémentaires. *Ann. de l'Éc. norm. sup.*, (3) x. pp. 9–42.]

Valuable mainly, but not exclusively, as an exposition and a continuation of the work of Weierstrass and of Darboux.

FROBENIUS, G. (1894).

[Ueber die Elementartheiler der Determinanten. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), 1894, pp. 31–44.]

Hitherto Frobenius' contributions to the theory of invariant-factors have appeared in memoirs dealing directly or indirectly with bilinear forms. Now he devotes to it alone a paper of a dozen pages, there being no reference to bilinears save in the historical introduction; and, what is more important, no mixing up of the technicalities of the two subjects.

At the outset he notes a possible further extension in the nature of the elements, which may now, without calling for any serious change of treatment, be integral magnitudes belonging to one and the same domain of rationality. Incidentally he gives 'elementary-invariant' as an alternative for Weierstrass' 'elementary-divisor.' Smith's term 'major' is adopted, being translated into 'Super-determinante'; and for Smith's 'not divisible in excess (by p)' and his own still longer expression he substitutes 'regular (in regard to p)'—a convenient change leading to considerable shortening in the formulation of theorems. Of the ten theorems enunciated and established, the first to be noted is the fourth, namely, *The product,* $M_s M_t$, *of any* s*-line minor and any* t*-line minor is divisible by the product of the highest-common-factor of all the* (t−u)*-line minors of* M_t *and the highest-common-factor of all the* (s+u)*-line majors of* M_s, t *being* $>$ s *and* u $\not>$ t−s. The fifth theorem is an easy deduction from the first and second, namely, *If a minor* M *and one of its minors* N *be both regular, there exists a regular minor which is a minor of* M *and a major of* N. In the next section (§ 2), after showing that *by mere transposition of rows and of columns a determinant may be so changed as to have the first of each of its sets of minors regular,* he takes up the case where the basic determinant is axisymmetric. Here his first established result is that *If in an axisymmetric determinant the first* ρ*-line minor be regular, and none of its coaxial primary majors be regular, then the first* $(\rho+2)$*-line minor is regular.* From this he is led to the theorem that *By mere transposition of rows and of columns an axisymmetric determinant may be changed into another such that if its first* ρ*-line minor be non-regular, then both the first*

$(\rho-1)$-line minor and the first $(\rho+1)$-line minor are regular, and $e_{\rho+1} = e_\rho$. A simple corollary to this but entitled 'Theorem VI.' is that *If in an axisymmetric determinant transformed as in the preceding* $e_{\sigma+1} > e_\sigma$, *then the first σ-line minor is regular.* As a consequence of these it easily follows that *we can have the first* r*-line minor always regular.* The theorems of the next section (§ 3) we have already had under notice, two being due to Smith and one to Frobenius himself. As the two former, however, are now stated in regard to a single prime factor p, a second enunciation of them is not uncalled for. In their new form they are: 1°. *If an* s*-line minor* M *be divided by the highest-common-factor of its primary minors, the quotient contains* p *in the power* E_s *at least; and, if* M *is regular, in that power exactly.* 2°. *If the highest-common-factor of the primary majors of an* $(s-1)$*-line minor* N *be divided by* N, *the numerator of the reduced quotient contains* p *in the power* E_s *at most; and, if* N *is regular, in that power exactly.* The next section (§ 4) is devoted to a determinantal equality which is quite independent of the subject of invariant-factors, being concerned simply with the summation of a peculiar inflated aggregate of products of pairs of determinants. In this all the first factors of the products are minors of any fixed rectangular array of $|a_{1n}|$, every minor of every order, even the 0^{th}, being taken; the second factor is in every case the cofactor of the first in $|a_{1n}|$; the sign prefixed to any product is + or − according as the first factor of the product is of even or odd order; and the sum of the whole is the determinant got from $|a_{1n}|$ by making all the elements of the fixed array zeros.* The contents of the last section are two additional demonstrations, one of Theorem IX. by Frobenius himself and one suggested by Hensel of Theorem IV.

HENSEL, K. (1894).

[Ueber reguläre Determinanten, und die aus ihnen abgeleiteten Systeme. *Crelle's Journ.*, cxiv. pp. 25–30.]

Without extending the usage of 'regular,' Hensel's interesting result is that *If a given minor be equal to the highest-common-factor*

* In Frobenius' statement, which involves a considerable number of symbols, there is a dangerous misprint of *r* for *s* in the third line.

of its set, then the highest-common-factors of its own sets of minors and majors in ascending order are equal to $\nabla_1, \nabla_2, \ldots, \nabla_n$ *respectively.* He also deduces and formulates the analogous theorem where the equality, instead of being absolute, is limited to the highest power of a prime factor ; and having his eye, like Stickelberger and Frobenius, on the difficulty in Weierstrass' memoir, ends up with the now familiar double theorem—*Any minor that is regular has at least one primary minor and one primary major that are similarly regular.*

HENSEL, K. (1894).

[Ueber die Elementartheiler componirter Systeme. *Crelle's Journ.*, cxiv. pp. 109–115.]

The theorem here set for proof and attributed to Frobenius (but note the latter's view above) is that *The necessary and sufficient condition that one given matrix may be a multiple of another is that each invariant-factor of the former be a multiple of the corresponding invariant-factor of the latter.* It may be worth noting that a proof is incidentally given (p. 113) also of the first theorem of Frobenius' collection, Hensel's special merit lying in the care bestowed on his demonstrations.

SAUVAGE, L. (1895).

[Note sur le équations en λ de la géométrie. *Nouv. Annales de Math.*, (3) xiv. pp. 369–385.]

The last eight pages of this contain an elementary sketch of invariant-factors and their applications as conceived by Weierstrass.

PINCHERLE, S. (1897).

[Mémoire sur le calcul fonctionnel distributif. *Math. Annalen*, xlix. pp. 325–381.]

This comprehensive memoir contains a chapter (§§ 29–40) bearing the heading " Variétés invariantes pour une opération donnée," and dealing with matter closely connected with invariant-factors. The latter subject, however, is only referred to in a footnote, where further the author merely says that the reader will see for himself

how naturally and easily the new theory leads to the concept of "Weierstrass' elementary-divisors." *

MUTH, P. (1899).

[Theorie und Anwendung der Elementartheiler. xvi+236 pp. Leipzig.]

This, the first text-book on invariant-factors, is almost as extensive as the last edition of Baltzer's similar work dealing with the whole subject of determinants. By far the greater portion of it is occupied with the 'applications,' the subject of bilinear forms naturally receiving most attention. Almost from the very start, indeed, the latter subject is seriously entered on, the second chapter or section (§ 2, pp. 20–42) being devoted to the so-called 'symbolical' treatment of these forms, that is to say, to Cayleyan matrices.

The first chapter (pp. 1–19) contains an exposition of the theory alone. The four opening pages present it from the purely Weierstrassian standpoint, the determinants being of the type considered by Sylvester in his first foreshadowings of the subject. Attention is then transferred to determinants suitable for treatment in Smith's manner, that is to say, determinants whose elements are integers or integral functions of one and the same set of variables; and to these the rest of the chapter is given up. In the third chapter (§ 3, pp. 43–57) the subjects considered are determinants having integers for elements and bilinears having integers for coefficients; in the fourth (§ 4, pp. 58–63) the said elements and coefficients are integral functions of one and the same variable; and in the fifth (§ 5, pp. 63–69) they are binary quantics of one and the same degree. The remaining thirteen chapters deal almost entirely with the so-called applications, that is to say, with the discussion of subjects whose treatment is facilitated by the use of invariant-factors in the specification of conditions and in divers other ways.

As far as the theory is concerned, there is little of fresh interest to point to. The material is brought together in an order con-

* The memoir being intended, doubtless, to spread a knowledge of the author's 'calculus' is to a considerable extent a re-exposition of results communicated to the academies of Bologna, Turin and Rome in 1875, 1876.

sidered suitable for purposes of exposition and demonstration; the proofs are in parts expanded and in general made easier reading; and there is a conscientious supply of notes to indicate the sources. Under these heads the author has done good service.

Bibliographical Note.

From what precedes it will readily be understood that in addition to the papers here reported on there are others which may possibly prove helpful, although directly dealing only with subjects to which invariant-factors are merely auxiliary. The titles of about thirty of these are herewith added: it will be seen that by far the greater number concern bilinear and quadratic forms. A few which do not use invariant-factors have been included for the sake of continuity.

1858. Weierstrass, K. Ueber ein die homogenen Function zweiten Grades betreffendes Theorem . . . *Monatsb. . . . Akad. d. Wiss.* (Berlin), 1858, pp. 207–220; or *Math. Werke*, i. pp. 233–246.

1868. Kronecker, L. Ueber Schaaren quadratischer Formen. *Monatsb. . . . Akad. d. Wiss.* (Berlin), 1868, pp. 339–346; or *Werke*, i. pp. 165–174.

1873. Jordan, C. Sur les polynômes bilinéaires. *Comptes Rendus . . . Acad. des Sci.* (Paris), lxxvii. pp. 1487–1491.

1874. Kronecker, L. Ueber Schaaren von quadratischen und bilinearen Formen. *Monatsb. . . . Akad. d. Wiss.* (Berlin), 1874, pp. 59–76, 149–156, 206–232; or *Werke*, i. pp. 351–372, 373–381, 382–399, 400–403, 404–413. Also, *Comptes Rendus. . . Acad. des Sci.* (Paris), lxxviii. pp. 1181–1182.

1874. Jordan, C. Mémoire sur les formes bilinéaires. *Journ.* (*de Liouville*) *de Math.*, (2) xix. pp. 35–54.

1874. Darboux, G. Mémoire sur la théorie algébrique des formes quadratiques. *Journ.* (*de Liouville*) *de Math.*, (2) xix. pp. 347–396.

1874. Jordan, C. Mémoire sur la réduction et la transformation des systèmes quadratiques. *Journ.* (*de Liouville*) *de Math.*,

(2) xix. pp. 397–422. Also, *Comptes Rendus . . . Acad. des Sci.* (Paris), lxxviii. pp. 614–617, 1763–1766.

1874. Kronecker, L. Ueber die congruenten Transformationen der bilinearen Formen. *Monatsb. . . . Akad. d. Wiss.* (Berlin), 1874, pp. 397–447 ; or *Werke*, i. pp. 423–483.

1887. Maurer, L. Zur Theorie der linearen Substitutionen. Dissert. 26 pp. Strassburg.

1887. Voss, A. Ueber bilineare Formen. *Göttinger Nachrichten*, 1887, pp. 424–433.

1889. Weyr, E. O theorii forem bilinearných. 111 pp. V Praze. German translation in *Monatshefte f. Math. u. Phys.*, i. (1890) pp. 163–236.

1889. Voss, A. Ueber die conjugirte Transformation einer bilinearen Form in sich selbst. *Sitzungsb. . . . Akad. d. Wiss.* (München), xix. pp. 175–211.

1889. Voss, A. Ueber die mit einer bilinearen Form vertauschbaren bilinearen Formen. *Sitzungsb. . . . Akad. d. Wiss.* (München), xix. pp. 283–300.

1889. Cosserat, E. Sur les formes bilinéaires. *Annales . . . Faculté des Sci.* (Toulouse), iii. m. pp. 1–12.

1890. Voss, A. Ueber die cogredienten Transformationen einer bilinearen Form in sich selbst. *Abhandl. . . . Akad. d. Wiss.* (München), $xvii_2$. pp. 235–356.

1890. Frobenius, G. Theorie der biquadratischen Formen. *Crelle's Journ.*, cvi. pp. 125–188.

1890. Kronecker, L. Algebraische Reduktion der Schaaren quadratischen Formen. *Sitzungsb. . . . Akad. d. Wiss.* (Berlin), 1890, pp. 1225–1237, 1375–1388 ; 1891, pp. 9–17, 33–44.

1891. Netto, E. Zur Theorie der linearen Substitutionen. *Acta Math.*, xvii. pp. 265–280.

1895. Calo, B. Dimostrazione algebrica del teorema di Weierstrass sulle forme bilineare. *Annali di Mat.*, (2) xxiii. pp. 159–179.

1895. Taber, H. Note on the automorphic linear transformation of a bilinear form. *American Acad. Proceed.*, xxxi. pp. 181–193.

1895. Landsberg, G. Ueber Fundamentalsysteme und bilineare Formen. *Crelle's Journ.*, cxvi. pp. 331–349.

1896. Taber, H. On the group of real linear transformations whose invariant is an alternate bilinear form. *American Acad. Proceed.*, xxxi. pp. 336–337.

1896. Taber, H. Notes on the theory of bilinear forms. *Bull. American Math. Soc.*, (2) iii. pp. 156–164.

1897. Muir, T. A reinvestigation of the problem of the automorphic linear transformation of a bipartite quadric. *American Journ. of Math.*, xx. pp. 215–228.

1898. Loewy, A. Ueber die Characteristik einer reellen quadratischen Form von nicht verschwindender Determinante. *Math. Annalen*, lii. pp. 588–592.

1899. Burnside, W. On the reduction of a linear substitution to its canonical form. *Proceed. London Math. Soc.*, xxx. pp. 180–194.

1899. Baker, H. F. Note to the foregoing paper, pp. 195–198.

1900. Bromwich, T. J. I'a. Canonical reduction of linear substitutions and bilinear forms, with a dynamical application. *Proceed. London Math. Soc.*, xxxii. pp. 79–118.

1900. Bromwich, T. J. I'a. Note on Weierstrass' reduction of a family of bilinear forms. *Proceed. London Math. Soc.*, xxxii. pp. 158–163.

1900. Bromwich, T. J. I'a. On a canonical reduction of bilinear forms, with special consideration of congruent reductions. *Proceed. London Math. Soc.*, xxxii. pp. 321–352.

CHAPTER XXII.

THE LESS COMMON SPECIAL FORMS, FROM 1880 TO 1899.

THE number of writings dealing with those special determinant forms that are not included in the preceding chapters is strikingly great, being approximately double the corresponding number for the preceding twenty-year period, even when allowance is made for the fact that the special forms of Chapters XX. and XXI. of Vol. III. are not so segregated in the present volume.

BRUNO, F. FAÀ DI (1880).

[Notes on modern algebra. *American Journ. of Math.*, iii. pp. 154–165.]

[Trois notes sur la théorie des formes. 3. Théorème général sur les déterminants fonctionnels. *Math. Annalen*, xviii. pp. (280–288) 286–288.]

The second note of the first paper, which is reproduced as the third note of the second paper, concerns a theorem which we may formulate thus: *If* $|\,a_{1n}\,|$ *and the operator* ω *be such that*

$$a_{rs} = \omega(a_{r,s-1}) \quad \textit{and} \quad \omega(a_{rn}) = 0,$$

then $\omega|\,a_{1n}\,| = 0$. It has to be noted, however, that in the proof which he indicates Bruno makes a further assumption regarding ω, namely, that the application of it to a determinant produces a sum of n determinants exactly like differentiation in Jacobi's theorem. His first example is

$$\begin{vmatrix} xyz & xy+yz+zx & x+y+z & 1 \\ yzt & yz+zt+ty & y+z+t & 1 \\ ztx & zt+tx+xz & z+t+x & 1 \\ txy & tx+xy+yt & t+x+y & 1 \end{vmatrix}, \quad \text{or A say,}$$

where ω is $\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}+\frac{\partial}{\partial t}$, and where therefore by the theorem

$$\sum \frac{\partial A}{\partial x} = 0.$$

Another theorem is then used to deduce from this that A is the difference-product of x, y, z, t,—a result readily seen otherwise (*Hist.*, iii. pp. 141–142). The second example is Schläfli's $(m+1)$-line determinant equal to $(pq'-p'q)^{\frac{1}{2}m(m+1)}$. Here, however, evaluation is not reached without utilizing four different operators, the determinant being formable by operating with

$$p'\frac{\partial}{\partial p}+q'\frac{\partial}{\partial q} \quad \text{on the first column,}$$

or $$p\frac{\partial}{\partial p'}+q\frac{\partial}{\partial q'} \quad ,, \quad \text{last} \quad ,,$$

or $$q\frac{\partial}{\partial p}+q'\frac{\partial}{\partial p'} \quad ,, \quad \text{first row,}$$

or $$p\frac{\partial}{\partial q}+p'\frac{\partial}{\partial q'} \quad ,, \quad \text{last} \quad ,,$$

The procedure, too, is lengthy. Finally, the theorem is used to show that a solution of the equation

$$x\frac{\partial D}{\partial x}+y\frac{\partial D}{\partial y}+z\frac{\partial D}{\partial z} = 0$$

is

$$D = \begin{vmatrix} (\log x)^2 & \log x & 1 \\ (\log y)^2 & \log y & 1 \\ (\log z)^2 & \log z & 1 \end{vmatrix},$$

since the determinant is formable as above by means of the operator

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}.$$

MERINO, M. (1881): SCOTT, R. F. (1881).

[Sobre una propriedad de las determinantes de tercer grado. *Cronica Científica* . . . (Barcelona), iv. pp. 133–137.]

[Mathematical notes. *Messenger of Math.*, x. pp. 142–149.]

The first paper here deals mainly with certain exercises in Dostor's text-book, the tone being in general captious.

The theorems dealt with in Scott's third note (pp. 145–146) concern determinants whose elements are covariants of two sets of binary quantics, the same covariant for each element. At the close of the paper there is also given without proof and without reference to his note of 1879 on double alternants, the equality

$$\left|+(x_1-y_1)^{-1}(x_2-y_2)^{-1}\ldots(x_n-y_n)^{-1}+\right| = \frac{n\{1\cdot3\cdot5\ldots(n-2)\}^2}{2^n} \text{ for } n \text{ odd}$$

$$= 0 \text{ for } n \text{ even},$$

where the x's are the roots of $x^n-1=0$ and the y's are the roots of $y^n+1=0$.*

CASORATI, F. (1881).

[Osservazioni sui modi comunemente usati nella trattazione di parecchie questioni fondamentali dell' analisi infinitesimale. *Atti . . . Accad. dei Lincei*: (3) *Transunti*, v. pp. 216–217.]

The result established here is that the necessary and sufficient condition for the vanishing of the determinant

$$\begin{vmatrix} c_1^x F_1(x) & c_2^x F_2(x) & \ldots. & c_n^x F_n(x) \\ c_1^{x+1} F_1(x+1) & c_2^{x+1} F_2(x+1) & \ldots. & c_n^{x+1} F_n(x+1) \\ \ldots & \ldots & \ldots & \ldots \\ c_1^{x+n-1} F_1(x+n-1) & c_2^{x+n-1} F_2(x+n-1) & \ldots. & c_n^{x+n-1} F_n(x+n-1) \end{vmatrix}$$

is the vanishing identically of one of the functions F, these being rational and integral and the c's all different. The mode of proof is gradational, the theorem used in passing from one case to the next being Hermite's condensation-theorem of 1849 (*Hist.*, ii. p. 46).

CATALAN, E. C. (1881).

[Une application des déterminants. *Mélanges math.*, ii. pp. 356–359; or *Mém. . . . de Liège*, (2) xiii. pp. 356–359.]

The application, which concerns products of sums of squares, depends on being able to find a determinant whose square has zeros in all its non-diagonal places.

* This result regarding a *permanent* is the first for the period under review, and for convenience the others on the same subject are brought together in its neighbourhood (see pp. 457–460).

GRAM, J. P. (1881).

[Ueber die Entwickelung reeller Functionen in Reihen, mittelst der Methode der kleinsten Quadrate. *Crelle's Journ.*, xciv. pp. 41–73.]

Note is merely taken of this paper * as containing the foreshadowings of the special determinant known much later as the *Gramian.*

LÉVY, L. (1881).

[Sur la possibilité de l'équilibre électrique. *Comptes Rendus . . . Acad. des Sci.* (Paris), xciii. pp. 706–708.]

The examination of the problem in question leads to a special set of linear equations, the consideration of which brings the author to announce that *If the non-diagonal elements of a determinant be all positive, and each diagonal element be negative and exceed in absolute value the sum of the absolute values of the other elements of the row to which it belongs, the determinant cannot be equal to* 0. The mode of proof is gradational.

MUIR, T. (1882).

[On a class of permanent symmetric functions. *Proceed. R. Soc. Edinburgh*, xi. pp. 409–418.]

The symmetric functions in question are those whose terms are formed from a square array of elements in the same way as the terms of a determinant, save that they are all taken with a positive sign. The name ' permanent ' is adopted for them, and the notation

$$\overset{+}{\left|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right|}, \text{ shortened into } \overset{+}{|} a_1 b_2 c_3 \overset{+}{|}.$$

As an initial fact it is pointed out that $\overset{+}{|} a_1 b_2 c_3 \overset{+}{|}$ is equal to

$$(a_1 i_1 + a_2 i_2 + a_3 i_3)(b_1 i_1 + b_2 i_2 + b_3 i_3)(c_1 i_1 + c_2 i_2 + c_3 i_3)$$

* What may be called a first edition of it appeared in 1879 with the title : Om Raekkeadviklinger, bestemte ved Hjaelp af de mindste kvadraters Methode. Dissert. 122 pp. Kjøbenhavn.

if $i_1 i_2 i_3 = 1$ and i_1, i_2, i_3 be symbols subject to the laws of ordinary algebra, except that $i_1^2 = i_2^2 = i_3^2 = 0$; also, that it is almost the same thing to say that $\overset{+}{|}\, a_1 b_2 c_3 \,\overset{+}{|}$ is the coefficient of xyz in the expansion of

$$(a_1x+a_2y+a_3z)(b_1x+b_2y+b_3z)(c_1x+c_2y+c_3z).$$

Properties resembling properties of determinants are then referred to, and exemplified by cases of the analogue of Laplace's expansion-theorem; for example,

$$\begin{aligned}\overset{+}{|}\, a_1b_2c_3d_4 \,\overset{+}{|} &= a_1 \overset{+}{|}\, b_2c_3d_4 \,\overset{+}{|} + a_2 \overset{+}{|}\, b_1c_3d_4 \,\overset{+}{|} + \dots . \\ &= \overset{+}{|}\, a_1b_2 \,\overset{+}{|} \cdot \overset{+}{|}\, c_3d_4 \,\overset{+}{|} + \overset{+}{|}\, a_1b_3 \,\overset{+}{|} \cdot \overset{+}{|}\, c_2d_4 \,\overset{+}{|} + \dots .\end{aligned}$$

Next are established three theorems of a different type. The first is that *The product of two determinants of the* nth *order is expressible as an aggregate of* n ! *permanents of the same order*; the second is a rediscovery of Horner's theorem of 1865; and the third is that *The product of a permanent and a determinant both of the* nth *order is expressible as an aggregate of* n ! *determinants of the same order*, for example,

$$\begin{aligned}\overset{+}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}} \cdot \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} &= \begin{vmatrix} a_1x_1 & a_2x_2 & a_3x_3 \\ b_1y_1 & b_2y_2 & b_3y_3 \\ c_1z_1 & c_2z_2 & c_3z_3 \end{vmatrix} - \begin{vmatrix} a_1x_1 & a_2x_2 & a_3x_3 \\ b_1z_1 & b_2z_2 & b_3z_3 \\ c_1y_1 & c_2y_2 & c_3y_3 \end{vmatrix} \\ &\quad - \begin{vmatrix} a_1y_1 & a_2y_2 & a_3y_3 \\ b_1x_1 & b_2x_2 & b_3x_3 \\ c_1z_1 & c_2z_2 & c_3z_3 \end{vmatrix} + \begin{vmatrix} a_1y_1 & a_2y_2 & a_3y_3 \\ b_1z_1 & b_2z_2 & b_3z_3 \\ c_1x_1 & c_2x_2 & c_3x_3 \end{vmatrix} \\ &\quad + \begin{vmatrix} a_1z_1 & a_2z_2 & a_3z_3 \\ b_1x_1 & b_2x_2 & b_3x_3 \\ c_1y_1 & c_2y_2 & c_3y_3 \end{vmatrix} - \begin{vmatrix} a_1z_1 & a_2z_2 & a_3z_3 \\ b_1y_1 & b_2y_2 & b_3y_3 \\ c_1x_1 & c_2x_2 & c_3x_3 \end{vmatrix}\end{aligned}$$

This last is viewed as the most important, and is illustrated by applications in the theory of alternants. Cayley's simple instance of 1857 being recalled, it is shown that *If any determinant of the third order with non-zero elements vanishes identically, the product of the permanent and determinant whose elements are the reciprocals of the elements of the said determinant is equal to the determinant whose*

elements are the squares of those reciprocals. Finally, as a curious example for evaluation there is given

$$\left|\begin{matrix}+\\ 2\cos\frac{1\cdot1\cdot2\pi}{2n+1}\cdot 2\cos\frac{2\cdot2\cdot2\pi}{2n+1}\cdot\ldots\ldots 2\cos\frac{n\cdot n\cdot 2\pi}{2n+1}\end{matrix}\right|^{+},$$

which is known otherwise to be equal to -1 for odd values of n.

MUIR, T. (1897).

[A relation between permanents and determinants. *Proceed. R. Soc. Edinburgh,* xxii. pp. 134–136.]

The relation in question is, in the case of the fourth order,

$$|a_1b_2c_3d_4| - \sum a_1|b_2c_3d_4| + \sum^{+}|a_1b_2|^{+}|c_3d_4| - \sum^{+}|a_1b_2c_3|^{+}\cdot d_4 + {}^{+}|a_1b_2c_3d_4|^{+} = 0,$$

where the Σ is taken to mean 'sum of all products of coaxial minors like'; for example,

$$\sum^{+}|a_1b_2|^{+}|c_3d_4|$$

denotes the sum of six products, the first factors of which are

$$^{+}|a_1b_2|^{+},\quad ^{+}|a_1c_3|^{+},\quad ^{+}|a_1d_4|^{+},\quad ^{+}|b_2c_3|^{+},\quad ^{+}|b_2d_4|^{+},\quad ^{+}|c_3d_4|^{+},$$

and the second factors

$$|c_3d_4|,\quad |b_2d_4|,\quad \ldots\ldots$$

The verification in any particular case is effected on noting that the vanishing of the coefficients of a_1, a_2, a_3, ... is entailed by the previous case.

FERBER, F. (1899).

[Sur un symbole analogue aux déterminants. *Bulletin . . . Soc. math. de France,* xxvii. pp. 285–288.]

What is here brought forward as a fresh object of study with a fresh name and a fresh notation is the class of permanents that have arrays like those of alternants. The fundamental result of

course is that such a permanent, if its rows be all different and its columns all different, represents a simple symmetric function, but that if it have h columns * identical it represents a multiple of a symmetric function by h! Its suitability for the representation of $(a+b+c+\ldots)^m$ and of the aleph functions is pointed out; also a further application when the elements composing a term are restricted to be written in the order of the rows to which they belong.

VOIGT, W. (1882).

[Allgemeine Formeln für die Bestimmung der Elasticitätsconstanten . . . *Wiedemann's Annalen d. Phys. u. Chem.*, (2) xvi. pp. 273–321, 398–416.]

Here in dealing with a question in physics the author is faced (p. 314) with a determinant whose array is composed of n^2 arrays of the type †

$$\begin{matrix} a & b \\ -b & a \end{matrix},$$

and whose value is ascertained to be the sum of two squares; for example, when n is 2,

$$\begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ e & f & g & h \\ -f & e & -h & g \end{vmatrix} \equiv \left\{ \begin{vmatrix} a & c \\ e & g \end{vmatrix} - \begin{vmatrix} b & d \\ f & h \end{vmatrix} \right\}^2 + \left\{ \begin{vmatrix} a & d \\ e & h \end{vmatrix} + \begin{vmatrix} b & c \\ f & g \end{vmatrix} \right\}^2.$$

* The author might have noted that when h *rows* are alike $h!$ is still a factor, but the function is no longer necessarily symmetric. A hint at such a case as

$$\begin{vmatrix} a & a & a^2 \\ a & a & a^2 \\ b & b & b^2 \end{vmatrix} \equiv 2\,(a^2b^2 + 2a^3b)$$

might also have been given.

† In the original the typical array is $\begin{vmatrix} a & b \\ b & -a \end{vmatrix}$. Another helpful change is to denote a minor of the array

$$\begin{matrix} a & b & c & d \\ e & f & g & h \end{matrix}$$

by the numbers of its columns, so that the result takes the form

$$(13-24)^2 + (14+23)^2.$$

MUIR, T. (1882): GENESE, R. W. (1882).

[Questions 7217, 7218. *Educ. Times*, xxxv. p. 313; or *Math. from Educ. Times*, xxxix. pp. 106–107.]

The first result here is

$$\begin{vmatrix} x & . & . & x_1^2 & x_1x_2 & x_2^2 \\ . & y & . & y_1^2 & y_1y_2 & y_2^2 \\ . & . & z & z_1^2 & z_1z_2 & z_2^2 \\ . & z & y & 2y_1z_1 & y_1z_2+y_2z_1 & 2y_2z_2 \\ z & . & x & 2z_1x_1 & z_1x_2+z_2x_1 & 2z_2x_2 \\ y & x & . & 2x_1y_1 & x_1y_2+x_2y_1 & 2x_2y_2 \end{vmatrix} = |xy_1z_2|^3.$$

The determinant in the second is more general in that it has for its fifth column $l, m, n, 2p, 2q, 2r$, the cofactors of which are evaluated and found each of them to contain the factor $|xy_1z_2|$.

SCOTT, R. F. (1882).

[Notes on determinants. *Messenger of Math.*, xii. pp. 105–118.]

The determinant considered in the first two paragraphs here is

$$\begin{vmatrix} x_1 & \alpha_1+\beta_2 & \alpha_1+\beta_3 & \dots \\ \alpha_2+\beta_1 & x_2 & \alpha_2+\beta_3 & \dots \\ \alpha_3+\beta_1 & \alpha_3+\beta_2 & x_3 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n,$$

which, by raising it to the $(n+2)^{th}$ order, is shown to be equal to

$$\begin{vmatrix} . & 1 & 1 & 1 & 1 & \dots \\ 1 & . & -\beta_1 & -\beta_2 & -\beta_3 & \dots \\ 1 & -\alpha_1 & x_1-\alpha_1-\beta_1 & . & . & \dots \\ 1 & -\alpha_2 & . & x_2-\alpha_2-\beta_2 & . & \dots \\ 1 & -\alpha_3 & . & . & x_3-\alpha_3-\beta_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

The fact is then used that this is equal to the quotient of the first two-line minor of its adjugate by its own last n-line minor, the final result reached being

$$\left\{1+\sum\frac{\alpha_h+\beta_h}{x_h-\alpha_h-\beta_h}-\sum\frac{(\alpha_h-\alpha_k)(\beta_h-\beta_k)}{(x_h-\alpha_h-\beta_h)(x_k-\alpha_k-\beta_h)}\right\}\cdot\Pi(x_h-\alpha_h-\beta_h).$$

After a natural digression to n-dimensional determinants, the writer takes up (§ 6) the consideration of one or two geometrical matters, and is thereby led on to the evaluation of another determinant, namely,

$$\begin{vmatrix} (a_1-\beta_1)^2 & (a_1+\beta_2)^2 & \dots & (a_1+\beta_n)^2 & 1 \\ (a_2+\beta_1)^2 & (a_2-\beta_2)^2 & \dots & (a_2+\beta_n)^2 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ (a_n+\beta_1)^2 & (a_n+\beta_2)^2 & \dots & (a_n-\beta_n)^2 & 1 \\ 1 & 1 & \dots & 1 & . \end{vmatrix}$$

$$= 2^{2n-3}(-1)^{n-1}\left\{\sum \frac{n-3}{a_h\beta_h} - \sum\left(\frac{1}{a_h\beta_k}-\frac{1}{a_k\beta_h}\right)\right\} a_1a_2\dots a_n\cdot\beta_1\beta_2\dots\beta_n,$$

where, as before, h is to receive the values 1, 2, . . . , n, and k is greater than the companion h but not greater than n.

It is important to note that when in the former of these two determinants we make either all the a's or all the β's vanish, the degenerate form obtained is Sardi's of 1868.

CAPELLI, A. (1882, 1886).

[Fondamenti di una teoria generale delle forme algebriche. *Mem. . . . Accad. dei Lincei*, (3) xii. pp. 529–598.]

[Ueber die Zurückführung der Cayley'schen Operation Ω auf gewöhnliche Polaroperationen. *Math. Annalen*, xxix. pp. 331–338.]

The first of these papers foreshadows and the second fully establishes an interesting general theorem regarding the determinantal operators of Cayley's original memoir of 1845 on hyperdeterminants. If the n^2 variables be

$$\begin{matrix} x_1 & x_2 & \dots & x_n \\ y_1 & y_2 & \dots & y_n \\ z_1 & z_2 & \dots & z_n \\ \dots & \dots & \dots & \dots \\ w_1 & w_2 & \dots & w_n \end{matrix}$$

and we denote the 'elementary polar operator'

$$q_1\frac{\partial}{\partial p_1}+q_2\frac{\partial}{\partial p_2}+\dots+q_n\frac{\partial}{\partial p_n} \quad \text{by} \quad D_{pq},$$

then the theorem is to the effect that the application of Cayley's operator

$$\left| \frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial y_2} \quad \dots \quad \frac{\partial}{\partial w_n} \right|$$

to a function and the multiplication of the result by $| x_1 y_2 z_3 \dots w_n |$ is equivalent to the application of the single operator

$$\begin{vmatrix} \mathrm{D}_{xx} & \mathrm{D}_{xy} & \mathrm{D}_{xz} & \dots \\ \mathrm{D}_{yx} & 1+\mathrm{D}_{yy} & \mathrm{D}_{yz} & \dots \\ \mathrm{D}_{zx} & \mathrm{D}_{zy} & 2+\mathrm{D}_{zz} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n,$$

on the understanding that the elements composing each term of the expansion of the last determinant are to be taken in the order of the columns to which they belong.

It is also shown that an alternative form of single operator may be derived from this by making the 1st column reversed the n^{th} row, the 2nd column reversed the $(n-1)^{\text{th}}$ row, and so on.*

MUIR, T. (1883).

[On a peculiar development of a special determinant of the sixth order. *Messenger of Math.*, xiii. pp. 95–100.]

The determinant in question is

$$\begin{vmatrix} a_1 & a_2 & a_3 & \mathrm{A}a_1 & \mathrm{A}a_2 & \mathrm{A}a_3 \\ b_1 & b_2 & b_3 & \mathrm{B}b_1 & \mathrm{B}b_2 & \mathrm{B}b_3 \\ c_1 & c_2 & c_3 & \mathrm{C}c_1 & \mathrm{C}c_2 & \mathrm{C}c_3 \\ d_1 & d_2 & d_3 & \mathrm{D}d_1 & \mathrm{D}d_2 & \mathrm{D}d_3 \\ e_1 & e_2 & e_3 & \mathrm{E}e_1 & \mathrm{E}e_2 & \mathrm{E}e_3 \\ f_1 & f_2 & f_3 & \mathrm{F}f_1 & \mathrm{F}f_2 & \mathrm{F}f_3 \end{vmatrix}.$$

It is first expanded in the form

$$| a_1 b_2 c_3 | \, | d_1 e_2 f_3 | \, (\mathrm{DEF}-\mathrm{ABC}) - | a_1 b_2 d_3 | \, | c_1 e_2 f_3 | \, (\mathrm{CEF}-\mathrm{ABD})$$
$$+ | a_1 b_2 e_3 | \, | c_1 d_2 f_3 | \, (\mathrm{CDF}-\mathrm{ABE}) - \dots \dots \dots \dots \dots \dots \dots,$$

or, say,

$$\pi_1(\mathrm{DEF}-\mathrm{ABC}) - \pi_2(\mathrm{CEF}-\mathrm{ABD}) + \dots + \pi_\omega(\mathrm{BCD}-\mathrm{AEF}),$$

* The student of the algebra of quantics may be referred to a third paper of Capelli's in *Math. Annalen*, xxxvii. (1890) pp. 1–37.

and then advantage is taken of the existence of linear relations like

$$\pi_1 + \pi_5 - \pi_6 + \pi_7 = 0$$

to eliminate five of the π's,—a procedure which can be followed in a variety of ways,— giving results of the type

$$\begin{aligned}
&|a_1b_2c_3|\,|d_1e_2f_3|\,(F-C)(E-B)(D-A)\\
-&|a_1b_2e_3|\,|c_1d_2f_3|\,(F-A)(E-D)(C-B)\\
+&|a_1c_2f_3|\,|b_1d_2e_3|\,(F-D)(E-A)(C-B)\\
-&|a_1d_2e_3|\,|b_1c_2f_3|\,(F-D)(E-B)(C-A)\\
-&|a_1d_2f_3|\,|b_1c_2e_3|\,(F-C)(E-D)(B-A).
\end{aligned}$$

In exactly similar fashion it is shown that

$$\begin{vmatrix}
a_1 & a_2 & a_3 & Aa_1 & Aa_2\\
b_1 & b_2 & b_3 & Bb_1 & Bb_2\\
c_1 & c_2 & c_3 & Cc_1 & Cc_2\\
d_1 & d_2 & d_3 & Dd_1 & Dd_2\\
e_1 & e_2 & e_3 & Ee_1 & Ee_2
\end{vmatrix} = \begin{aligned}
&-|a_1b_2e_3|\,|c_1d_2|\,(E-D)(C-B)\\
&+|a_1d_2e_3|\,|b_1c_2|\,(E-B)(D-C)\\
&+|b_1c_2d_3|\,|a_1e_2|\,(E-D)(B-A)\\
&-|b_1d_1e_2|\,|a_1c_2|\,(E-A)(D-C)\\
&+|c_1d_2e_3|\,|a_1b_2|\,(E-A)(D-B),
\end{aligned}$$

and attention is drawn to the first germ of an identity of the kind (*Hist.*, ii. pp. 451–456).

ANDRÉIEF, C. (1883).

[Note sur une relation entre les intégrales définies des produits des fonctions. *Mém. de la Soc. des Sci. . . . Bordeaux*, (3) ii. pp. 1–14.]

Interest in this for us lies in the fact that its main result concerns the determinant whose $(r, s)^{\text{th}}$ element is

$$\int_a^b f_r(x)\,\phi_s(x)\,dx,$$

namely,

$$\left|\int f_1\phi_1 dx \;\ldots\; \int f_n\phi_n dx\right|$$

$$= \frac{1}{n!}\iint\ldots\left\{\left|f_1(x_1)\;\ldots\;f_n(x_n)\right|\cdot\left|\phi_1(x_1)\;\ldots\;\phi_n(x_n)\right|\right\}dx_1\ldots dx_n,$$

the limits being the same for every integral involved.

MUIR, T. (1883).

[Question 7525. *Educ. Times*, xxxvi. p. 318 ; or *Math. from Educ. Times*, xli. p. 27.]

The theorem here established is that *the number of positive terms in the ordinary development of a determinant having negative elements in the diagonal and positive elements elsewhere is*

$$\tfrac{1}{2} \cdot \mathrm{n}! - (-2)^{\mathrm{n}-2}(\mathrm{n}-2).$$

In the proof use is made of the fact that

$$\mathrm{C}(-1, 1, 1, \ldots .)_n = (-2)^{n-2}(n-2).$$

STUDNIČKA, F. J. (1883).

[Neuer Beweis des Satzes dass das Produkt der Summe von acht Quadratzahlen mit der Summe von acht Quadratzahlen sich als Summe von acht Quadratzahlen darstellen lasse. *Sitzungsb. . . . Ges. d. Wiss.* (Prag), 1883, pp. 475–481.]

The proof contains nothing of fresh interest as regards determinants. Genocchi's supposed generalization * is accepted as valid.

SYLVESTER, J. J. (1884).

[Question 7567. *Educ. Times*, xxxvii. pp. 27, 111 ; or *Math. from Educ. Times*, xli. p. 53.]

The determinant which appears here in connection with a geometrical subject is

$$\begin{vmatrix} a+b+c & l+m\gamma^2+n\gamma & x+y\gamma+z\gamma^2 \\ x+y\gamma^2+z\gamma & a+b\gamma+c\gamma^2 & l+m+n \\ l+m\gamma+n\gamma^2 & x+y+z & a+b\gamma^2+c\gamma \end{vmatrix},$$

where γ is a primitive 3rd root of 1. The expansion given of it is

$$a^3+b^3+c^3+l^3+m^3+n^3+x^3+y^3+z^3$$
$$-3abc-3lmn-3xyz-3alx-3bmy-3cxz$$
$$-3a(mz+ny)-3b(nx+lz)-3c(ly+mx).$$

* *Annali di Mat.*, (1) iii. (1860) pp. 202–205.

SYLVESTER, J. J. (1884): CAYLEY, A. (1885).

[Sur l'équation en matrices $px = xq$. *Comptes Rendus . . . Acad. des Sci.*, xcix. pp. 67–71; or *Coll. Math. Papers*, iv. pp. 176–180.]

[On the matrical equation $qQ - Qq' = 0$. *Messenger of Math.*, (2) xiv. pp. 176–178; or *Coll. Math. Papers*, xii. pp. 311–313.]

Of the numerous notes * written on Cayleyan matrices by Sylvester in 1884 this is the only one containing a result that is viewable as having a purely determinantal side. The question directly involved is the existence of a matrix μ so related to two given matrices m_1, m_2 that †

$$m_1\mu = \mu m_2.$$

The matrices

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \quad \begin{pmatrix} c_1 & c_2 \\ d_1 & d_2 \end{pmatrix} \quad \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

being taken for m_1, m_2, μ, and the indicated multiplications being performed, it is at once found that the required condition is

$$\begin{vmatrix} a_1-c_1 & b_1 & -c_2 & . \\ a_2 & b_2-c_1 & . & -c_2 \\ -d_1 & . & a_1-d_2 & b_1 \\ . & -d_1 & a_2 & b_2 \quad d_2 \end{vmatrix} = 0.$$

For us, however, the important point at present is that Sylvester next evaluates the four-line determinant, and shows that it is identical with the resultant of the equations

$$x^2 - (a_1+b_2)x + |a_1b_2| = 0, \qquad x^2 - (c_1+d_2)x + |c_1d_2| = 0.$$

There is thus obtained for us a new form for the resultant of the two Lagrangian determinantal equations

$$\begin{vmatrix} a_1-x & a_2 \\ b_1 & b_2-x \end{vmatrix} = 0, \qquad \begin{vmatrix} c_1-x & c_2 \\ d_1 & d_2-x \end{vmatrix} = 0.$$

The same result occurs in the same way in Cayley's paper.

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* *Comptes Rendus . . . Acad. des Sci.* (Paris), xcix. pp. 13–15, 67–71, 115–116, 117–118, 409–412, 432–436, 473–476, 502–505, 527–529, 555–558, 621–631.

† See *Hist.*, iii. p. 47.

CESÀRO, E. (1885).

[Quistione 46. *Giornale di Mat.*, xxiii. pp. 19, 374–376.]

The determinant here spoken of is not explicitly given ; it appears in the form

$$\begin{vmatrix} u_{11} & u_{12} & \dots & u_{1,n-1} & 1 \\ u_{21} & u_{22} & \dots & u_{2,n-1} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ u_{n1} & u_{n2} & \dots & u_{n,n-1} & 1 \end{vmatrix}$$

with the annexed statement that the square of the difference of any two rows is 2. It is shown, however, that as a consequence of this datum the multiplication of the alternative form

$$\begin{vmatrix} u_{11}-u_{21} & u_{12}-u_{22} & \dots & u_{1,n-1}-u_{2,n-1} \\ u_{21}-u_{31} & u_{22}-u_{32} & \dots & u_{2,n-1}-u_{3,n-1} \\ \dots & \dots & \dots & \dots \\ u_{n-1,1}-u_{n1} & u_{n-1,2}-u_{n2} & \dots & u_{n-1,n-1}-u_{n,n-1} \end{vmatrix}$$

by itself gives

$$\begin{vmatrix} 2 & -1 & . & . & \dots \\ -1 & 2 & -1 & . & \dots \\ . & -1 & 2 & -1 & \dots \\ . & . & -1 & 2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_{n-1},$$

which is equal to n.

MÉRAY, CH. (1885).

[Décomposition des polynômes entiers a plusieurs variables en éléments linéaires. *Annales sci. de l'Éc. norm. sup.*, (3) ii. pp. 289–302.]

The problem formulated in the title here is viewable as a purely determinantal problem, the particular case to which the author devotes almost the whole of his space being the finding of an expression for

$$\begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & y_1z_1 & z_1x_1 & x_1y_1 \\ x_2^2 & y_2^2 & z_2^2 & y_2z_2 & z_2x_2 & x_2y_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x_6^2 & y_6^2 & z_6^2 & y_6z_6 & z_6x_6 & x_6y_6 \end{vmatrix}, \quad \text{or } \Delta_{3:2} \text{ say,}$$

in terms of the three-line minors of

$$\left\| \begin{matrix} x_1 & x_2 & \dots & x_6 \\ y_1 & y_2 & \dots & y_6 \\ z_1 & z_2 & \dots & z_6 \end{matrix} \right\|.$$

Putting pqr for the minor whose columns are the p^{th}, q^{th}, r^{th} columns of the array, he obtains for the expression an aggregate of 30 terms, of which the first two pairs are

$$\tfrac{1}{4} \left\{ \begin{matrix} -145 \cdot 136 \cdot 246 \cdot 235 + 236 \cdot 245 \cdot 135 \cdot 146 \\ +236 \cdot 245 \cdot 134 \cdot 156 - 145 \cdot 136 \cdot 256 \cdot 234 \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{matrix} \right\}.$$

The proof given of so interesting a result ought to be replaceable by something much simpler and more readily convincing. This is the more desirable, as a general theorem is foreshadowed in which the number of variables in every row of the given determinant, $\Delta_{p:r}$ say, is p and the degree of each element is r. The order of the determinant is then $(p+r-1)_r$, and the number of p-line determinants in each term of the development is $(p+r-1)_p$.

We note for ourselves that a partial simplification may be made in the expression * of the development of $\Delta_{3:2}$, the equality being

$$\begin{aligned} 4\Delta_{3:2} = {} & 123 \overset{+}{\begin{vmatrix} 145 & 245 & 345 \\ 146 & 246 & 346 \\ 156 & 256 & 356 \end{vmatrix}} - 124 \overset{+}{\begin{vmatrix} 135 & 235 & 435 \\ 136 & 236 & 436 \\ 156 & 256 & 456 \end{vmatrix}} \\ & + 125 \overset{+}{\begin{vmatrix} 134 & 234 & 534 \\ 136 & 236 & 536 \\ 146 & 246 & 546 \end{vmatrix}} - 126 \overset{+}{\begin{vmatrix} 134 & 234 & 634 \\ 135 & 235 & 635 \\ 145 & 245 & 645 \end{vmatrix}} \\ & + 134 \cdot 156 (235 \cdot 246 + 236 \cdot 245) \\ & + 135 \cdot 146 (-234 \cdot 256 + 236 \cdot 245) \\ & - 136 \cdot 145 (234 \cdot 256 + 235 \cdot 246). \end{aligned}$$

* In Méray's expression, where 0, 1, 2, 3, 4, 5 take the place of our 1, 2, 3, 4, 5, 6,

-045 of line 3 should be -054,
241 „ 11 „ 243.

In the absence of a readily applicable rule of formation such misprints are troublesome.

It is also worth observing that if all the determinants on the right be expanded in the ordinary way, and the indicated multiplications be performed, the number of terms at first obtained is $6^4 \times 30$, so that for the 720 terms of $\Delta_{3:2}$ we are faced at first with $\frac{1}{4} \cdot 6^4 \cdot 30$, that is 9000 more than 720.

RUSSELL, W. H. L. (1885).

[On the reduction of algebraical determinants. *Report . . . British Assoc. for the Adv. of Sci.*, lv. pp. 910–911.]

The determinants referred to have elements that are all rational and integral functions of x, and an unimportant method is indicated of obtaining for such a determinant an equivalent sum of determinants of lower order.

SYLVESTER, J. J. (1885): MUIR, T. (1885).

[Questions 8275, 8321. *Educ. Times*, xxxviii. pp. 349, 379; or *Math. from Educ. Times*, xlv. pp. 85–86.]

The result here established is

$$\begin{vmatrix} \lambda+\alpha a+\alpha'a' & \alpha b+\alpha'b' & \alpha c+\alpha'c' & \dots \\ \beta a+\beta'a' & \lambda+\beta b+\beta'b' & \beta c+\beta'c' & \dots \\ \gamma a+\gamma'a' & \gamma b+\gamma'b' & \lambda+\gamma c+\gamma'c' & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n$$

$$= \lambda^{n-2}\{\lambda^2+\lambda(\mathrm{A}+\mathrm{B})+\mathrm{AB}-\mathrm{CD}\},$$

where

$$\mathrm{A} = (a, b, c, \dots \diagdown a', b', c', \dots), \quad \mathrm{B} = (a, \beta, \gamma, \dots \diagdown a', \beta', \gamma', \dots)$$
$$\mathrm{C} = (a, b, c, \dots \diagdown a', \beta', \gamma', \dots), \quad \mathrm{D} = (a, \beta, \gamma, \dots \diagdown a', b', c', \dots).$$

It is mainly dependent on the fact that the determinant with λ omitted is the product of

$$\begin{vmatrix} a & a' & 0 & 0 & \dots \\ \beta & \beta' & 0 & 0 & \dots \\ \gamma & \gamma' & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n \text{ and } \begin{vmatrix} a & a' & 0 & 0 & \dots \\ b & b' & 0 & 0 & \dots \\ c & c' & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n,$$

and thus bears on the subject of the latent roots of the product of two n-by-2 arrays.

CAPELLI, A. (1886).

[Sopra un teorema che si collega . . . *Rendic. del Circolo Mat.* (Palermo), i. pp. 133–138.]

The theorem in question belongs strictly to the Theory of Quantics, being that *if* f_1, f_2, f_3, F *be integral functions of the elements of the last three rows of* $|x_1y_2z_3t_4|$, *a relation of the form*

$$\left(\sum x\frac{\partial}{\partial y}\right)f_1+\left(\sum x\frac{\partial}{\partial z}\right)f_2+\left(\sum x\frac{\partial}{\partial t}\right)f_3 = |x_1y_2z_3t_4|\cdot F$$

cannot exist without F *being* 0. In the proof of it, however, interesting use is made of the operator

$$|x_1y_2z_3t_4|\cdot\left|\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_2}\frac{\partial}{\partial z_3}\frac{\partial}{\partial t_4}\right|,$$

and it thus deserves passing notice.

TORELLI, G. (1886, 1891).

[Alcune relazioni fra le forme invariantive di un sistema di binarie. *Rendic. . . . Accad. . . .* (Napoli), xx. pp. 125–134.]

[Appunti sulla teoria delle forme binarie. *Annali . . . Ist. Tecn. e Naut.* (Napoli), anno 1891, 10 pp.]

The determinant used in the first of these papers differs from that of Rosanes of 1872 (*Hist.*, iii. pp. 481–484) merely in that the binary quantics involved are not necessarily all of the same degree ; and in the second paper there is not even this difference, the degree of each quantic being the same as the order of the determinant. The results reached involve the multiplication of two such determinants, but their main concern is with the operation of transvection in the Algebra of Quantics.

SCHRADER, W. (1887).

[Beiträge zur Theorie der Determinanten. vi+156 pp. Halle a. S.]

Schrader devotes a section (pp. 116–123) to n-line determinants whose $(r, s)^{\text{th}}$ element is $f(x+r-1, y+s-1)$ and f is an integral function of the m^{th} degree. The process of transformation which he employs is the repeated diminution of a row by the row preceding it, and thereafter the like repeated diminution in the case

of the columns. Save in one negligible instance, n is taken greater than m. *The function being*

$$a_0x^m + a_1x^{m-1}y + \ldots + a_my^m,$$

the $(m+1)$*-line determinant is found to be*

$$(-1)^{\frac{1}{2}m(m+1)} \cdot (m!!)^2 \cdot a_0a_1 \ldots a_m ;$$

and most of the other examples given are specializations of this. It is noted, too, that *determinants of higher order vanish*, and that *if the function be increased by another of lower degree the value of the determinant is unaltered.*

The next section (pp. 123–127) deals first with the case where the foregoing binary quantic is increased by a^{x+y}, next with some special difference-products, and finally with determinants whose analogous basic function is

$$a(a+d)(a+2d) \ldots (a+\overline{x+y-1}\cdot d).$$

We need only note that

$$\begin{cases} \text{when} & a_{rs} \equiv a + \{(x+y+r-1)\delta\}^{s-1}, \\ \text{then} & |a_{1n}| = \delta^{\frac{1}{2}n(n-1)} \cdot (n-1)!!, \end{cases}$$

and

$$\begin{cases} \text{when} & a_{rs} \equiv (r+s-1)!, \\ \text{then} & |a_{1n}| = n!!(n-1)!!. \end{cases}$$

Of the next section (pp. 127–136) rather more than half consists of rediscoveries regarding determinants whose elements are combinatorial numbers, that is to say, integers of the type

$$k(k-1) \ldots (k-h+1)/h!.$$

The remainder is occupied with a form whose elements are reciprocals of integers, the results, when a correction is made, being that

hen $a_{rs} \equiv \dfrac{1}{a+(r+s-2)\beta}$,

ıen $|a_{1n}| = \dfrac{\{(n-1)!!\}^2 \cdot \beta^{n(n-1)}}{a(a+\beta)^2 \ldots (a+\overline{n-2}\cdot\beta)^{n-1} \cdot (a+\overline{n-1}\cdot\beta)^n (a+n\beta)^{n-1} \ldots}$

and

hen $a_{rs} \equiv \dfrac{1}{(a+\overline{r+s-2}\cdot\beta)(a+\overline{r+s-1}\cdot\beta) \ldots (a+\overline{r+s+m-3}\cdot\beta)}$,

ıen $|a_{1n}|$

$$= \frac{m^{n-1}(m+1)^{n-2} \ldots (m+n-2)^1 \cdot (n-1)!!\, \beta^{n(n-1)}}{a(a+\beta)^2 \ldots (a+\overline{n-2}\cdot\beta)^{n-1} \cdot \mathrm{M} \cdot (a+\overline{m+n-1}\cdot\beta)^{n-1} \ldots (a+\overline{m+2n-3}\cdot\beta)^1},$$

where $\mathrm{M} = \{(a+\overline{n-1}\cdot\beta)(a+n\beta) \ldots (a+\overline{n+m-2}\cdot\beta)\}^n$.

On putting $m = 1$ the second degenerates into the first, which is itself a generalization of Ligowski's of 1861. Since in each the $(r, s)^{\text{th}}$ element is a function of $r+s$, the determinants are persymmetric; for example, when $\alpha = 1$, $\beta = 1$, $m = 2$, $n = 4$, we have

$$\begin{vmatrix} \frac{1}{1\cdot 2} & \frac{1}{2\cdot 3} & \frac{1}{3\cdot 4} & \frac{1}{4\cdot 5} \\ \frac{1}{2\cdot 3} & \frac{1}{3\cdot 4} & \frac{1}{4\cdot 5} & \frac{1}{5\cdot 6} \\ \frac{1}{3\cdot 4} & \frac{1}{4\cdot 5} & \frac{1}{5\cdot 6} & \frac{1}{6\cdot 7} \\ \frac{1}{4\cdot 5} & \frac{1}{5\cdot 6} & \frac{1}{6\cdot 7} & \frac{1}{7\cdot 8} \end{vmatrix} = \frac{1^4\cdot 2^3\cdot 3^2\cdot 4^1\cdot 6\cdot 2\cdot 1}{1\cdot 2^2\cdot 3^3(4\cdot 5)^4\cdot 6^3\cdot 7^2\cdot 8^1} = \frac{1}{2^9\cdot 3^3\cdot 5^4\cdot 7^2}$$

The seemingly complicated denominator is very easily obtainable from the secondary diagonal of the determinant and the parallel minor diagonals.

DRUDE, P. (1887): BALTZER, R. (1887).

[Ein Satz aus der Determinantentheorie. *Nachrichten . . . Ges. d. Wiss.* (Göttingen), pp. 118–122.]

[Ueber einen Satz aus der Determinantentheorie. *Nachrichten . . . Ges. d. Wiss.* (Göttingen), pp. 389–391.]

These deal with Voigt's determinant of 1882 in regard to its expressibility as a sum of two squares. The second of the two proofs is the preferable, as being purely determinantal. In essence it depends on performing the operations

$$\text{row}_1 + \text{row}_2\sqrt{-1}, \quad \text{row}_3 + \text{row}_4\sqrt{-1}, \quad \text{row}_5 + \text{row}_6\sqrt{-1}, \ \ldots .$$

followed by

$$\text{col}_1 + \text{col}_2\sqrt{-1}, \quad \text{col}_3 + \text{col}_4\sqrt{-1}, \quad \text{col}_5 + \text{col}_6\sqrt{-1}, \ \ldots ,$$

for these produce a determinant which breaks up into two determinants, one of the form $\mathrm{M} + \mathrm{N}\sqrt{-1}$ and the other of the form $\mathrm{M} - \mathrm{N}\sqrt{-1}$. For example, when n is 3 and the 1st, 3rd, 5th rows of the given determinant are

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6, \end{matrix}$$

the resolution arrived at is

$$\begin{vmatrix} a_2+a_1i & a_4+a_3i & a_6+a_5i \\ b_2+b_1i & b_4+b_3i & b_6+b_5i \\ c_2+c_1i & c_4+c_3i & c_6+c_5i \end{vmatrix} \cdot \begin{vmatrix} a_2-a_1i & a_4-a_3i & a_6-a_5i \\ b_2-b_1i & b_4-b_3i & b_6-b_5i \\ c_2-c_1i & c_4-c_3i & c_6-c_5i \end{vmatrix},$$

and the final result

$$(246-235-145-136)^2 + (245+236+146+135)^2,$$

where the notation is that of p. 468 above.

GEGENBAUR, L. (1887).

[Note über Determinanten. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), xcvi. (2, A) pp. 5–7.]

[Ueber eine specielle Determinante. *Sitzungsb. . . . Akad. d. Wiss.* (Wien), xcvi. (2, A) pp. 489–490.]

The peculiar determinant here brought forward is best described in connection with that which suggested it, Voigt's of 1882. The latter we spoke of as being composed of two-line cells of the type

$$\begin{matrix} a & b \\ -b & a. \end{matrix}$$

Gegenbaur's is similarly composed, the cells being of the type

$$\begin{matrix} a & b \\ -b & a-b\,; \end{matrix}$$

for example,

$$\begin{vmatrix} a & b & c & d \\ -b & a-b & -d & c-d \\ e & f & g & h \\ -f & e-f & -h & g-h \end{vmatrix}.$$

It is manifestly a function of determinants of the array composed of the odd-numbered rows, and his assertion is that this function takes the form

$$P^2 + 3Q^2,$$

where P and Q are aggregates of such determinants.

In the second paper he passes on to a similar but more complicated type, still avoiding demonstration.

RAIMONDI, R. (1888).

[Un teorema sui determinanti di differenze. *Giornale di Mat.*, xxvi. pp. 185–187.]

The theorem in question is that *if* $a_1, a_2, a_3, \ldots$ *be a series of quantities whose* n^{th} *differences are constant, equal to* c *say, then the determinant of* n+1 *consecutive columns of the array*

$$\begin{array}{cccc} a_1 & a_2 & a_3 & \ldots \\ \delta_{1,1} & \delta_{2,1} & \delta_{3,1} & \ldots \\ \delta_{1,2} & \delta_{2,2} & \delta_{3,2} & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \delta_{1,n-1} & \delta_{2,n-1} & \delta_{3,n-1} & \ldots \\ c & c & c & \ldots \end{array}$$

is equal to

$$(-1)^{\frac{1}{2}n(n+1)}c^{n+1}.$$

For example, the array being

$$\begin{array}{cccccccc} 1 & 4 & 9 & 16 & 25 & 36 & \ldots \\ 3 & 5 & 7 & 9 & 11 & \ldots\ldots \\ 2 & 2 & 2 & 2 & \ldots\ldots\ldots, \end{array}$$

we have

$$\begin{vmatrix} 9 & 16 & 25 \\ 7 & 9 & 11 \\ 2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 9 & 7 & 9 \\ 7 & 2 & 2 \\ 2 & . & . \end{vmatrix} = \begin{vmatrix} 9 & 7 & 2 \\ 7 & 2 & . \\ 2 & . & . \end{vmatrix} = -2^3.$$

KLEYBER, J. A. (1888).

[Theory of the adjustment of series of observations. (In Russian.) *Phys.-math. Soc. Kasan*, vi. pp. 148–239.]

In connection with the use made of determinants in this long paper an appendix is added (pp. 234–239) dealing elementarily with axisymmetric and persymmetric determinants and with even-ordered determinants that have a zero in every $(r, s)^{th}$ place where $r+s$ is odd, and that consequently are expressible as the product of two determinants.*

* Muir's text-book (1882), § 81. The case where the factors of the product are both continuants should also be recalled (see above p. 402).

RUSSELL, A. (1888).

[Question 9556. *Educ. Times*, xli. p. 213; or *Math. from Educ. Times*, l. pp. 109–110.]

The determinant here brought forward, namely,

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e \\ c+d & d+e & e+a & a+b & b+c \\ \frac{1}{cd} & \frac{1}{de} & \frac{1}{ea} & \frac{1}{ab} & \frac{1}{bc} \\ \frac{a}{b+e-a} & \frac{b}{c+a-b} & \frac{c}{d+b-c} & \frac{d}{e+c-d} & \frac{e}{a+d-e} \end{vmatrix},$$

or R say,

appears as the eliminant of five equations in

$$c-d, \quad d-e, \quad e-a, \quad a-b, \quad b-c,$$

and thus as a quasi equivalent to the first member of the last of the five, the four others being manifest identities.

It would have given additional interest to show that if the said first member be denoted by S, then

$$\mathrm{R} = \frac{ab(c^2-a^2)+bc(d^2-b^2)+cd(e^2-c^2)+de(a^2-d^2)+ea(b^2-e^2)}{abcde} \cdot \mathrm{S};$$

in other words, that the value of R is

$$\frac{1}{abcde} \cdot \sum^{0} \frac{a(c-d)}{b+e-a} \cdot \sum^{0} ab(c^2-e^2).$$

. . . . . (1888).

[*Cambridge Univ. Exam. Papers*, xvii. p. 553.]

Like the circulant and others the determinant here appearing is one whose every row is a permutation of the same n quantities, the particular law of permutation requiring that one of the quantities be confined to the diagonal, and that in completing each row the others be taken in a fixed order. It is resolvable into n rational factors; for example,

$$\begin{vmatrix} x & a_1 & a_2 & a_3 \\ a_1 & x & a_2 & a_3 \\ a_1 & a_2 & x & a_3 \\ a_1 & a_2 & a_3 & x \end{vmatrix} = (x-a_1)(x-a_2)(x-a_3)(x+a_1+a_2+a_3)$$

(see *Hist.*, iii. p. 480).

CAPELLI, A. (1889).

[Sopra certi sviluppi di determinanti. *Rendic. . . . Accad. . . . Sci.* (Napoli), (2) iii. pp. 58–63.]

The main result here is a development of the determinant

$$\begin{vmatrix} a_{11}+x & a_{12} & a_{13} & \dots \\ a_{21} & a_{22}+x+1 & a_{23} & \dots \\ a_{31} & a_{32} & a_{33}+x+2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}_n, \text{ or } \Gamma_n \text{ say,}$$

in the form

$$X_n + X_{n-1}A_1 + \dots + X_1A_{n-1} + A_n,$$

where the X's are functions of x and the A's are functions of the coaxial minors of $|a_{1n}|$; namely,

$$X_n = x(x+1)(x+2)\dots(x+n-2)(x+n-1),$$
$$X_{n-1} = x(x+1)(x+2)\dots(x+n-2),$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

and

$$A_1 = a_{11} + a_{22} + a_{33} + \dots.$$
$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}+1 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33}+1 \end{vmatrix} + \dots.$$
$$A_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22}+1 & a_{23} \\ a_{31} & a_{32} & a_{33}+2 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22}+1 & a_{24} \\ a_{41} & a_{42} & a_{44}+2 \end{vmatrix} + \dots.$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

The proof is dependent on the assumption from another paper that such a development with the A's as specified but the X's of unknown form is actually possible, all then remaining to be done being to determine the X's by extreme specialization of the a's. On this account the following transformation in the case of the 4th order will be found more instructive:

Either of two analogous developments may be taken as the starting point, namely, that according to powers of x, or that according to products of

$$x, \quad x+1, \quad x+2, \quad x+3.$$

Choosing the latter we have

$$\begin{aligned}\Gamma_4 &= x(x+1)(x+2)(x+3)\\ &\quad + x(x+1)(x+2)\cdot d_4 + x(x+1)(x+3)\cdot c_3 + \ldots .\\ &\quad + x(x+1)\cdot|\,c_3d_4\,| + x(x+2)\cdot|\,b_2d_4\,| + \ldots . + (x+2)(x+3)\cdot|\,a_1b_2\,|\\ &\quad + x\cdot|\,b_2c_3d_4\,| + (x+1)\cdot|\,a_1c_3d_4\,| + (x+2)\cdot|\,a_1b_2d_4\,| + (x+3)\cdot|\,a_1b_2c_3\,|\\ &\quad + |\,a_1b_2c_3d_4\,|,\end{aligned}$$

and calling the expressions of these lines on the right T_1, T_2, T_3, T_4, T_5, we see or can readily show that

$$\begin{aligned}= X_4\\ = X_3A_1 + x(x+1)\cdot(c_3+2b_2+3a_1) + x(2b_2+6a_1) + 6a_1,\\ = x(x+1)\cdot\big(|\,c_3d_4\,|+\ldots+|\,a_1b_2\,|\big) + x\big(|\,b_2d_4\,|+2\,|\,b_2c_3\,|+\ldots\big) + \big(2\,|\,a_1d_4\,|+\ldots\big),\\ = x\big(|\,b_2c_3d_4+\ldots\big) + \big(|\,a_1c_3d_4\,|+\ldots\big),\\ = + |\,a_1b_2c_3d_4\,|;\end{aligned}$$

therefore, by addition,

$$= X_4+X_3A_1 + X_2A_2 + X_1A_3 + A_4$$

as desired.

The rest of the paper is occupied with unimportant deductions.

KILLING, W. (1889).

[De determinante quodam disquisitiones mathematicae. Sch. Prog. 16 pp. Braunsberg.]

The n-line determinant in question we may define for ourselves as having all its elements linear homogeneous functions of one and the same set of n variables, subject to the condition that in each column the coefficients of the variables form a zero-axial skew array; for example, n being 3 and the variables x, y, z, the determinant is

$$\begin{vmatrix} ay+bz & dy+ez & gy+hz\\ -ax+cz & -dx+fz & -gx+kz\\ -bx-cy & -ex-fy & -hx-ky \end{vmatrix}.$$

Its value is 0, as the operation

$$x\,\mathrm{row}_1 + y\,\mathrm{row}_2 + z\,\mathrm{row}_3$$

shows, and the author devotes himself to the consideration of its minors. Most interest attaches to those of the $(n-1)^{\text{th}}$ order, in

regard to which the result in effect is : *The* rth *row of the adjugate contains the* rth *variable as a factor, the cofactor throughout any column being constant.*

PALMSTRØM, A. (1890).

[Determinantteoriens anvendelse på laeren om complexe størrelsers multiplikation. *Archiv for Math. og Naturvid.*, xiii. pp. 128–132.]

Manifestly

$$(a_1+ib_1)(a_2+ib_2) = \begin{vmatrix} a_1 & b_1 \\ b_2 & a_2 \end{vmatrix} + i \begin{vmatrix} b_1 & -a_1 \\ b_2 & a_2 \end{vmatrix},$$

and consequently

$$(a_1+ib_1)(a_2+ib_2)(a_3+ib_3) = \left\{ a_3 \begin{vmatrix} a_1 & b_1 \\ b_2 & a_2 \end{vmatrix} - b_3 \begin{vmatrix} b_1 & -a_1 \\ b_2 & a_2 \end{vmatrix} \right\} + i \left\{ b_3 \begin{vmatrix} a_1 & b_1 \\ b_2 & a_2 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & -a_1 \\ b_2 & a_2 \end{vmatrix} \right\}$$

$$= \begin{vmatrix} \cdot & a_1 & b_1 & \cdot \\ & b_2 & a_2 & b_3 \\ & -a_2 & b_2 & a_3 \end{vmatrix} + i \begin{vmatrix} & b_1 & -a_1 & \cdot \\ & b_2 & a_2 & b_3 \\ & -a_2 & b_2 & a_3 \end{vmatrix}$$

where the two determinants differ from one another only in the first row, and are got from the two determinants of the previous case by affixing as a border the elements

$$\cdot \quad b_3 \quad a_3 \quad b_2 \quad -a_2$$

to each. Similarly

$$(a_1+ib_1)(a_2+ib_2)(a_3+ib_3)(a_4+ib_4)$$

$$= \begin{vmatrix} a_1 & b_1 & \cdot & \cdot \\ b_2 & a_2 & b_3 & -a_3 \\ -a_2 & b_2 & a_3 & b_3 \\ \cdot & \cdot & b_4 & a_4 \end{vmatrix} + i \begin{vmatrix} b_1 & -a_1 & \cdot & \cdot \\ b_2 & a_2 & b_3 & -a_3 \\ -a_2 & b_2 & a_3 & b_3 \\ \cdot & \cdot & b_4 & a_4 \end{vmatrix},$$

the border now being

$$\cdot \quad \cdot \quad b_4 \quad a_4 \quad b_3 \quad -a_3 \quad \cdot$$

in reverse order : the border for the next case

$$\cdot \quad \cdot \quad \cdot \quad b_5 \quad a_5 \quad b_4 \quad -a_4 \quad \cdot \quad \cdot$$

in the original order ; and so on. Any of the equalities may be verified by performing the addition indicated on the right and

removing the factor $a_1 + ib_1$, then lowering the order and removing the factor $a_2 + ib_2$, and so on.

Since

$$\text{os } \alpha + i \sin \alpha)(\cos \beta + i \sin \beta) \ldots . = \cos(\alpha+\beta+\ldots) + i \sin(\alpha+\beta+\ldots),$$

the author also readily deduces from his result determinant equivalents for $\cos(\alpha+\beta+\ldots)$ and $\sin(\alpha+\beta+\ldots)$.

TISSOT, A. (1890).

[Questions 289, 291. *Journ. de Math. Spéc.*, (3) iv. pp. 144, 168.]

What we have here is a pair of interesting 3-line determinants connected with the quaternary quadric, their values being

$$D \cdot \sqrt{\Delta} \cdot (U)_{w=1}, \quad D^2 \cdot \sqrt{\Delta} \cdot (U)_{w=1},$$

where

$$U = ax^2 + by^2 + cz^2 + dw^2 + 2exy + 2fxz + 2gxw + 2hyz + 2iyw + 2jzw,$$

Δ is the discriminant of U, and D is the cofactor of d in Δ.

VIVANTI, G. (1890).

[Alcune formole relative all' operazione Ω. *Rendic. del Circolo Mat. di Palermo*, iv. pp. 261–268.]

This, unlike Capelli's comprehensive paper of the same year (see above, p. 463), deals solely with the operational determinant—which the author prefers to symbolize by $\Omega^{(n)}_{(x_{11}\ldots x_{nn})}$—and not with its application to the Algebra of Quantics. First, he establishes (pp. 261–265) a theorem for facilitating the performance of the operation on a product of functions $\phi_1\phi_2\phi_3 \ldots \phi_r$, the proof being doubly gradational, from n to $n+1$ and from r to $r+1$. Then, using as a lemma the extensional of Laplace's expansion-theorem, he submits to the operation the s^{th} power of a determinant having more than the n^2 elements implied in the operator. The outcome of this is the interesting theorem

$$\Omega^{(n)}_{(x_{11}\ldots x_{nn})} \left| x_{11} \ldots x_{n+m,\,n+m} \right|^s$$
$$= \frac{(s+n-1)!}{(s-1)!} \left| x_{11} \ldots x_{n+m,\,n+m} \right|^{s-1} \cdot \left| x_{n+1,\,n+1} \ldots x_{n+m,\,n+m} \right|.$$

KRONECKER, L. (1890).

[Reduction der Systeme von n^2 ganzzahligen Elemente. *Crelle's Journ.*, cvii. pp. 135–136.]

The so-called 'reduction' here effected is the transformation of the determinant, without change of value, into one having all its non-diagonal elements 0 and each diagonal element a factor of the next following, the permissible operations being (1) interchange of two parallel lines accompanied by change of sign in one of them, (2) increase of a line by any parallel line.* A solution by Frobenius in 1878 (*Crelle's Journ.*, lxxxvi. p. 157) is referred to, but not H. J. S. Smith's of 1861 (see above, pp. 439–440).

CAYLEY, A. (1891).

[Note on the involutant of two binary matrices. *Messenger of Math.*, (2) xx. pp. 136–137; or *Collected Math. Papers*, xiii. pp. 74–75.]

All that directly concerns us here is the equality

$$\begin{vmatrix} 1 & a & a' & aa'+bc' \\ . & b & b' & ab'+bd' \\ . & c & c' & ca'+dc' \\ 1 & d & d' & cb'+dd' \end{vmatrix} = \begin{vmatrix} 1 & a' & a & aa'+cb' \\ . & b' & b & ba'+db' \\ . & c' & c & ac'+cd' \\ 1 & d' & d & bc'+dd' \end{vmatrix},$$

which we can best establish for ourselves by noting that the difference of the two determinants is

$$\begin{vmatrix} 1 & a & a' & aa'+bc'+aa'+cb' \\ . & b & b' & ab'+bd'+ba'+db' \\ . & c & c' & ca'+dc'+ac'+cd' \\ 1 & d & d' & cb'+dd'+bc'+dd' \end{vmatrix},$$

and vanishes because the last column is reducible to

$$bc'+cb'-ad'-a'd$$

$$\cdot$$

$$\cdot$$

$$cb'+bc'-a'd-ad'$$

by the operation

$$\mathrm{col}_4 - (a+d)\,\mathrm{col}_3 - (a'+d')\,\mathrm{col}_2.$$

* This with (1) permits the increase or diminishing of a line by any integral multiple of another line.

LONGCHAMPS, G. DE (1891).

[Sur les determinants troués. *Journ. de Math. Spéc.*, (3) v. pp. 9–12, 29–32, 54–56, 85–87.]

The determinants in question are those with one or more zero elements, and the problem set is the finding of the number of non-zero terms in their final development. The main result obtained is Cunningham's of 1874 (*Hist.*, iii. pp. 466–467); but indications are also given of the mode of procedure to be followed when, besides having a zero diagonal, the determinant has one or more zeros in the adjacent minor diagonal. In the latter connection Dickson's paper of 1879 (*Hist.*, iii. p. 467) is not referred to: indeed, no previous writer is mentioned throughout the entire paper.

LAISANT, C. A. (1891).

[Sur les permutations limitées. *Comptes rendus . . . Acad. des Sci.* (Paris), cxii. pp. 1047–1049.]

The result here given is in effect that *the number of arrangements of* n *letters* $a_1, a_2, a_3, \ldots, a_n$ *when certain of them may be debarred from occupying certain places of the arrangement is*

$$\frac{\partial^n (s_1 s_2 s_3 \ldots s_n)}{\partial a_1 \partial a_2 \partial a_3 \ldots \partial a_n},$$

where s_r *is the sum of those letters that may occupy the* r^{th} *place.* Thus the number of terms in the determinant

$$\begin{vmatrix} . & a_2 & a_3 & a_4 & . \\ . & . & b_3 & b_4 & b_5 \\ c_1 & . & . & c_4 & c_5 \\ d_1 & d_2 & . & . & d_5 \\ e_1 & e_2 & e_3 & . & . \end{vmatrix}$$

is

$$\frac{\partial^5}{\partial a\, \partial b\, \partial c\, \partial d\, \partial e} \{(c+d+e)(a+d+e)(a+b+e)(a+b+c)(b+c+d)\}.$$

The reason of course is that each term of the product, on being differentiated, produces 1 or 0 according as it is or is not one of the desired arrangements.

SEGAR, H. W. (1892).

[Question 11434. *Educ. Times*, xlv. p. 82.]

The proposition submitted is that the n-line determinant whose $(r,s)^{\text{th}}$ element is

$$\frac{\partial}{\partial x^r}\left\{x^{\frac{1}{2}r+s-1}\cdot \mathrm{J}_r(\sqrt{x})\right\}$$

is equal to

$$\frac{(n-1)!!}{2^n}\, x^{\frac{1}{2}n(n-1)}\cdot \mathrm{J}_0(\sqrt{x})\cdot \mathrm{J}_1(\sqrt{x})\ \ldots\ \mathrm{J}_{n-1}(\sqrt{x}),$$

where J_r stands for the Bessel's function of the r^{th} order, and $(n-1)!!$ for $1!\,2!\,\ldots\,(n-1)!$

ECHOLS, W. H. (1893).

[On a general formula for the expansion of functions in series. *Bull. New-York Math. Soc.*, ii. pp. 135–144.]

[Note on the theory of functions of a real variable. *Annals of Math.* viii. pp. 65–99.]

These are important continuations, each in its own way, of the papers referred to under 'Alternants.' The most general determinant now brought forward for use in the specification of expansions is

$$\begin{vmatrix} f(x) & f_1(x) & \ldots. & f_n(x) & f_{n+1}(x) \\ f(y_1) & f_1(y_1) & \ldots. & f_n(y_1) & f_{n+1}(y_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ f(y_p) & f_1(y_p) & \ldots. & f_n(y_p) & f_{n+1}(y_p) \\ \mathrm{O}f(x_1) & \mathrm{O}f_1(x_1) & \ldots. & \mathrm{O}f_n(x_1) & \mathrm{O}f_{n+1}(x_1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \mathrm{O}^q f(x_q) & \mathrm{O}^q f_1(x_q) & \ldots. & \mathrm{O}^q f_n(x_q) & \mathrm{O}^q f_{n+1}(x_q) \\ \Phi & . & \ldots. & . & 1 \end{vmatrix},$$

where $\mathrm{O}^r f(x_i)$ stands for the result of performing an operation O r times on $f(x)$ and substituting x_i for x, and Φ is an expression so chosen in relation to the other elements that the determinant vanishes. An expansion of $f(x)$ thence evidently obtainable is of the form

$$f(x) = \mathrm{A}_1 f_1(x) + \mathrm{A}_2 f_2(x) + \ldots + \mathrm{A}_{n+1} f_{n+1}(x).$$

Carefully proved specializations of this are made under three headings, (1) where O does not occur, q being 0, (2) where O is d/dx, (3) where O is the Δ of Finite Differences. Only the first need be given as a specimen, namely,

$$\begin{vmatrix} f(x) & f_1(x) & \dots & f_n(x) & f_{n+1}(x) \\ f(x_1) & f_1(x_1) & \dots & f_n(x_1) & f_{n+1}(x_1) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f(x_n) & f_1(x_n) & \dots & f_n(x_n) & f_{n+1}(x_n) \\ \Phi(u) & . & \dots & . & 1 \end{vmatrix} = 0,$$

where

$$\Phi(u) = \begin{vmatrix} f(u) & \dots & f_n(u) \\ f(x_1) & \dots & f_n(x_1) \\ \cdot & \cdot & \cdot \\ f(x_n) & \dots & f_n(x_n) \end{vmatrix} \div \begin{vmatrix} f_1{}^n(u) & \dots & f_{n+1}^n(u) \\ f_1(x_1) & \dots & f_{n+1}(x_1) \\ \cdot & \cdot & \cdot \\ f_1(x_n) & \dots & f_{n+1}(x_n) \end{vmatrix}$$

and u is some unknown quantity lying between the greatest and least values of x, x_1, . . . , x_n. A simple case of this,

$$\begin{vmatrix} f(x) & 1 & x & \dots & x^{n-1} & x^n \\ f(a_1) & 1 & a_1 & \dots & a_1{}^{n-1} & a_1{}^n \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ f(a_n) & 1 & a_n & \dots & a_n{}^{n-1} & a_n{}^n \\ \dfrac{f^n(u)}{n!} & . & . & \dots & . & 1 \end{vmatrix} = 0,$$

is Lagrange's interpolation-formula.

HADAMARD, J. (1893).

[Sur le module maximum que puisse atteindre un déterminant. *Comptes Rendus . . . Acad. des Sci.* (Paris), cxvi. pp. 1500–1501.]

[Résolution d'une question relative aux déterminants. *Bull. des Sci. Math.*, (2) xvii. pp. 240–246.]

The first result here established is that *if* $s_r{}^2$ *be the sum of the squares of the moduli of the elements of the* r^{th} *row of a determinant* δ, *then*

$$mod\ \delta \leqq s_1 s_2 s_3 \dots,$$

an evident deduction being that *if the moduli of the elements be at most* 1, *the modulus of the determinant is at most* $n^{\frac{1}{2}n}$. Attention is then given to determinants whose elements actually have 1 for modulus and whose own value attains the maximum specified. The important property of such determinants is found to be that *their elements are inversely proportional to the corresponding elements of the adjugate*, and consequently Sylvester's ' thoughts on inversely orthogonal matrices ' (1867) are called in as an aid, and are profitably passed under review. In particular the case where the order of the determinant is composite benefits by this, Hadamard's improved solution for the 4th order being

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & e^{i\theta} & -e^{i\theta} \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -e^{i\theta} & e^{i\theta} \end{vmatrix}.$$

In the next place is considered the formation of ' maximum ' determinants with *real* elements, namely, 1 or -1. These exist when n is a power of 2, and are shown to be non-existent when n is not a multiple of 4. Setting himself the problem of formation in the case of the first two important multiples of 4, namely, 12 and 20, Hadamard is successful, the 12-line determinant obtained being

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - \\ 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & - \\ - & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 \\ - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 \\ - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - \\ 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - \\ 1 & - & 1 & - & 1 & - & - & - & 1 & - & 1 & 1 \\ 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 \\ 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & 1 \\ 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & - \\ 1 & 1 & - & - & - & 1 & - & 1 & - & - & 1 & 1 \end{vmatrix},$$

where $-$ is used for -1. This is unintentionally corroborated later by Scarpis (see below p. 495).

PASCH, M. (1893).

[Verschwindende Determinanten dritten Grades aus ternären linearen Formen. *Math. Annalen,* xliv. pp. 89–96.]

The opening result here is the simple theorem that *if all the primary minors of the determinant* $|f_{11}f_{22}f_{33}|$ *vanish, the* f's *being functions of any variables, then the determinant can be put in the form*

$$\begin{vmatrix} a\phi_1 & \beta\phi_1 & \gamma\phi_1 \\ a\phi_2 & \beta\phi_2 & \gamma\phi_2 \\ a\phi_3 & \beta\phi_3 & \gamma\phi_3 \end{vmatrix}, \quad i.e. \quad a\beta\gamma\phi_1\phi_2\phi_3 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix},$$

where a, β, γ, ϕ_1, ϕ_2, ϕ_3 *are integral functions.* The real interest of the paper, however, is not in connection with determinants.

MACMAHON, P. A. (1893).

[A certain class of generating functions in the theory of numbers. *Philos. Transac. R. Soc. London,* clxxxv. pp. 110–160.]

The results of this memoir, the first of which is most important in the Theory of Integers, are also very interesting as properties of determinants. The main theorem (pp. 114–119) is : *If* X_r *stand for* $a_{r1}x_1 + a_{r2}x_2 + \ldots + a_{rn}x_n$, *and*

$$\{(1-s_1X_1)(1-s_2X_2)\ldots(1-s_nX_n)\}^{-1}$$

be expanded in ascending powers and products of the x's, *the aggregate of all the terms of the expansion that contain no* x *without the corresponding* s—*in other words, the aggregate that involves only* s_1x_1, s_2x_2, . . . *and the* a's—*is equal to*

$$\begin{vmatrix} 1-a_{11}\cdot s_1x_1 & -a_{12}\cdot s_1x_1 & \ldots. & -a_{1n}\cdot s_1x_1 \\ -a_{21}\cdot s_2x_2 & 1-a_{22}\cdot s_2x_2 & \ldots. & -a_{2n}\cdot s_2x_2 \\ \ldots\ldots\ldots\ldots & & & \\ -a_{n1}\cdot s_nx_n & -a_{n2}\cdot s_nx_n & \ldots. & 1-a_{nn}\cdot s_nx_n \end{vmatrix}^{-1}.$$

The proof consists in showing that the whole expansion is equal to the portion in question multiplied by a series of which the first term is 1 and all the others are free of s_1x_1, s_2x_2, ... The second theorem (pp. 133–138) concerns determinants like Horner's (1865), whose elements are 'inversely-axisymmetric,' and is to the effect that *any such determinant is transformable into one of the same kind*

in which all the elements of the two minor diagonals adjacent to the main diagonal are 1. This depends on the fact that a determinant is not altered in substance nor its diagonal even in form if a row be multiplied and the corresponding column divided by the same quantity; for example, if we want the element a_{23} and its conjugate in

$$\begin{vmatrix} 1 & a_{12} & a_{13} & a_{14} \\ \frac{1}{a_{12}} & 1 & a_{23} & a_{24} \\ \frac{1}{a_{13}} & \frac{1}{a_{23}} & 1 & a_{34} \\ \frac{1}{a_{14}} & \frac{1}{a_{24}} & \frac{1}{a_{34}} & 1 \end{vmatrix},$$

to be replaced by units without disturbing the present diagonal, we multiply the 3rd row and divide the third column by a_{23}. The n-line inversely-axisymmetric determinant is thus seen to be expressible as a function of $\frac{1}{2}(n-1)(n-2)$ variables.*

SCHUMACHER, J. (1894).

[Ueber Determinanten. Sch.-Festschrift, pp. 55–75. Neustadt a. Haardt.]

There is here broached the subject of determinants whose rows are all of them permutations of the same set of quantities. Taking, for example, the six permutations of a, b, c, and examining the twenty determinants formable by segregating three of the six, he ascertains that there is only one distinct type in addition to the circulant. When the set of quantities is $a_1, a_2, \ldots, a_n$ and $n > 3$, he instances two types beside the circulant, namely, (1) where the k^{th} row is got from the first by interchanging a_k and a_1, (2) where the k^{th} row is got from the first by passing a_k over those preceding it to occupy the first place. The values in these two cases are respectively

$$\Sigma a \cdot (a_1 - a_2)(a_1 - a_3) \ldots (a_1 - a_n),$$
$$\Sigma a \cdot (a_1 - a_2)(a_2 - a_3) \ldots (a_{n-1} - a_n),$$

the former having been previously noted (see above, p. 475).

* Opportunity may be taken here to note that a very special determinant of this form is evaluated by A. Hirsch in *Math. Annalen*, lii. (1899) p. 151.

To the circulant * and certain applications of it a special section is given (pp. 66–74).

MUSSO, G. (1894).

[Sulle permutazioni relative ad una data. *Rivista di Mat.*, iv. pp. 109–119.]

Merely touches on the question of the number of terms in a zero-axial determinant.

TUCKER, R. (1894, 1895): CRAWFORD, G. E. (1895).

[Questions 12394, 12828. *Educ. Times*, xlvii. p. 273, xlviii. p. 304; or *Math. from Educ. Times*, lxiii. pp. 83–84.]

[Question 12684. *Educ. Times*, xlviii. p. 160; or *Math. from Educ. Times*, lxiv. p. 38.]

The results are:

$$\begin{vmatrix} \cos A - \cos 2A \cos(B-C) & 2\cos^2 A \cos B & 2\cos^2 A \cos C \\ 2\cos^2 B \cos A & \cos B - \cos 2B \cos(C-A) & 2\cos^2 B \cos C \\ 2\cos^2 C \cos A & 2\cos^2 C \cos B & \cos C - \cos 2C \cos(A-B) \end{vmatrix}$$

$$= 2\cos A \cos B \cos C \cdot \sin 2A \sin 2B \sin 2C.$$

$$\begin{vmatrix} de+a & . & . & ae+d & ad \\ . & de-b & . & be-d & bd \\ . & ce+b & be+c & . & be \\ ce-a & . & ae-c & . & ae \\ 1 & 1 & 1 & 1 & . \end{vmatrix}$$

$$= (e^2-1)(a+b)(c+d)\{cd(a+b)-ab(c+d)\}.$$

The latter we may profitably view in another form as the second of a set of three, the first of which is

$$\begin{vmatrix} ax+dy & . & . & dx+ay \\ . & bx-dy & . & dx-by \\ . & bx+cy & cx+by & . \\ ax-cy & . & cx-ay & . \end{vmatrix} = -xy(x^2-y^2)(a+b)(c+d)(ab-cd),$$

* Note should have been taken of this in Chap. XVI. above, p. 378.

the second arising from the first by bordering symmetrically with 1, 1, 1, 1, 0, and the third being got from the first two by addition.

BOOTH, W. (1895).

[Question 12952. *Educ. Times*, xlviii. p. 483.]

The interesting theorem here given concerns a quaternary quadric and its discriminant. It is that if

$$\mathrm{Q} \equiv ax^2 + by^2 + cz^2 + dw^2 + 2hxy + 2gxz + 2lxw + 2fyz + 2myw + 2nzw,$$

$$\mathrm{L} \equiv \tfrac{1}{2}\partial\mathrm{Q}/\partial x, \quad \mathrm{M} \equiv \tfrac{1}{2}\partial\mathrm{Q}/\partial y, \quad \mathrm{N} \equiv \tfrac{1}{2}\partial\mathrm{Q}/\partial z,$$

then

$$\begin{vmatrix} a\mathrm{Q}-\mathrm{L}^2 & h\mathrm{Q}-\mathrm{LM} & g\mathrm{Q}-\mathrm{LN} \\ h\mathrm{Q}-\mathrm{LM} & b\mathrm{Q}-\mathrm{M}^2 & f\mathrm{Q}-\mathrm{MN} \\ g\mathrm{Q}-\mathrm{LN} & f\mathrm{Q}-\mathrm{MN} & c\mathrm{Q}-\mathrm{N}^2 \end{vmatrix} = \mathrm{Q}^2 w^2 \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}.$$

In default of a proof we suggest, as a starting point, bordering with

$$\mathrm{L},\ \mathrm{M},\ \mathrm{N},\ 1,$$

and $0,\ 0,\ 0,\ 1,$

for a last row and column ; also, expanding in terms of determinants with monomial elements. Both lead to pleasing bits of transformation.

CAPELLI, A. (1895) : CAZZANIGA, T. (1896).

[Sur les déterminants dont les éléments principaux varient en progression arithmétique. *Nouv. Annales de Math.*, (3) xiv. pp. 62–63.]

[Sopra i determinanti di cui gli elementi principali variano in progressione aritmetica. *Rendic. . . . Ist. Lombardo* (Milano), (2) xxix. pp. 541–558.]

The subject of both papers here is an extension of Capelli's result of 1889, the new statement being got from the old by changing $1, 2, \ldots, n-1$ into $y, 2y, \ldots, (n-1)y$; for example,

$$\begin{vmatrix} a_1+x & a_2 & a_3 \\ b_1 & b_2+x+y & b_3 \\ c_1 & c_2 & c_3+x+2y \end{vmatrix} = x(x+y)(x+2y) + x(x+y)(a_1+b_2+c_3) + x\left\{\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2+y \end{vmatrix} + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3+y \end{vmatrix} + \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3+y \end{vmatrix}\right\} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2+y & b_3 \\ c_1 & c_2 & c_3+2y \end{vmatrix}.$$

The first paper contains little more than a mere enunciation and a call for proof; the second takes almost the form of an answer to the first. The proof given, however, is lengthy, the mode of it being the gradational.

NIELSEN, N. (1896): VALENTINER, H. (1898).

[En Determinantenformel. *Nyt Tidsskrift for Mat.*, B. vii. pp. 59–62; B. ix., pp. 15–16.]

The result here obtained is, after a slight correction, that the determinant having $(ra-s)^{-1}$ for its $(r, s)^{\text{th}}$ element is equal to

$$(-a)^{\frac{1}{2}n(n-1)} \cdot \{1^{n-1}2^{n-2}3^{n-3} \ldots (n-1)^1\}^2 \div \mathrm{P},$$

where P stands for the product of the denominators of all the elements, that is to say, for $(a-1)\cdot(a-2)\ldots(2a-1)\ldots(na-n)$. The determinant may be viewed as a case of Cauchy's double alternant.

GRANTE, C. (1896).

[Question 742. *L'Intermédiaire des Math.*, iii. pp. 33, 191.]

The vague request being made to form a determinant with its $(r, r)^{\text{th}}$ elements equal to 1 and $(r, s)^{\text{th}}$ elements equal to $h_r - h_s + 1$, there is supplied a determinant whose first row and first column are

$$1, a, b, \ldots \text{ and } 1, 2-a, 2-b, \ldots,$$

and whose every other element in the $(r, s)^{\text{th}}$ place is 1 less than the sum of the elements in the places $(r, 1)$, $(1, s)$.

It is not noted, however, that the latter determinant vanishes for all orders above the second.

BRAND, E. (1896).

[Une propriété des mineurs des déterminants nuls. *Journ. de Math. Spéc.*, (4) v. pp. 104–106.]

This is a faulty paper as regards printing or reasoning or both. It is professedly based on the theorem regarding any minor of the m^{th} compound of $|\, a_{1n} \,|$ (*Hist.*, iii. p. 177), the case considered being that where $|\, a_{1n} \,| = 0$. It may be noted that the vanishing axi-symmetric determinant with arithmetical elements which the author brings forward to illustrate and support one of his results serves more effectively the opposite purpose.

JOLY, C. I. (1896).

[Quaternion invariants of linear vector functions, and quaternion determinants. *Proceed. R. Irish Acad.*, (3) iv. pp. 1–15.]

To quaternion determinants, as an auxiliary subject, Joly devotes a couple of pages at the outset. As is natural, the same convention is adopted as by Spottiswoode in his paper of 1876 on determinants of 'alternate numbers,' namely, that the elements forming a term must be taken in the order of the rows, the sign being left to be determined by the order of the columns. In consequence such a determinant must vanish when two columns are identical, and need not when two rows are identical. Similarly, a determinant is not altered when one column is increased by a multiple of another, but the same does not hold in the case of rows. Next, it is noted that the ordinary rule of multiplication is restricted in its application to the product of a quaternion determinant by a scalar determinant. A paragraph is then devoted to determinants having all their rows identical, another to the geometry connected with vanishing determinants, and after that the main subject of the paper is entered on.

WELSCH, . (1896): MONTESSUS, R. DE (1896, 1899).

[Déterminants à terme général de la forme $h_i - h_j + 1$ $(i, j = 1, \ldots, n)$. *L'Intermédiaire des Math.*, iii. p. 191.]

[Questions 906, 935, 1445. *L'Intermédiaire des Math.*, iii. p. 201, p. 248; vi. pp. 28, 257–258.]

It is not noted that the determinant specified vanishes for all

orders above the second, as the operation $row_2 - row_1$, $row_3 - row_1$ at once shows; it may also be effectively viewed as a product (see above p. 132). The three questions referred to lead to nothing of any consequence.

CARO, R. (1897): A. C. (1897):
D'AVILLEZ, J. F. (1897): LAISANT, C. A. (1898).

[Determinante de $(a+b)^m$. *Archivo de Mat.*, ii. pp. 68–70.]

[Question 1125. *Mathesis*, (2) vii. p. 150; viii. pp. 144–145.]

[Sobre algunas applicaçõnes dos determinantes á geometria do triangulo. *Jorn. de Sci. mat., phys., e nat.* (Lisboa), (2) v. pp. 14–42.]

[Déterminant de quatre points d'un plan par rapport à un cinquième point. *Jorn. de Sci. mat., phys., e nat.* (Lisboa), (2) v. pp. 205–206.]

In the second of these the result involved is perhaps best put in the form

$$\begin{vmatrix} xy(x-a) & -y(x+a) & x+a \\ y+b & yz(y-b) & -z(y+b) \\ -x(z+c) & z+c & zx(z-c) \end{vmatrix}$$

$$= (xyz-1)\left[x^2y^2z^2-(\mathrm{P}-\mathrm{Q}-2)\,xyz-(\mathrm{P}+\mathrm{Q}+1)\right] - (abc-1)\,(x^2y^2z^2+4xyz-1),$$

where P, Q stand for $ayz + bzx + cxy$, $bcx + cay + abz$ respectively.

PINCHERLE, S. (1897).

[Mémoire sur le calcul fonctionnel distributif. *Math. Annalen*, xlix. pp. 325–381.]

The special determinant which occurs here (in addition to the wronskian) is only new in appearance. In fact, by reversing the order of the columns, and dividing them in their new order by 1!, 2!, 3!, ... respectively, we obtain

$$\left| (n)_0 \cdot (n+1)_1 \cdot (n+2)_2 \cdot \ldots \right| = 1,$$

the opening result of Chapter XX. of our third volume (pp. 447–448).

MUIR, T. (1897).

[The automorphic linear transformation of a quadric. *Transac. R. Soc. Edinburgh*, xxxix. pp. 209–230.]

The peculiar determinant

$$\begin{vmatrix} a & h+\nu & g-\mu & \dots \\ h-\nu & b & f+\lambda & \dots \\ g+\mu & f-\lambda & c & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

which had already appeared in a geometrical article * of Salmon's, takes a prominent place in the very different kind of discussion which we have now reached. It may be defined as the determinant whose matrix is the sum of an axisymmetric matrix and a zero-axial skew matrix. By reason of its importance considerable space (pp. 220–230) is given to the examination of it. The first noteworthy point connected with the development is that the terms of it are always of an even degree in the elements of the skew matrix; from which it follows of course that they are of an even or odd degree in the elements of the axisymmetric matrix according as the given determinant is even or odd-ordered. The next point is that the expression of the development is in terms of minors of the axisymmetric matrix and minors of the Pfaffian of the skew matrix, the word minor being used in the extended sense, that is to say, so as to include on the one extreme the elements, and on the other the determinant itself. In the third place, space being limited, the umbral notation is almost a necessity; and, this being so, as well as for the sake of explanation, it is best to give an example in two notations:

$$\begin{vmatrix} a & h+\nu & g-\mu & r+\tau \\ h-\nu & b & f+\lambda & q-\sigma \\ g+\mu & f-\lambda & c & p+\rho \\ r-\tau & q+\sigma & p-\rho & d \end{vmatrix}$$

$$= \begin{vmatrix} a & h & g & r \\ h & b & f & q \\ g & f & c & p \\ r & q & p & d \end{vmatrix} + \begin{array}{c|cccccc|} & \rho & \sigma & \lambda & \dots & \nu & \\ \hline & \begin{vmatrix} a & h \\ h & b \end{vmatrix} & \begin{vmatrix} a & g \\ h & f \end{vmatrix} & \dots & \dots & \dots & \rho \\ & \begin{vmatrix} a & h \\ g & f \end{vmatrix} & \dots & \dots & \dots & \dots & \sigma \\ & \dots & \dots & \dots & \dots & \dots & \dots \end{array} + \begin{vmatrix} & \lambda & \mu & \nu \\ & & & \rho \\ & & & \sigma \\ & & & \tau \end{vmatrix}^2,$$

* *Geometry of Three Dimensions* (1882), § 80 c, or 5th edit. (1912), § 80 e, pp. 70–72.

$$= \begin{vmatrix} 1&2&3&4 \\ 1&2&3&4 \end{vmatrix} + \begin{array}{|c|} \hline \| 34 \| \; \| 24 \| \; \| 23 \| \; \| 14 \| \; \| 13 \| \; \| 12 \| \\ \hline \begin{vmatrix} 12 \\ 12 \end{vmatrix} \; \begin{vmatrix} 12 \\ 13 \end{vmatrix} \quad \cdots\cdots \\ \begin{vmatrix} 13 \\ 12 \end{vmatrix} \quad \cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \hline \end{array} \begin{array}{|c|} \hline \| 34 \| \\ \hline \cdots \\ \cdots \\ \hline \end{array} + \| 1234 \|^2$$

The development in the next case is

$$\begin{vmatrix} 1&2&3&4&5 \\ 1&2&3&4&5 \end{vmatrix}$$

$$+ \begin{array}{|ccc|} \hline \| 45 \| & \| 35 \| & \cdots \; \| 12 \| \\ \hline \begin{vmatrix} 123 \\ 123 \end{vmatrix} & \begin{vmatrix} 123 \\ 124 \end{vmatrix} & \cdots \quad \begin{vmatrix} 123 \\ 345 \end{vmatrix} \\ \begin{vmatrix} 124 \\ 123 \end{vmatrix} & \cdots\cdots\cdots & \\ \cdots\cdots\cdots & & \\ \hline \end{array} \begin{array}{|c|} \hline \| 45 \| \\ \hline \| 35 \| \\ \cdots \\ \hline \end{array}$$

$$+ \begin{array}{|cc|} \hline \| 2345 \| & \| 1345 \| \quad \cdots\cdots \\ \hline \begin{vmatrix} 1 \\ 1 \end{vmatrix} & \begin{vmatrix} 1 \\ 2 \end{vmatrix} \quad \cdots\cdots \\ \begin{vmatrix} 2 \\ 1 \end{vmatrix} & \cdots\cdots \quad \cdots\cdots \\ \cdots\cdots\cdots & \\ \hline \end{array} \begin{array}{|c|} \hline \| 2345 \| \\ \hline \| 1345 \| \\ \cdots, \\ \hline \end{array}$$

which ought to make the general law clear, the square arrays being recognized as compound arrays of the given axisymmetric array.

It must suffice merely to add that the development of the primary minors of the given determinant is also gone into (§ 12), that the adjugate is shown to be of the same type as the original, and that sources of fresh identities are pointed out (§ 13).

STUDNIČKA, F. J. (1897).

[Neuer Beitrag . . . *Sitzungsb. . . . Ges. d. Wiss.* (Prag), no. 16, 16 pp.]

In the section (pp. 8–12) not occupied with alternants the main result is the equality of two special persymmetric determinants

$$\begin{vmatrix} (n)_k & (n)_{k-1} & (n)_{k-2} & \cdots \\ (n)_{k+1} & (n)_k & (n)_{k-1} & \cdots \\ (n)_{k+2} & (n)_{k+1} & (n)_k & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{m-k+1} = \begin{vmatrix} (n+m-k)_{m-k+1} & (n+m-k-1)_{m-k} & \cdots \\ (n+m-k+1)_{m-k+2} & (n+m-k)_{m-k+1} & \cdots \\ (n+m-k+2)_{m-k+3} & (n+m-k+1)_{m-k+2} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}_k.$$

Essentially, however, it is a rediscovery, being a variant of Zeipel's condensation-theorem of 1865,*—that is to say, his theorem asserting the equality of an $(r+1)$-line and a p-line determinant.

MUIR, T. (1898).

[The relations between the coaxial minors of a determinant of the fourth order. *Transac. R. Soc. Edinburgh*, xxxix. pp. 323–339.]

The second part of this paper (§§ 7–11) deals with the determinant

$$\begin{vmatrix} dl & cq & br & aqrl+2bcd \\ cp & dm & ar & brpm+2cda \\ bp & aq & dn & cpqn+2dab \\ al & bm & cn & dlmn+2abc \end{vmatrix}, \quad \text{or W say,}$$

seen to be a function of $a, b, c, d, l, m, n, p, q, r$, and to be separable into two determinants, U and V say. The curious result established is that all three determinants are resolvable into factors. In the case of W and U this is done by showing that they can be made to take the form of the determinant known to be resolvable into

$$(x+y+z+w)(x-y-z+w)(x-y+z-w)(x+y-z-w)\ ;$$

in fact, if this product be temporarily denoted by $\Phi(x, y, x, w)$, then

$$\mathrm{W} = \frac{1}{l^2m^2n^2}\Phi(dlmn-abc,\ cn\lambda\mu,\ bm\nu\lambda,\ al\mu\nu),$$

---

* The facts regarding Zeipel's determinant

$$\left| (m)_p (m+1)_{p+1} \ldots (m+r)_{p+r} \right|$$

are perhaps best put as follows: There are at least four forms of it of the $(r+1)^{\text{th}}$ order (*Hist.*, iii. pp. 448–451), which for the moment we may call A_1, A_2, A_3, A_4. Then his condensation-theorem gives a form of the p^{th} order, and this being similar to A_1 in type, we have a set B_1, B_2, B_3, B_4 all of the p^{th} order. Consequently there are sixteen ways of writing the condensation-theorem. What Studnička arrives at is $A_2 = B_4$.

For the case $m, p, r = 5, 2, 2$ (or $n, k, m = 5, 2, 4$) we have already printed A_1, A_2, A_3, B_1: the remaining equivalents are

$$\begin{vmatrix} 7_2 & 8_3 & 9_4 \\ 6_1 & 7_2 & 8_3 \\ 5_0 & 6_1 & 7_2 \end{vmatrix}, \quad \begin{vmatrix} 6_3 & 6_4 \\ 7_3 & 7_4 \end{vmatrix}, \quad \begin{vmatrix} 6_3 & 6_4 \\ 6_2 & 6_3 \end{vmatrix}, \quad \begin{vmatrix} 6_3 & 7_4 \\ 7_3 & 8_4 \end{vmatrix}.$$

where

$$\lambda^2,\ \mu^2,\ \nu^2 = pl\left(1-\frac{a^2}{mnp}\right),\quad qm\left(1-\frac{b^2}{nlq}\right),\quad rn\left(1-\frac{c^2}{lmr}\right),$$

and $$U = \Phi(d\sqrt{lmn},\ c\sqrt{npq},\ b\sqrt{mrp},\ a\sqrt{lqr}).$$

In the case of V the procedure is different, the last column being readily transformable into a column of units, and leads to

$$V = 2(caq-dbm)(abr-cdn)(bcp-adl).$$

SCARPIS, U. (1898).

[Sui determinanti di valore massimo. *Rendic. . . . Istituto Lombardo,* (2) xxxi. pp. 1441–1446.]

Putting the last specialized result of Hadamard's paper of 1893 in the self-evident form,—*An* n-*line determinant whose elements are* 1 *or* -1 *will have the value* $n^{\frac{1}{2}n}$ *if the product of any row by any other row vanishes,*—Scarpis in his search for such determinants arrives at the theorem that *if there be one of order* q *and* $q-1$ *be a prime, then there exists an infinity of others of the order* $q(q-1)\ 2^{\lambda}$. As an example he takes the case where q is 4, and shows how to form the determinant of the 12th order.

The essential identity of the latter with Hadamard's might have been noted. As a matter of fact, if the transposition

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 3 & 2 & 6 & 4 & 5 & 10 & 11 & 12 & 8 & 9 & 7 \end{pmatrix}$$

be made in the order of the columns, and

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 10 & 12 & 11 & 4 & 6 & 5 & 7 & 9 & 8 \end{pmatrix}$$

in the order of the rows, the result is Hadamard's form.

METZLER, W. H. (1899).

[On a determinant each of whose elements is the product of k factors. *American Math. Monthly,* vii. pp. 151–153.]

Taking an equality from Muir's text-book (p. 117), Metzler first generalizes by obtaining in similar fashion the product of a deter-

minants of the β^{th} order by β determinants of the α^{th} order; for example,

$$\begin{vmatrix} a_1P_1 & a_1Q_1 & a_1R_1 & a_2X_1 & a_2Y_1 & a_2Z_1 \\ b_1P_1 & b_1Q_1 & b_1R_1 & b_2X_1 & b_2Y_1 & b_2Z_1 \\ c_3P_2 & c_3Q_2 & c_3R_2 & c_4X_2 & c_4Y_2 & c_4Z_2 \\ d_3P_2 & d_3Q_2 & d_3R_2 & d_4X_2 & d_4Y_2 & d_4Z_2 \\ e_5P_3 & e_5Q_3 & e_5R_3 & e_6X_3 & e_6Y_3 & e_6Z_3 \\ f_5P_3 & f_5Q_3 & f_5R_3 & f_6X_3 & f_6Y_3 & f_6Z_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & . & . & . & . \\ b_1 & b_2 & . & . & . & . \\ . & . & c_3 & c_4 & . & . \\ . & . & d_3 & d_4 & . & . \\ . & . & . & . & e_5 & e_6 \\ . & . & . & . & f_5 & f_6 \end{vmatrix} \cdot \begin{vmatrix} P_1 & . & P_2 & . & P_3 & . \\ Q_1 & . & Q_2 & . & Q_3 & . \\ R_1 & . & R_2 & . & R_3 & . \\ . & X_1 & . & X_2 & . & X_3 \\ . & Y_1 & . & Y_2 & . & Y_3 \\ . & Z_1 & . & Z_2 & . & Z_3 \end{vmatrix}$$

$$= -|a_1b_2| \cdot |c_3d_4| \cdot |e_5f_6| \cdot |P_1Q_2R_3| \cdot |X_1Y_2Z_3|.$$

He then takes γ determinants of the newly found type, and $\alpha\beta$ determinants of the γ^{th} order, and, constructing therewith two determinants as before, performs multiplication, the result being a determinant of the $(\alpha\beta\gamma)^{\text{th}}$ order with trifactorial elements and equal to the product of $\beta\gamma$ determinants of the α^{th} order, $\gamma\alpha$ determinants of the β^{th} order, and $\alpha\beta$ determinants of the γ^{th} order. Imagining this process to be continued until a determinant is reached with elements that are the product of k factors, he courageously sets about formulating the result.

It has to be recalled that the proof, given by Sterneck in 1895 (see above, p. 62), of Zehfuss' theorem proceeds on the same lines as this; and that, indeed, Metzler here effects a generalization of Zehfuss' theorem as well as of the simple equality given by Muir.

PEIRCE, J. M. (1899).

[Determinants of quaternions. *Bull. American Math. Soc.*, v. pp. 335–338.]

As this must have been written without knowledge of Joly's paper of 1896, there is naturally considerable repetition in connection with the elementary theorems. A fairly large share of space is devoted to determinants having all their rows identical.

STUDNIČKA, F. J. (1899).

[O součtových determinantech vůbec a figurovaných zvlást, *Rozpravy České Akad.*, (2) ix. no. 4, 8 pp.]

The determinant here called ' summatory '—and in a special case ' figurate '—is a rediscovery (see *Hist.*, iii. pp. 102–3, 493). Some simple results are given regarding the so-called ' figurate,' in which the $(r, s)^{\text{th}}$ element is the combinatory number $(r+s-2)_{s-1}$.

STUDNIČKA, F. J. (1899).

[Nové doplňky nauky determinantní jakož i upotřebení jejího. *Věstník České Akad.* . . . , viii. pp. 59–67, 197–208.]

The last and the longest part (pp. 201–208) of this clearly written paper concerns Hermite's determinants with complex elements. It is unfortunate, however, that the author views them as being for the first time introduced to notice : there is consequently little or nothing to be recorded.

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